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WEYL-GROUPS, SUPERCURRENTS AND COVARIANT LATTICES - II

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ABSTRACT

A recent construction of bosonic realizations of world sheet supersymmetry is applied to a class of string theories with left and right world sheet supersymmetries ((1,1) theories). A simple formula for the number of generations in asymmetric (1,1) lattice theories is derived. For the subclass of lattices that can be obtained by single shifts, we enumerate all (1,1) theories and obtain their complete massless spectra. We show how every realization of world sheet supersymmetry implies a θ -function identity, and give some of these identities.

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1. Introduction

In a previous paper [1], hereinafter referred to as **I**, we described a broad class of supercurrents for string theories arising from covariant lattices[†]. This opens the way to the construction of many new theories with (p, q) supersymmetry on the world sheet. The vast majority of these theories will have just one supersymmetry for the left-moving sector (the bosonized *NSR* model). Here, however, we will restrict our attention to those models that have at least $(1, 1)$ supersymmetry. In heterotic strings there is no physical reason for imposing any world sheet supersymmetry on the left sector, nevertheless such models have been discussed extensively, and most particularly the $(2, 2)$ models. The reason is partly historical, since such models arise in Calabi-Yau compactification, and partly practical because such models almost always have a spectrum with generations of chiral (16) 's of $SO(10)$ or (27) 's of E_6 . As we will see, $(1, 1)$ models have several other elegant properties that make them easier to analyse than the models without world sheet supersymmetry for the left-movers.

As shown in **I** and in [2], these theories can all be thought of as Weyl-twisted asymmetric orbifolds, but in our experience it is considerably simpler to describe them in terms of lattices. In particular it is far more straightforward to compute the spectrum of the theory. Moreover, as discussed in **I**, it is elementary to understand the rôle of the “mod 2” conditions of [3], and how they can be ignored by merely doubling the order of the shift.

In this paper we examine the massless spectra of a large class of $(1, 1)$ theories. In section 2 we start by considering the problem of counting the chiral generations in any $(1, 1)$ theory obtained by a shift-vector applied to some other $(1, 1)$ theory. We derive a simple algorithm for obtaining the number of chiral generations of the new theory from the character valued partition function of the original theory. The theories that can be generated by the shift vector method include the set of Weyl twisted asymmetric orbifolds, and it is clear that our algorithm for determining chiral generations will generalize to $(1, 1)$ asymmetric orbifolds with arbitrary twists. In section 3 we consider the special case where the original lattice (prior to application of the shift) is $(E_8 \times E_8)_L \times \Gamma_{6,6} \times (E_8)_R$, and obtain a simple generalization of the symmetric orbifold result [4] for the number of chiral generations.

[†] This procedure provides us with a natural choice for a shift vector that can be used to convert a torus compactified ten-dimensional heterotic string into a non-trivial four-dimensional covariant lattice theory.

Although as yet no *simple* formula exists for obtaining the rest of the massless spectrum in the covariant lattice approach, it is straightforward though somewhat arduous to compute it directly. In section 4 we do this for the entire class of theories obtained from $(E_8 \times E_8)_L \times \Gamma_{6;6} \times (E_8)_R$ by means of a single shift. Level matching is automatically satisfied for any combinations of Weyl twists on left and right. Therefore classifying all these theories amounts to finding all even self-dual lattices $\Gamma_{6;6}$ that contain Lie algebra lattices on the left and right. We show how this can be done by generalizing the Euclideanization trick of refs. [5], [6]. With one exception, all these lattices turn out to be of the form discussed by Englert and Neveu [7]. Out of the 178 possible covariant lattices obtained by applying all possible single shifts, we find 21 different massless spectra with space-time supersymmetry (we also find a number of models without space-time supersymmetry, two of which have no tachyons). For the space-time supersymmetric models the number of massless generations is either 0, 12, 24, 36 or 48 and these numbers are realized in several distinct combinations of numbers of left- and right-handed fermions. The precise spectra are tabulated in section 4.

Finally, in section 5 we assemble a number of facts from this paper and the previous one to establish some sets of θ -function identities that are a direct consequence of the realization of world sheet supersymmetry. In appendix A we supplement the discussion of Weyl groups in **I** with a number of facts relevant to the computations of this paper. In **I** we also gave the list of shift vectors corresponding to non-degenerate Weyl twists on simple, simply-laced Lie algebras of rank ≤ 8 , in appendix B we complete this list by giving the remaining such shifts for simple, simply-laced Lie algebras of rank > 8 .

2. Chiral generations

We will restrict our attention to those four-dimensional covariant lattice theories whose gauge group contains a D_5 factor, and we will content ourselves with the counting, at every mass level, of the difference between the number of right-handed fermions in the (s) conjugacy class of D_5 and the number of right-handed fermions in the (c) conjugacy class. (By CPT this is of course the negative of the same count taken on the left-handed fermions). At the massless level this corresponds to counting the number of (16) 's minus $(\overline{16})$'s, and if the D_5 is embedded in E_6 it is equivalent to counting (27) 's minus $(\overline{27})$'s of E_6 . It will become clear below that our methods will generalize to other space-time dimensions, and

can almost certainly be applied to other methods of constructing strings, and possibly to other gauge groups. In the even lattice formulation of covariant lattice theories, the NSR-fermions plus ghost are also described by a D_5 lattice, which upon restriction to light-cone gauge becomes a D_1 ($SO(2)$) helicity lattice^{*}.

If we describe the two D_5 's in terms of fermions, then the partition function we wish to compute is

$$P(\tau, \bar{\tau}) = \frac{1}{2} \text{Tr} [(-1)^{f_L} (-1)^{f_R} q^H \bar{q}^{\bar{H}}] \quad (2.1)$$

where the trace is restricted to the Ramond sector of the D_5 gauge group and to the Ramond sector of the D_1 transverse space-time rotation group. The operators f_L and f_R in (2.1) are the fermion numbers in these D_5 and D_1 factors. The overall factor of $\frac{1}{2}$ is because the trace counts the states in the following representations of the $D_5 \times D_1$ product:

$$((s) - (c)) \otimes ((s) - (c)) = (s) \otimes (s) + (c) \otimes (c) - ((c) \otimes (s) + (s) \otimes (c)) \quad (2.2)$$

which, by CPT, is twice the desired quantity. For Lorentz invariance (and other reasons) the string is required to be supersymmetric in the left-moving sector and so by the usual index theorem arguments [8] $P(\tau, \bar{\tau})$ must be a holomorphic function of τ . Moreover the complete partition function of the theory must have modular weight $(-1, -1)$, and since the function $P(\tau, \bar{\tau})$ is a restriction of this function to the coefficient of the Ramond vacua in the $D_5 \times D_1$ directions, it follows that $P(\tau, \bar{\tau})$ must itself be a function of modular weight $(4, 0)$. (This is because fixing the spinor components in the chiral Ramond sector is equivalent to appropriately differentiating the character valued partition function and setting the character parameters to zero [8]). However, the tachyon pole on the left is at worst q^{-2} , and since we are restricting to the chiral Ramond sector of the D_5 , we necessarily isolate a part of the partition function that contains a factor of $(\theta_1'(0|\tau))^5$. This means that as $\tau \rightarrow \infty$, $P(\tau)$ has a divergence of at most $q^{-\frac{3}{4}}$, but since $P(\tau)$ is a holomorphic modular

* We adopt the usual convention [6] of writing D_1 for a $U(1)$ whose charges are normalized to integers or half-integers. The classes (s) and (c) are those with charge $\frac{1}{2} \bmod 2$ and $-\frac{1}{2} \bmod 2$ respectively, the (v) conjugacy class has odd integer charges and (0) has even integer charges. In the following q refers to the left-moving gauge sector of the heterotic string.

function of weight 4 it must therefore be given by

$$P(\tau) = a G_4(\tau) \quad (2.3)$$

for some constant a , and where $G_4(\tau)$ is the Eisenstein function:

$$G_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \quad (2.4)$$

The entire chiral D_5 content of the space-time fermion spectrum is determined by one constant, and dictated by this single modular function.

In the covariant lattice formulation the theory is defined by an even self-dual lattice, $\Gamma_{22;14}$. The vectors $(\vec{v}_L, \vec{v}_R)$ on this lattice are always required to be such that the last five components of $\vec{v}_R$ belong to the $(s), (c), (o)$, or (v) conjugacy classes of D_5 . This D_5 is not to be confused with the one above: this D_5 is the one that encodes all the helicity and bosonized superghost information of the string. The lattice is also further constrained so as to admit a supercurrent on the nine “internal” dimensions of the right-moving lattice orthogonal to the D_5 of helicity and superghosts [9].

If we also require that there is a D_5 gauge group then the lattice must also have a D_5 factor on the left. Consider the character valued partition function of this lattice, $P_{\Gamma}(\vec{\lambda}_L, \vec{\lambda}_R; \tau, \bar{\tau})$, where the first five components, λ_L^i and λ_R^i , $i = 1 \dots 5$, of $\vec{\lambda}_L$ and $\vec{\lambda}_R$ define the characters of the left and right D_5 factors. Apart from the usual modular anomaly [8], P_{Γ} is a modular function of weight $(11, 7)$. If one expands this function about $\vec{\lambda}_L \equiv \vec{\lambda}_R \equiv 0$, the expansion will contain a term:

$$(2\pi)^{10} \lambda_L^1 \dots \lambda_L^5 \lambda_R^1 \dots \lambda_R^5 |\theta'_1(0|\tau)|^{10} F(\tau, \bar{\tau}) \quad (2.5)$$

Because this term is linear in the first five components of $\vec{\lambda}_L$ and $\vec{\lambda}_R$, the quadratic modular anomaly does not affect this term, and since $\vec{\lambda}_L$ and $\vec{\lambda}_R$ are defined to have modular weight $(1, 0)$ and $(0, 1)$ respectively, the function $|\theta'_1|^{10} F$ has modular weight $(16, 12)$. To truncate

away the unphysical ghost part of the lattice and add in the partition function of the physical bosonic oscillators (24 on the left and 12 on the right), one must remove a factor of $(\theta'_1(0|\bar{\tau}))^4$ and divide by $(\eta(\tau))^{24} (\eta(\bar{\tau}))^{12}$. This is of course equivalent to dividing by the weight (12, 12) modular function $|\eta(\tau)|^{48}$. Hence the function of interest is a modular function of weight (4, 0) given by

$$P(\tau, \bar{\tau}) = \frac{|\theta'_1(0|\tau)|^{10}}{|\eta(\tau)|^{48}} F(\tau, \bar{\tau}) = \frac{F(\tau, \bar{\tau})}{|\eta(\tau)|^{18}} \quad (2.6)$$

and world sheet supersymmetry implies that this function is holomorphic.

We note in passing that if the left-hand side of $\Gamma_{22;14}$ contains a factor of E_8 as well as a D_5 gauge factor then the E_8 root lattice contributes a simple multiplicative factor of $G_4(\tau)$ to $F(\tau, \bar{\tau})$. Since $\frac{P(\tau, \bar{\tau})}{G_4(\tau)}$ is a constant, this means that this constant is determined entirely from the projection of Γ onto the (9; 9) Lorentzian lattice orthogonal to the $(E_8 \times D_5)_L \times (D_5)_R$ factors of Γ .

We now consider how to obtain the function $P(\tau, \bar{\tau})$ for a theory that is obtained from another covariant lattice theory by applying a shift. (For details of the technique see [1],[2], [10].) Indeed, suppose that the theory on $\Gamma_{22;14}$ can be obtained from a lattice $\Gamma'_{22;14}$ by applying the shift $\vec{\sigma} = (\vec{\sigma}_L, \vec{\sigma}_R)$. For simplicity we assume that $\vec{\sigma}$ has even Lorentzian length. Let N be the order of $\vec{\sigma}$ on Γ' , and let $P_{\Gamma'}(\vec{\lambda}_L, \vec{\lambda}_R, \tau, \bar{\tau})$ be the character valued partition function of Γ' . Define a function $Q_{\Gamma'}$ by

$$\begin{aligned} Q_{\Gamma'}(\vec{\rho}, \vec{\lambda}, \tau, \bar{\tau}) &= \exp[i\pi(\tau(\vec{\rho}_L)^2 - \bar{\tau}(\vec{\rho}_R)^2) + 2\pi i(\vec{\rho}_L \cdot \vec{\lambda}_L - \vec{\rho}_R \cdot \vec{\lambda}_R)] \\ &\times P_{\Gamma'}(\vec{\lambda}_L + \vec{\rho}_L \tau, \vec{\lambda}_R + \vec{\rho}_R \bar{\tau}, \tau, \bar{\tau}) \end{aligned} \quad (2.7)$$

Explicitly in terms of the lattice sum this is simply:

$$\begin{aligned} Q(\vec{\rho}, \vec{\lambda}, \tau, \bar{\tau}) &= \sum_{(\vec{v}_L, \vec{v}_R) \in \Gamma'} \exp[i\pi(\vec{v}_L + \vec{\rho}_L)^2 \tau - i\pi(\vec{v}_R + \vec{\rho}_R)^2 \bar{\tau} \\ &\quad + 2\pi i(\vec{\lambda}_L \cdot (\vec{v}_L + \vec{\rho}_L) - \vec{\lambda}_R \cdot (\vec{v}_R + \vec{\rho}_R))] \end{aligned} \quad (2.8)$$

To obtain the character valued partition function for the new lattice, Γ , we first define

$$X(k, l, \vec{\lambda}, \tau, \bar{\tau}) \equiv Q(k\vec{\sigma}, l\vec{\sigma} + \vec{\lambda}, \tau, \bar{\tau}) \quad (2.9)$$

for $k, l = 0, 1, \dots, N-1$. It is then elementary to see that

$$P_{\Gamma}(\vec{\lambda}, \tau, \bar{\tau}) = \sum_{k=0}^{N-1} P_{\Gamma,k}(\vec{\lambda}, \tau, \bar{\tau}) \quad (2.10)$$

where

$$P_{\Gamma,k}(\vec{\lambda}, \tau, \bar{\tau}) = \frac{1}{N} \sum_{l=0}^{N-1} X(k, l, \vec{\lambda}, \tau, \bar{\tau}) \quad (2.11)$$

The sum in (2.11) projects out all vectors that have non-integral dot-product with $\vec{\sigma}$ while (2.10) assembles P_{Γ} shifted sector by shifted sector.

By using Gaussian resummation, and using the fact that Γ is even and self-dual, one can show that

$$Q\left(\vec{\rho}, \vec{\lambda}, -\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \tau^4 |\tau|^{14} \exp[2\pi i(\vec{\rho}_L \cdot \vec{\lambda}_L - \vec{\rho}_R \cdot \vec{\lambda}_R)] Q(-\vec{\lambda}, \vec{\rho}, \tau, \bar{\tau}) \quad (2.12)$$

and by more straightforward means one sees that

$$Q(\vec{\rho}, \vec{\lambda}, \tau + 1, \bar{\tau} + 1) = \exp[-i\pi((\vec{\rho}_L)^2 - (\vec{\rho}_R)^2)] Q(\vec{\rho}, \vec{\lambda} + \vec{\rho}, \tau, \bar{\tau}) \quad (2.13)$$

Now using the fact that $\vec{\sigma}$ has even Lorentzian length it follows that under modular transformations

$$X(k, l, \vec{\lambda}, \tau + 1, \bar{\tau} + 1) = X(k, k + l, \vec{\lambda}, \tau, \bar{\tau}) \quad (2.14)$$

$$X\left(k, l, \frac{\vec{\lambda}_L}{\tau}, \frac{\vec{\lambda}_R}{\bar{\tau}}, -\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \tau^4 |\tau|^{14} \exp\left[-\frac{i\pi}{\tau}(\vec{\lambda}_L)^2 + \frac{i\pi}{\bar{\tau}}(\vec{\lambda}_R)^2\right] X(-l, k, \vec{\lambda}, \tau, \bar{\tau}) \quad (2.15)$$

Now suppose that the original lattice $\Gamma'_{22;14}$ also defines a string theory with a D_5 gauge group, and that the shift vector $\vec{\sigma}$ has no components in the direction of both the D_5 gauge factor and the D_5 of the helicity and superghosts. In particular this guarantees that $\Gamma_{22;14}$ also has such D_5 factors inherited directly from Γ' . Moreover if we expand the

function $P_\Gamma(\vec{\lambda}, \tau, \bar{\tau})$ given by (2.10) and (2.11) about $\vec{\lambda} \equiv 0$ as in (2.5), we can isolate the contributions of each $X(k, l, \vec{\lambda}, \tau, \bar{\tau})$ to the function $F(\tau, \bar{\tau})$ and hence to the function $P(\tau, \bar{\tau})$ of (2.6). Denote these contributions to $P(\tau, \bar{\tau})$ by $Y(k, l, \tau, \bar{\tau})$. Since $\vec{\sigma}$ is orthogonal to the D_5 factors, (2.14) and (2.15) imply that the $Y(k, l, \tau, \bar{\tau})$ transform into each other under modular transformations. Indeed

$$Y(k, l, \tau + 1, \bar{\tau} + 1) = Y(k, k + l, \tau, \bar{\tau}) \quad (2.16)$$

$$Y\left(k, l, -\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \tau^4 Y(-l, k, \tau, \bar{\tau}) \quad (2.17)$$

Finally define

$$R(k, l, \tau, \bar{\tau}) = \frac{Y(k, l, \tau, \bar{\tau})}{G_4(\tau)} \quad (2.18)$$

Under an arbitrary element of the modular group R transforms according to

$$R\left(k, l, \frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) \equiv R(cl + dk, al + bk, \tau, \bar{\tau}) \quad (2.19)$$

Suppose that the original lattice, Γ' , has a supercurrent whose constraint vectors have integer dot product with $\vec{\sigma}$ [1]. Equivalently, suppose that the supercurrent on Γ' has local operator product with the operator $\exp[2\pi i \vec{\sigma} \cdot \vec{X}]$. It then follows from the usual supersymmetry argument that each of the functions $R(k, l, \tau, \bar{\tau})$ must be separately independent of $\bar{\tau}$. It follows from (2.19) that the functions R are modular invariants of the subgroup $\Gamma(N)$ of the modular group Γ [11]. This observation might help in the classification of the most general such R , but we will content ourselves with assuming that

1. The lattice Γ' factorizes to $\Gamma' = E_8 \times \Gamma'_{14;14}$, and hence $\Gamma'_{14;14}$ must be even and self-dual.
2. The shift $\vec{\sigma}$ vanishes inside the E_8 factor above, and so the E_8 factor is preserved on $\Gamma_{22;14} = E_8 \times \Gamma_{14;14}$.
3. The left-hand side of $\Gamma_{14;14}$ also admits a supercurrent in exactly the same sense that the right-hand side admits a supercurrent. *Note:* we are not assuming that the left and right supercurrents are constructed in the same way.

As we will see in later sections, there is a large number of examples satisfying these assumptions. The purpose in making these assumptions is that they imply that $R(k, l, \tau, \bar{\tau})$ is constant, and hence (2.19) implies that

$$R(k, l) = R(cl + dk, al + bk) \quad (2.20)$$

for any integers a, b, c and d such that $ad - bc = 1$. In particular if k and l are both non-zero and m is the highest common factor of k and l (denoted (k, l)) then we may choose c and d so that $cl + dk = m$, and to satisfy $ad - bc = 1$ we simply take $a = \frac{k}{m}$, $b = -\frac{l}{m}$. Hence

$$R(k, l) = R(m, mn); \quad m = (k, l), \quad n = \frac{1}{m^2}(k^2 - l^2) \quad (2.21)$$

The virtue of this identity is that it is elementary to deduce the contribution of a lattice vector $(\vec{v}_L, \vec{v}_R) \in \Gamma'$ to $R(m, mn)$, whereas the contribution to $R(k, l)$ would, in general, involve complicated phases. Define $\Delta_L = (\vec{v}_L + m\vec{\sigma}_L)^2$ and $\Delta_R = (\vec{v}_R + m\vec{\sigma}_R)^2$ then $2mn(\vec{\sigma}_L \cdot \vec{v}_L - \vec{\sigma}_R \cdot \vec{v}_R) \equiv n(\Delta_L - \Delta_R) \pmod{2}$ and thus the contribution of $(\vec{v}_L, \vec{v}_R)$ to the coefficient of $q^{\Delta_L} \bar{q}^{\Delta_R}$ in $X(m, mn, \vec{\lambda}, \tau, \bar{\tau})$ is

$$\exp[2\pi i(\vec{\lambda}_L \cdot \vec{v}_L - \vec{\lambda}_R \cdot \vec{v}_R)] \exp[i\pi n(\Delta_L - \Delta_R)] \quad (2.22)$$

Now, since $R(k, l)$ is a constant the only terms of importance in $X(k, l)$ are those with $\Delta_L = \Delta_R = 2$. (It is not $\Delta_L = \Delta_R = 0$ because of the factor of $|\eta(\tau)|^{-48}$ in Y and hence in R). It is then trivial to verify that a vector $\vec{v} \in \Gamma'$ contributes to $R(m, mn)$ if and only if $\vec{v}$ is a spinor in both D_5 factors and

$$(\vec{v}_L + m\vec{\sigma}_L)^2 = (\vec{v}_R + m\vec{\sigma}_R)^2 = 2 \quad (2.23)$$

Moreover this contribution is $+1$ if $\vec{v}$ is in the $((s), (s))$ or $((c), (c))$ class and -1 for the $((c), (s))$ or $((s), (c))$ classes.

One useful consequence of this observation is that $R(m, mn)$ is, in fact, independent of n , and hence we can choose $n = 0$. Therefore

$$R(k, l) = R(m, 0) = R(0, m), \quad m = (k, l) \quad (2.24)$$

where the last equality follows from using (2.20) and the modular transformation $\tau \rightarrow -\frac{1}{\tau}$. It is this last identity that is of central importance in counting the number of generations. (We note that even though we assumed, after (2.20), that k and l were both non-zero, the second identity in (2.24) is still valid).

So to summarize, the number of generations is given by

$$\frac{1}{2N} \sum_{k,l=0}^{N-1} R(k, l) = \frac{1}{2N} \sum_{k,l=0}^{N-1} R(0, (k, l)) \quad (2.25)$$

3. Shifted Narain Lattices

Our starting point will be a Narain-style compactification to four dimensions of the ten-dimensional $E_8 \times E_8$ heterotic string. The corresponding even, self-dual lattice is $\Gamma'_{22,14} = E_8 \times E_8 \times \Gamma'_{6,6} \times E_8$, where the first two E_8 factors are on the left lattice, the last E_8 factor is on the right, and $\Gamma'_{6,6}$ is an even, self-dual lattice in $6 + 6$ dimensions.

For the lattice $\Gamma'_{22,14}$ the D_5 of helicity and bosonized ghosts is embedded as $D_5 \times D_3 \subset E_8$ on the right. We take the gauge D_5 to be embedded in the same way in one of the E_8 factors on the left. Introduce a shift vector, with even Lorentzian length, of the form

$$\vec{\sigma} = (\vec{0}, \vec{0}, \vec{\theta}, \vec{\mu}; \vec{\nu}, \vec{\phi}, \vec{0}) \quad (3.1)$$

where the first three entries refer to the $E_8 \times D_5 \times D_3$ factors on the left, the last two entries refer to $D_3 \times D_5$ on the right, and $\vec{\mu}$ and $\vec{\nu}$ are shift components on the left and right parts of $\Gamma'_{6,6}$. We will assume, as we did at the end of the last section, that there are supercurrents on the right and left sides of $\Gamma'_{22,14}$ (excluding the unbroken, 'hidden' E_8 gauge sector) such that $\exp[2\pi i \vec{\sigma} \cdot \vec{X}]$ has local operator products with both supercurrents.

We note that one way of accomplishing this is as follows. Assume that $\Gamma'_{6;6}$ contains, on each side, the root vectors of some semi-simple, simply-laced Lie algebra of rank 6. Denote these Lie algebras by $\mathcal{G}_L$ and $\mathcal{G}_R$ respectively. (We do not assume that $\mathcal{G}_L = \mathcal{G}_R$). Choose $\vec{\mu}$ (respectively $\vec{\nu}$) to be any shift vector corresponding to a Weyl transformation, g_L , (respectively, g_R) of $\mathcal{G}_L$ (respectively $\mathcal{G}_R$). The vectors $\vec{\theta}$ and $\vec{\phi}$ are then chosen so that $e^{\pm 2\pi i \theta_j}$ and $e^{\pm 2\pi i \phi_j}$ are the eigenvalues of g_L and g_R on the Cartan sub-algebra. As will be seen in the next section it is always possible to choose the vectors $\vec{\theta}, \vec{\phi}, \vec{\mu}$ and $\vec{\nu}$ so that $\vec{\sigma}$ has even Lorentzian length. The required supercurrents are then obtained by the construction in **I**.

Irrespective of how we obtain the supercurrents, we have satisfied the conditions cited in the last section, and so the functions $R(k, l)$ defined there are constants. Moreover, for the lattices presently under consideration we have

$$R(k, l) = |\eta(\tau)|^{-18} \left\{ \prod_{j=1}^3 \left[e^{i\pi k(\theta_j - \phi_j)} \theta \left[\begin{matrix} k\theta_j - \frac{1}{2} \\ l\theta_j - \frac{1}{2} \end{matrix} \right] (0|\tau) \overline{\theta \left[\begin{matrix} k\phi_j - \frac{1}{2} \\ l\phi_j - \frac{1}{2} \end{matrix} \right] (0|\tau)} \right] \right\} \times \left\{ S(k(\vec{\mu}, \vec{\nu}), l(\vec{\mu}, \vec{\nu}), \tau, \bar{\tau}) \right\} \quad (3.2)$$

where, as usual [12]

$$\theta \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right] (0|\tau) = \sum_{n=-\infty}^{\infty} \exp [i\pi(n + \alpha)^2 \tau + 2\pi i(n + \alpha)\beta] \quad (3.3)$$

and

$$S(\vec{\rho}, \vec{\lambda}, \tau, \bar{\tau}) = \sum_{(\vec{w}_L, \vec{w}_R) \in \Gamma'_{6;6}} \exp [i\pi(\vec{w}_L + \vec{\rho}_L)^2 \tau - i\pi(\vec{w}_R + \vec{\rho}_R)^2 \bar{\tau} + 2\pi i(\vec{w}_L + \vec{\rho}_L) \cdot \vec{\lambda}_L - 2\pi i(\vec{w}_R + \vec{\rho}_R) \cdot \vec{\lambda}_R] \quad (3.4)$$

Observe that

$$\begin{aligned}
 R(0, m) = |\eta(\tau)|^{-18} & \left\{ \prod_{j=1}^3 \left[\theta_1(m\theta_j|\tau) \overline{\theta_1(m\phi_j|\tau)} \right] \right\} \\
 & \times \left\{ \sum_{(\vec{w}_L, \vec{w}_R) \in \Gamma'_{6,6}} \exp \left[i\pi(\vec{w}_L)^2 \tau - i\pi(\vec{w}_R)^2 \bar{\tau} + 2\pi i m (\vec{w}_L \cdot \vec{\mu} - \vec{w}_R \cdot \vec{\nu}) \right] \right\}
 \end{aligned} \tag{3.5}$$

Obviously the lowest power of q and $\bar{q}$ that can appear in this expression occurs when $\vec{w}_L = \vec{w}_R = \vec{0}$, and since

$$\theta_1(\nu|\tau) = 2 \sin(\pi\nu) \eta(\tau) q^{\frac{1}{6}} \prod_{n=1}^{\infty} (1 - q^{2n} e^{2\pi i \nu}) (1 - q^{2n} e^{-2\pi i \nu}) \tag{3.6}$$

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) \tag{3.7}$$

it follows that the lowest power of q and $\bar{q}$ in (3.5) is $q^0 \bar{q}^0$ and the coefficient is

$$N_m = 64 \prod_{j=1}^3 \sin(\pi m \theta_j) \sin(\pi m \phi_j) \tag{3.8}$$

However $R(0, m)$ is a constant and so must be precisely equal to N_m . Now observe that

$$\begin{aligned}
 |N_m| &= 64 \prod_{j=1}^3 |\sin(\pi m \theta_j) \sin(\pi m \phi_j)| \\
 &= [\det(1 - g_L^m) \det(1 - g_R^m)]^{\frac{1}{2}}
 \end{aligned} \tag{3.9}$$

Define integers k_j and l_j such that

$$0 \leq k_j + m\theta_j < 1; \quad 0 \leq l_j + m\phi_j < 1 \tag{3.10}$$

Then from (3.8) and (3.9) one obtains

$$R(0, m) = (-1)^{\sum_{j=1}^3 (k_j + l_j)} [\det(1 - g_L^m) \det(1 - g_R^m)]^{\frac{1}{2}} \quad (3.11)$$

The factor involving the determinants is quite natural, and the phase merely reflects whether the spinor weight

$$(k_1 - \frac{1}{2}, k_2 - \frac{1}{2}, k_3 - \frac{1}{2}; l_1 - \frac{1}{2}, l_2 - \frac{1}{2}, l_3 - \frac{1}{2}) \quad (3.12)$$

belongs to the conjugacy classes $((s), (s)) + ((c), (c))$ or $((s), (c)) + ((c), (s))$ of $(D_3)_L \times (D_3)_R$. Since each D_3 is embedded in E_8 via $D_3 \times D_5 \subset E_8$, the conjugacy classes of $(D_5)_L \times (D_5)_R$ must be the same, and hence the phases in (3.11) are nothing other than those required in (2.2).

It is interesting to note that another consequence of the identity $R(0, m) = R(m, 0)$ is the fact that if g_L^m and g_R^m are Weyl twists with no eigenvalue equal to 1, (i.e. non-degenerate), then the number of vectors $(\vec{w}_L, \vec{w}_R)$ on $\Gamma'_{6,6}$ such that $(\vec{w}_L + m\vec{\mu})^2$ and $(\vec{w}_R + m\vec{\nu})^2$ attain their minimum value is precisely $|R(0, m)|$ and is given by (3.9). To see this merely write out the expression for $R(m, 0)$ using (3.2) and (3.4). Observe that choosing integers k_j and l_j so that $\alpha_j = k_j + m\theta_j$ and $\beta_j = l_j + m\phi_j$ satisfy $0 \leq \alpha_j, \beta_j < 1$, is the same as choosing k_j and l_j to minimize $(k_j + m\theta_j - \frac{1}{2})^2$ and $(l_j + m\phi_j - \frac{1}{2})^2$. Then use the identity (see I):

$$\min (m\vec{\mu} + \vec{w}_L)^2 = \sum_{j=1}^3 \alpha_j(1 - \alpha_j) \quad (3.13)$$

$$\min (m\vec{\nu} + \vec{w}_R)^2 = \sum_{j=1}^3 \beta_j(1 - \beta_j) \quad (3.14)$$

and the result follows. Appendix A contains a longer but elementary group theoretical proof of this result.

As an elementary example take $\Gamma'_{6,6}$ to be the Englert-Neveu lattice [7] obtained from E_6 . That is, $(\vec{w}_L, \vec{w}_R) \in \Gamma'_{6,6}$ if and only if $\vec{w}_L$ and $\vec{w}_R$ are both weights of E_6 and $\vec{w}_L - \vec{w}_R$ is a root. Let g_L be the $E_6(a_2)$ Carter element and g_R be the $A_5 \times A_1$ Coxeter element in the Weyl group of E_6 . The eigenvalues on the CSA of E_6 are $e^{\pm \frac{i\pi}{3}}, e^{\pm \frac{i\pi}{3}}, e^{\pm \frac{2\pi i}{3}}$ for g_L and $e^{\pm \frac{i\pi}{3}}, e^{\pm \frac{2\pi i}{3}}, e^{\pm i\pi}$ for g_R . Choose θ_j and ϕ_j such that $e^{\pm 2\pi i \theta_j}$ and $e^{\pm 2\pi i \phi_j}$ coincide with these eigenvalues. If one chooses $0 \leq \theta_j, \phi_j < 1$ and chooses $\vec{\mu}$ and $\vec{\nu}$ to be shift vectors of minimal length corresponding to g_L and g_R then [1]

$$(\vec{\mu})^2 = \sum_{j=1}^3 \theta_j(1 - \theta_j) \quad (\vec{\nu})^2 = \sum_{j=1}^3 \phi_j(1 - \phi_j) \quad (3.15)$$

and hence

$$(\vec{\sigma}_L)^2 = \sum_{j=1}^3 \theta_j \quad (\vec{\sigma}_R)^2 = \sum_{j=1}^3 \phi_j \quad (3.16)$$

Therefore, in the foregoing example we obtain a shift of even Lorentzian length by choosing $\vec{\theta} = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$, $\vec{\phi} = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$. It is also elementary to check that this shift vector has order 6 on $\Gamma'_{22;14}$.

By direct computation, or by observing that $|Z(A_5 \times A_1)| = 12$ and $|Z(E_6)| = 3$, we see that

$$\det(1 - g_L) = 12 \quad \det(1 - g_R) = 3 \quad (3.17)$$

However, g_R^2 , g_R^3 and g_R^4 are degenerate, while $\det(1 - g_L^5) = \det(g_L^{-1}) \times \det(g_L - g_L^6) = \det(1 - g_L) = 12$, and similarly $\det(1 - g_R^5) = 3$. Now for $(5\vec{\theta}, 5\vec{\phi})$ the choice of k_j, l_j satisfying (3.10) is $(0, 0, -3; 0, -1, -2)$ and hence the sign in (3.11) is positive for $m = 5$. Thus we have

$$R(0, 1) = R(0, 5) = 6; \quad R(0, m) = 0, \quad m = 2, 3, 4 \quad (3.18)$$

Using (2.24) and (2.25) we obtain the number of generations to be

$$\frac{1}{12} \sum_{k,l=0}^5 R(k, l) = 12$$

4. Construction of (1,1) models

4.1 GENERALITIES

Although in I we discussed many possibilities for constructing new covariant lattices, no examples were given. We correct that omission here by examining a class of examples rather systematically. This class consists of four-dimensional string theories with (at least) one world sheet supersymmetry for left- and right-movers. As explained in the previous section, we start with a torus-compactified heterotic string, defined by a covariant lattice $\Gamma_{22;14} = E_8 \times E_8 \times \Gamma'_{6,6} \times E_8$. To this lattice we apply a single shift of the form (3.1). We denote the components of this shift vector on the left and right lattice in the following way

$$\begin{aligned}\vec{s}_L &= (\theta_1, \theta_2, \theta_3, \vec{\mu}) \\ \vec{s}_R &= (\phi_1, \phi_2, \phi_3, \vec{\nu})\end{aligned}\tag{4.1}$$

We require both $\vec{s}_L$ and $\vec{s}_R$ to correspond to non-degenerate Weyl group elements, *i.e.* θ_i and ϕ_i should not be integer. The physical reasons for this requirement are somewhat different for left and right-movers. If $\vec{s}_R$ were degenerate, there would be no chance of getting any chiral fermions. If $\vec{s}_L$ were degenerate, E_8 would be broken to E_8, D_8, E_7, D_7 or D_6 , of which only D_7 has complex representations. For non-degenerate $\vec{s}_R$ we get the far more interesting gauge groups D_5 or E_6 . (Even for degenerate $\vec{s}_L$ some fermions may be in complex representations of gauge groups arising from $\Gamma'_{6,6}$, but we will not examine these here.)

We will restrict ourselves to lattices obtained by applying a single shift. One could also consider successive shifts by $(0, \vec{s}_R)$ and $(\vec{s}_L, 0)$ instead of $(\vec{s}_L, \vec{s}_R)$, and in some cases one can replace $\vec{s}_R$ or $\vec{s}_L$ by several successive shifts. The resulting lattices are in general different from the ones obtained by applying the sum of the successive shifts.

The shifted lattices can be divided into four classes according to the way E_8 breaks on the left and right lattice. We label these possibilities as follows

type A	(E_6, E_6)	
type B	(D_5, E_6)	
type C	(E_6, D_5)	
type D	(D_5, D_5)	(4.2)

where for example (D_5, E_6) means that the left E_8 breaks to D_5 and the right one to E_6 (in fact the rank of E_8 is preserved, so that there is always an additional rank 2 or 3 factor which is not specified here). Types A and B are supersymmetric and have E_6 and D_5 gauge groups respectively.

In the literature these four types A, B, C and D are usually called (2,2), (2,1), (1,2) and (1,1) compactifications respectively. Apart from the unfortunate interchange of left and right in comparison with our conventions, the terminology is slightly misleading because it is meant to imply more than just the number of left- and right-moving supersymmetries. One usually means by $N = 2$ not just the realization of an $N = 2$ algebra, but also a “charge integrality” condition of the charges of its $U(1)$ current. This condition corresponds (in any string theory, not just those constructed with lattices) precisely to the extension of D_5 to E_6 .

The notion of a world sheet supersymmetry for the left-movers can be understood most easily by using the “bosonic string map” developed in [13],[6], [14]. In four dimensions, this map takes a left- or right-moving superstring partition into a bosonic string partition function by replacing the D_5 space-time factor by a $D_5 \times E_8$ (or D_{13}) gauge factor. Conversely, if one has a bosonic string partition function with a $D_5 \times E_8$ gauge factor, the inverse map can be used to turn it into a D_5 space-time factor. In this way, all (1,1) theories described above can be mapped to consistent type-II theories. The left-moving world sheet supersymmetry is then required for consistency. If the lattice is of type A, the resulting type-II theory has $N = 2$ supersymmetry; for type B and C it has $N = 1$ and for type D one gets $N = 0$.

4.2 LEVEL MATCHING

As discussed in I, the shift vector $\vec{s} = (\vec{s}_L, \vec{s}_R)$ should satisfy a level matching condition. If N is the smallest integer so that $N\vec{s} \in \Gamma'_{22,14}$, then this condition reads

$$\vec{s}^2 = \frac{2k}{N} \quad , \quad k \in \mathbb{Z}$$

This condition is trivially satisfied if the left and right shifts are the same. We will now show that it is satisfied even if they are different.

First we determine the order of $\vec{s}$ in terms of the orders N_L and N_R of the left and right Weyl group elements represented by $\vec{\mu}$ and $\vec{\nu}$. As discussed in I, the orders of these vectors on a lattice can be twice N_L or N_R . To determine if the order doubles, one computes the inner product of the shift vector with the lattice vectors, which are always weights of the algebra whose Weyl-group one is considering. If $\vec{w} \cdot \sigma = \frac{k}{N}$, $k \in Z$ for all weights that appear, then there is no doubling (in general, $\vec{w} \cdot \sigma$ is always equal to $\frac{k}{2N}$). On a Lorentzian lattice, where weights are paired as $(\vec{w}_L, \vec{w}_R)$, the quantity to compute is

$$\rho = \vec{\mu} \cdot \vec{w}_L - \vec{\nu} \cdot \vec{w}_R$$

In I we obtained the following relation for $\vec{w} \cdot \vec{\sigma}$

$$\vec{w} \cdot \vec{\sigma} \equiv -\frac{1}{2}\vec{w}^2 - \vec{r} \cdot \vec{\sigma} \pmod{1}$$

where $\vec{r}$ is a root. Because $\vec{r} \cdot \sigma$ is always $\frac{k}{N}$, we find that for any $(\vec{w}_L, \vec{w}_R)$ that can appear on an even self-dual Lorentzian lattice, $\rho = \frac{k}{N_{LR}}$, $k \in Z$, where N_{LR} is the least common multiple of N_L and N_R . So there are no additional factors of two.

Now we should still check the order of $\vec{\theta}$ and $\vec{\phi}$ on $(E_8)_L$ and $(E_8)_R$. Since $\theta_i = \frac{k_i}{N_L}$, the order of $\vec{\theta}$ on the E_8 lattice is N_L if $\sum k_i = 0 \pmod{2}$ and $2N_L$ otherwise, and analogously for $\vec{\phi}$. We denote the order of $\vec{\theta}$ on $(E_8)_L$ (or of $\vec{\phi}$ on $(E_8)_R$) as $\hat{N}_L$ ($\hat{N}_R$). Since the left and right E_8 lattices are uncorrelated, the combined order is $\hat{N}_{LR}$, the least common multiple of $\hat{N}_L$ and $\hat{N}_R$. This is divisible by N_{LR} , so that $\hat{N}_{LR}$ is the order of $\vec{s}$ on $\Gamma'_{22;14}$. Thus one only needs to consider the torus eigenvalues $\vec{\theta}$ and $\vec{\phi}$ to determine the order.

For a given non-degenerate element of the Weyl group one can choose four shift vectors that are, in principle, inequivalent. All choices $\vec{\mu}$ and $\vec{\nu}$ are equivalent, but the vectors $\vec{\theta}$ and $\vec{\phi}$ can be changed by adding 1 to one of their entries. The most convenient choice for $\vec{\mu}$ is one that has minimal norm, and we will assume that this choice has been made. One of the two choices of $\vec{\theta}$ is equivalent (under addition of E_8 roots) to the choice $0 < \theta_i \leq \frac{1}{2}$. In that case,

$$\vec{\mu}^2 = \sum_{i=1}^3 \theta_i(1 - \theta_i),$$

so that

$$\vec{s}_L^2 = \sum_{i=1}^3 \theta_i$$

We call this vector the *primary* shift vector. The second possibility for $\vec{s}_L$ is obtained by replacing $\vec{\theta}$ by $\vec{\theta}'$, with $\theta'_1 = \theta_1 - 1$, $\theta'_2 = \theta_2$ and $\theta'_3 = \theta_3$. This shift vector, $\vec{s}'_L$, satisfies

$$(\vec{s}'_L)^2 = \sum_{i=1}^3 \theta'_i + 2\theta_1,$$

and we will call this the *secondary* shift vector.

One sees immediately that

$$\hat{N}_{LR} \vec{s}_L^2 = 0 \pmod{2},$$

and similarly for $(\vec{s}'_L)^2$ so that the condition for level matching is satisfied for the left sector. The same arguments hold for the right sector, so that the combined shift vector $\vec{s}$ satisfies level matching. Notice that if N_L (N_R) is *odd*, then $\hat{N}_L$ ($\hat{N}_R$) for the two possible shift vectors differs by a factor of two.

If one of the twist eigenvalues θ_i is half-integer, the two choices of $\vec{s}_L$ are in fact equivalent because the resulting theories are chirality-flipped versions of each other.

4.3 CLASSIFICATION OF LORENTZIAN LIE ALGEBRA LATTICES

Different twists can of course only be combined if one can find a Lorentzian lattice for which they are automorphisms. Thus we would like to classify all Lorentzian even self-dual Lie algebra lattices $\Gamma'_{6;6}$ (such lattices have a 6-dimensional root-system of a simply-laced Lie algebra on the right as well as the left). This can be achieved by means of the following construction.

Let Γ be a Euclidean even self-dual lattice of dimension $p + q$, with a basis in which the lattice points are of the form $(x_1, \dots, x_p; y_1, \dots, y_q)$ ($p \neq 0, q \neq 0$), denoted as $(\vec{x}, \vec{y})$. We will require the basis to be such that there exists p linearly independent vectors of the form $(\vec{x}, 0)$ and q linearly independent vectors of the form $(0, \vec{y})$. Denote the collection of

all vectors of this form as $(\Delta_x, 0)$ and $(0, \Delta_y)$ respectively. Then Δ_x and Δ_y are integral lattices. Their duals can be decomposed in cosets:

$$\Delta_\alpha^* = \sum_{k=0}^{N_\alpha-1} (\vec{t}_\alpha(k) + \Delta_\alpha), \quad \alpha = x, y,$$

where $\vec{t}_\alpha$ is a set of coset representatives, with $\vec{t}_\alpha(0) = \vec{0}$. The volume of the unit cell of Δ_α is related to the number of cosets N_α

$$\text{vol } \Delta_\alpha = \sqrt{N_\alpha} \quad , \quad \text{vol } \Delta_\alpha^* = \frac{1}{\sqrt{N_\alpha}}$$

Since all vectors on Γ must have integer inner products with all vectors in $(\Delta_x, 0)$ and $(0, \Delta_y)$, all vectors must be of the form

$$(\vec{t}_x(k_1) + \Delta_x, \vec{t}_y(k_2) + \Delta_y).$$

If there are N such cosets (k_1, k_2) on the lattice Γ , then closure of the set under addition dictates that N must be a divisor of N_x as well as N_y , i.e. $N_x = l_x N, N_y = l_y N, l_x, l_y \in \mathbb{Z}$. The fact that Γ is unimodular implies that $N^2 = N_x N_y$, so that $l_x = l_y = 1$, and hence $N_x = N_y = N$. Then we can order the sets of coset representatives in such a way that

$$\Gamma = \sum_{k=0}^{N-1} \{ (t_x(k), t_y(k)) + (\Delta_x, \Delta_y) \} \tag{4.3}$$

Now choose a second even self-dual lattice Γ' of dimension $p + q'$, decomposed in a similar way

$$\Gamma' = \sum_{k=0}^{N-1} \{ (t'_x(k), t'_y(k)) + (\Delta'_x, \Delta'_y) \}$$

Suppose $t'_x(k) = t_x(k)$ and $\Delta'_x = \Delta_x$. Now consider the lattice

$$\Gamma_{LSD} = \sum_{k=0}^{N-1} \{(t_y(k), t'_y(k)) + (\Delta_y, \Delta'_y)\}.$$

Then Γ_{LSD} is Lorentzian even self-dual with respect to a metric $(-)^q(+)^{q'}$. To show that this is true one checks that Γ_{LSD} is integral and unimodular.

We apply this to the special case of Lie algebra lattices. Let Γ be an even self-dual Euclidean lattice with a decomposition of the form (4.3), with Δ_y a root lattice of a simply-laced Lie algebra $\mathcal{G}$. If N is the order of the center of G , and if Δ_x is non-trivial, we call $\Delta(\mathcal{G}) \equiv \Delta_x$ a *complement* of $\mathcal{G}$. Included in this definition is a rule for matching the conjugacy classes of $\mathcal{G}$ with those of $\Delta(\mathcal{G})$. Clearly $\Delta(\mathcal{G})$ is by no means unique, but for our purposes all choices will be equivalent, and the most convenient choice is the one of minimal dimension. It is easy to see that up to this equivalence the following relations hold

$$\Delta(\Delta(\mathcal{G})) = \mathcal{G}$$

and

$$\Delta(\mathcal{G}_1 \times \dots \times \mathcal{G}_l) = \Delta(\mathcal{G}_1) \times \dots \times \Delta(\mathcal{G}_l)$$

Using the general result proved above, one can easily show that if $\mathcal{G}_L \times \mathcal{G}_R$ is Lorentzian even self-dual if and only if $\mathcal{G}_L \times \Delta(\mathcal{G}_R)$ is Euclidean even self-dual, where “ $\times$ ” denotes a decomposition of the form (4.3). Thus to construct all Lorentzian even self-dual lattices of the form $\mathcal{G}_L \times \mathcal{G}_R$, one looks for all Euclidean even self-dual lattices of the correct dimension which contain $\Delta(\mathcal{G}_R)$. This is a practical method only if the dimension of the Euclidean lattice is 24 or smaller, but this turns out to be sufficient for our purpose.

We have used this “Euclideanization” trick before for D_n factors [5],[6],[10]. A D_n root lattice can always be embedded regularly in the even self-dual lattice D_{8k} , for $8k > n$. Thus

$\Delta(D_n) = D_{8k-n}$. For other simple, simply-laced algebras the results are as follows

$$\begin{aligned}\Delta(A_1) &= E_7 \\ \Delta(A_2) &= E_6 \\ \Delta(A_3) &= D_5 \\ \Delta(A_4) &= A_4 \\ \Delta(A_5) &= A_1 \times A_2\end{aligned}$$

These are all obtained from regular sub-algebras of E_8 , and one can get the relation between the conjugacy classes from the decomposition of the (248) of E_8 . By inverting these relations one gets the complements of E_7 and E_6 .

Of the rank ≤ 6 algebras only A_6 is still missing. Its smallest complement is $(A_6)^3$, and can be obtained from the $(A_6)^4$ Niemeier lattice [15], [16].

The following example illustrates the Euclideanization technique. Suppose we want to construct all Lorentzian even self-dual lattices $\Gamma_{6,6}$ such that $\mathcal{G}_R = (A_2)^3$. Any such lattice can be mapped to a 24-dimensional Euclidean lattice containing $(E_6)^3$, and there are only two such lattices, $(E_8)^3$ and $(E_6)^4$. The first yields $\mathcal{G}_L = (A_2)^3$, and the second $\mathcal{G}_L = E_6$. The two Lorentzian lattices are in fact the Englert-Neveu lattices derived from $(A_2)^3$ and E_6 (such lattices are obtained by pairing identical conjugacy classes on left and right [7], and will be referred to as “EN-lattices” in the following). In fact all lattices except one are of this type. The exception is obtained by choosing $\mathcal{G}_R = (D_2)^3$. The Niemeier lattices containing its complement $(D_6)^3$ are $(E_8)^3, E_8 \times D_{16}, D_{24}, (D_{12})^2, (D_6)^4$ and $(D_8)^3$. Not all resulting Lorentzian lattices are different. One gets the expected EN-lattices $(D_2)^3, D_4 \times D_2$ and D_6 , but the last one yields a second $(D_2)^3$ lattice. It is manifestly different, because it does not contain any vectors of norm $(\frac{1}{2}, \frac{1}{2})$.

4.4 DISCUSSION OF EXAMPLES

Inequivalent single shifts

We have shown that for all Lorentzian even self-dual Lie algebra lattices of dimension six or less, one must have $\mathcal{G}_L = \mathcal{G}_R \equiv \mathcal{G}$, and all but one are EN-lattices. The possible Weyl-twists are then simply all possible combinations of non-degenerate Weyl twists of $\mathcal{G}$

and its regular sub-algebras. For each such combination there are four shift vectors (or less if one of the θ_i 's or ϕ_i 's is half-integer). The explicit form of $\vec{\mu}$ and $\vec{\nu}$ was given in I for the proper^{*} elements of the Weyl-groups. To get the remaining ones (which are proper in some sub-algebra $\mathcal{H} \subset \mathcal{G}$) one has to perform a basis transformation from the simple root basis of $\mathcal{H}$ to that of $\mathcal{G}$.

Perhaps surprisingly, shift vectors obtained from different Weyl conjugacy classes can sometimes be equivalent. This can be understood as follows. Consider the G -lifts $\tilde{w}_1$ and $\tilde{w}_2$ of two elements w_1 and w_2 of the Weyl group of $\mathcal{G}$. Both $\tilde{w}_1$ and $\tilde{w}_2$ can be conjugated into the maximal torus of G :

$$t_1 = k_1^{-1} \tilde{w}_1 k_1; \quad t_2 = k_2^{-1} \tilde{w}_2 k_2.$$

Two shifts are equivalent if the corresponding torus elements are conjugate. Clearly if t_1 and t_2 are conjugate, then so are $\tilde{w}_1$ and $\tilde{w}_2$. But the element of G that conjugates $\tilde{w}_1$ into $\tilde{w}_2$ is not necessarily itself a lift of an element of the Weyl-group.

In most cases two shift-vectors representing two different Weyl group conjugacy classes have different order and/or norm, in which case the problem does not arise. For simply-laced algebras of rank 6 or less there is just one exception, namely the $D_4(a1) \times D_2$ and $D_3 \times D_3$ conjugacy classes of the Weyl group of D_6 . Their shift vectors turn out to be Weyl rotations of each other, so that the corresponding torus elements are conjugate.

Spectra

The number of a priori different shift vectors is 81 for the $E_6 \times E_6$ lattice, 49 for $D_6 \times D_6$ and 48 for all other lattices combined. For all these shift vectors we have determined the complete mass spectrum. If two theories have an identical spectrum we regard them as equivalent, although there is a logical possibility that they differ either in the gauge groups that we do not consider, or in their massive spectra.

The 178 shift vectors yield 59 different spectra. 13 of type A, 8 of type B and C and 30 of type D. The latter, and 6 of the eight type C models are tachyonic, and we omit most

^{*} By a proper element of a Weyl group, $W(\mathcal{G})$, of a Lie algebra, $\mathcal{G}$, we mean an element that is not an element of the Weyl group of any regular sub-algebra of $\mathcal{G}$.

of the details of their spectra. The E_6 and D_5 representations of the complete massless spectrum of the type A,B and the non-tachyonic type C models is given in table 1. In all cases, N_1 is the number of (27)'s or (16)'s, N_2 the number of $(\overline{27})$'s or $(\overline{16})$'s and N_G is the number of extra gauge-bosons, not including the $U(1)$ -factors that complete the rank of the gauge group to 22. Of course we always have a graviton, a $B_{\mu\nu}$ and a dilaton, as well as (for types A and B) all their supersymmetric partners. Furthermore N_3 denotes the number of $N = 1$ scalar multiplets that are E_6 or D_5 singlets (types A,B) or the number of scalars that are E_6 singlets (type C); N_4 is the number of scalar multiplets in the (10) of D_5 (type B) or the number of scalars in the (27) of E_6 ; finally, N_5 is the number of left-handed E_6 singlet fermions in type C models.

Remarkably, the number of E_6 or D_5 generations (i.e. $|N_1 - N_2|$) is always a multiple of 12, and takes the values 0, 12, 24 and 36 for type A, 0, 12, 36 and 48 for types B and C and 0, 24, 48, 72, 96, 144 and 192 for type D. This can be compared with the results of [17], who considered symmetric orbifolds with Coxeter twists. Although we apply twists to different lattices (with a specific value for the radius and B_{ij} background fields), the formula for the number of generations, (2.25) (combined with (2.24) and (3.11)) reduces to the symmetric orbifold formula if we use identical twists on left and right. Our results for $N_1 - N_2$ do indeed agree with those of [17], and we extend them in two directions, namely by non-Coxeter twists and asymmetric twists. In particular, it extends the observation of [17] and [18] that the number of generations in Weyl-twisted orbifolds seems to be always a multiple of 12.

All of the 24 and 36 type A models can be obtained by means of symmetric twists (although many can also be obtained by asymmetric ones), while the 12 and 0 generation type A models require asymmetric twists. Type B and C models are asymmetric by definition. A closely related question is whether the spectrum changes when the left and right lattices are interchanged. Such an interchange maps type B to type C and vice-versa. It does not change the spectrum of any type A model except the ones with 0 generations, which transform into each other. Interestingly, the 12 generation model is symmetric, but can only be constructed asymmetrically (at least with single shifts).

In general, shifting different lattices gives different results (the exceptions are A_{24}^4 , which is obtained from the E_6 and the A_5A_1 lattice, and A_{24}^3 , obtained from D_6 and A_3A_3 . In both cases the second Lie algebra is a sub-algebra of the first), while different shifts on the

same lattice yield very often identical results.

Gauge groups

Gauge bosons — other than those in the CSA — are due to lattice vectors $(\vec{v}_L, 0)$, $\vec{v}_L^2 = 2$. In symmetrically shifted lattices, the “0” on the right sector can only occur in the unshifted sector, and hence all additional gauge bosons come from that sector. They correspond precisely to the roots of $\mathcal{K}_{tot}$ defined in I, where $\mathcal{K}_{tot} = \mathcal{K}_w \times \mathcal{K}_F$. The first factor, $\mathcal{K}_w$, is the invariance algebra of the shift vector. It can be determined by writing the shift vector in its minimal form, rotating it into the positive Weyl chamber and computing its inner products with the simple roots α_i and the root $-\psi$, where ψ is the highest root. If these inner products vanish (mod 1), the corresponding roots, and the root system they generate, are part of $\mathcal{K}_w$, and in fact one can show that it is all of $\mathcal{K}_w$ [19], [20]. These roots can be read off from the extended Dynkin diagram by writing the inner products (reduced to the interval $[0,1)$ by adding integers) next to nodes, and taking the sub-graph consisting of the nodes with inner product 0, and the bonds connecting them (for proper elements, it is sufficient to consider the unextended Dynkin diagram). The second factor, $\mathcal{K}_F$, consists of the unbroken generators of $(E_8)_L$, and can be read off directly from the torus eigenvalues.

Asymmetrically shifted lattices can have additional roots from the twisted sectors. Since (in general) $\vec{s}_L$ and $\vec{s}_R$ have different orders, the latter may complete its cycle while the former is non-trivial. Then the shifted left-moving sector may have norm 2 vectors matched with 0 on the right, i.e. roots. Of course this can only enlarge the gauge group; using the procedure for symmetric shifts one can therefore determine at least a subgroup of the complete gauge group. If this does not allow a complete determination of the gauge group (by means of its dimension) one has to compute the roots explicitly. This was necessary in only one case, A_0^2 .

Anomalies

Because these examples are modular invariant, the fermion spectrum is of course anomaly free in the Green-Schwarz sense [21],[8]. Thus the only possible anomalies are due to a $U(1)$ -factor with non-vanishing trace, and these anomalies are cancelled by the Green-Schwarz mechanism. The presence of such anomalies changes the theory rather drastically: the $U(1)$ gauge boson acquires a mass [22], and the perturbative supersymmetric vacuum is destabilized by Fayet-Iliopoulos type symmetry breaking [23]. In most cases we get extra $U(1)$

factors, whose charges can be worked out explicitly, if needed. Fortunately there is a simple reason why such anomalous $U(1)$'s cannot occur in the theories we are considering. This argument holds for any string theory satisfying the conditions formulated in [8] (that is, any theory which has a character-valued chiral partition function with the transformation property specified in [8]), and which also has a level 1 E_8 factor (“hidden sector”) in its gauge-group.

The crucial point is that the fermion-loop anomaly must have a very special structure for the Green-Schwarz mechanism to work. In four dimensions it must be proportional to

$$(Tr R^2 - Tr F^2) Tr F_{U(1)},$$

where $Tr F^2$ is a trace over the entire gauge group. In particular it must contain a trace over the E_8 factor. But this requires the existence of massless fermions with $U(1)$ charges in a non-trivial E_8 representation. Since the lowest non-trivial E_8 representation, the (248), is precisely massless for zero charge, this is manifestly impossible.

Anomalous $U(1)$ factors in four-dimensional string compactifications were first discussed in [24], [25]. The first of these papers discussed the $SO(32)$ theory, where the above does of course not apply, whereas the second observed that anomalous $U(1)$'s may occur for $SO(32)$ as well as $E_8 \times E_8$, but only if the spin-connection is embedded non-trivially in both E_8 factors. These results illustrate (and are explained by) the foregoing.

(2,2) models

If the conjecture of [26] is correct, then table 1 represents 13 different points in the moduli space of some Calabi-Yau manifolds. These examples are in no essential way different from those constructed in [26], and in fact some of these lattice models can also be obtained by tensoring $N = 2$ superconformal field theories. If there exists a CY-interpretation, the Hodge numbers $h_{1,1}$ and $h_{2,1}$ of those manifolds are given by N_1 and N_2 (or vice-versa, since we cannot determine the sign of the Euler number). The Euler numbers χ can take the values $|\chi| = 0, 24, 48$ and 72. Interestingly, the examples discussed in [17], which are obtained by applying the same Coxeter twists to different lattices, have the same Euler number (as they should), but in general different Hodge numbers.

There are several cases where $h_{1,1}$ and $h_{2,1}$ are the same, but the spectrum is still different, even if one removes the E_6 -singlet gauge-bosons and the corresponding number of scalars (needed to make the gauge bosons massive), as suggested in [26]. It remains an open question whether theories which differ only by E_6 -singlets correspond to different CY-manifolds, to no manifolds at all or whether there is any relation between the number of singlets from CY-compactification and from exactly solvable theories. Such a relation does not seem very plausible in view of the fact that there is no such relation for the E_6 singlet gauge bosons. (The existence of such a relation in 6 dimensions can be explained entirely by anomaly cancellation.)

Table 1 does not exhaust the list of type-A lattices. Such lattices can also be constructed directly out of certain building blocks. The simplest such building blocks are triplets of D_1 lattices to realize the $(D_2)^3$ supercurrent [27], [6] or one-dimensional lattices $\mathcal{U} = \{\frac{m}{6}\sqrt{3}, m \in \mathbb{Z}\}$ to realize the $(A_2)^3$ supercurrent [28],[9],[10]. In this way one can construct a type-A theory with $(D_2)^3$ supercurrent, which has 48 generations; one can also obtain a type-A theory with $(A_2)^3$ supercurrent with 84 generations. For the mixed case, i.e. supercurrents $(A_2)^2 D_2$ and $A_2(D_2)^2$ one can show that no chiral supersymmetric theory of any type (not even (2,0)) can exist.

Since we did not get the 48 and 84 generation theories with single shifts, the obvious conclusion is that they must result from multiple shifts. Indeed, two very similar theories were constructed in [18] by means of multiple twists (these theories are not identical to ours, because the rank of the gauge group differs). The 84 generation model is in fact identical to that of [26]. The building block $\mathcal{U}$ is nothing but a realization of the $k = 1$ element of the $N = 2$ discrete series. Any theory constructed by tensoring $k = 1$ superconformal blocks can be written in terms of lattices constructed out of $\mathcal{U}$ building blocks. In four dimensions, we know three type-A theories built out of 9 $\mathcal{U}$ factors on the left and right lattice: the 84 generation model, A_{36}^1 and A_{36}^2 . The second one is well known, see e.g. [3],[28]. To complete this discussion of overlaps between various constructions, we note that the 84 generation model may be related to a CY-manifold constructed in [29][★]; it cannot be obtained with the standard fermionic constructions [30], [31] (and neither can most of the models in table 1) because the supercurrent does not have a fermion-trilinear structure. The 48 generation model on the other hand does have a simple fermionic construction.

★ This suggestion is due to C.A. Lütken.

We have also constructed single-shifted six-dimensional theories. There are three supercurrents that keep gravitini in the unshifted sector, namely $(A_2)^2, (D_2)^2$ and $D_4(a_1)$. The massless spectrum of six-dimensional type-A theories (defined by replacing E_6 by E_7 in (4.2)) is characterized completely by the additional gauge group. We find just three possibilities, namely $U(1)^5$, $SU(2) \times U(1)^4$ and $SU(2)^5$. The first of these is identical to the "1⁶" tensor product of [26], and is obtained from the $(A_2)^2$ EN-lattice. The second gauge group appears in two ways, namely by a Coxeter twist on the $(D_2)^2$ lattice, and by a symmetric $D_4(a_1)$ twist on the D_4 lattice. The third gauge group results from an asymmetric $D_4(a_1) \times (D_2)^2$ on the D_4 lattice, as well as from the symmetric $(D_2)^2$ twist on the same lattice.

5. Supersymmetric theta function identities

It is interesting to note that there are a number of identities between theta functions that can be derived from the construction of supercurrents described in I. Such supersymmetric theta function identities were essentially employed in section 3 to determine the number of chiral generations.

Suppose that $\mathcal{G}$ is a semi-simple, simply-laced Lie algebra of rank $2M$, and suppose that g is any non-degenerate element of the Weyl group, $W(\mathcal{G})$, of $\mathcal{G}$. Let $\Lambda_r(\mathcal{G})$ and $\Lambda_w(\mathcal{G})$ denote the root and weight lattices of $\mathcal{G}$, and let Λ be a sub-lattice of $\Lambda_w(\mathcal{G})$ that contains $\Lambda_r(\mathcal{G})$. Let $\vec{\mu}$ be any shift vector that corresponds to g , and let N be its order on the lattice, Λ . Choose a set of $\theta_j, j = 1, \dots, M$, such that $e^{\pm 2\pi i \theta_j}$ are the eigenvalues of g on the CSA of $\mathcal{G}$. Define a vector $\vec{\sigma}$ by

$$\vec{\sigma} = (\theta_1, \theta_2, \dots, \theta_M, \vec{\mu}) \quad (5.1)$$

Then by the construction of I it is always possible to define a supercurrent that has local operator products with $\exp[2\pi i \vec{\sigma} \cdot \vec{X}]$ and with all states defined by Λ . Consider a bosonic string on the lattice $\Gamma = \Lambda_w(D_M) \times \Lambda$ and define the character-valued partition function:

$$P(\vec{\lambda}, \tau) = \text{Tr} \left[e^{2\pi i \vec{\lambda} \cdot \vec{J}_0} q^H \right] \quad (5.2)$$

where the trace is taken over all string states that are spinors of D_M , and $\vec{J}_0$ are the charges corresponding to the currents $\partial \vec{X}(z)$ that define the maximal torus of $D_M \times \mathcal{G}$. Let

$\partial X^1(z), \dots, \partial X^M(z)$ be the canonically normalized currents for the maximal torus of D_M , and introduce a vector $\vec{u}$ such that

$$\vec{u} \cdot \partial \vec{X}(z) = \partial X^1(z) + \partial X^2(z) + \dots + \partial X^M(z). \quad (5.3)$$

For any weight $\vec{w}$ of $\mathcal{G}$, define a vector $\vec{w}' = (\vec{0}, \vec{w})$ on Γ and define the functions

$$\begin{aligned} R(k, l, \vec{w}, \tau) &= i \exp \left[i\pi\tau(k\vec{\sigma} + \vec{w}')^2 + 2\pi i(k\vec{\sigma} + \vec{w}') \cdot l\vec{\sigma} \right] \\ &\quad \times P \left((k\vec{\sigma} + \vec{w})\tau + (l\vec{\sigma} + \frac{1}{2}\vec{u}), \tau \right) \\ &\equiv (\eta(\tau))^{-3m} \left\{ \prod_{j=1}^M \left[e^{i\pi k(\theta_j - \frac{1}{2})} \theta \left[\frac{k\theta_j - \frac{1}{2}}{l\theta_j - \frac{1}{2}} \right] (0|\tau) \right] \right\} \\ &\quad \times \left\{ \sum_{\vec{v} \in \Lambda} q^{(\vec{v} + k\vec{u} + \vec{w})^2} e^{2\pi i(\vec{v} + k\vec{\mu} + \vec{w}) \cdot \vec{\mu}} \right\} \end{aligned} \quad (5.4)$$

Since the supercharge commutes with the operators $\exp[2\pi i \vec{\sigma} \cdot \vec{J}_0]$ and $\exp[2\pi i \vec{w}' \cdot \vec{J}_0]$ and anti-commutes with $\exp[i\pi \vec{u} \cdot \vec{J}_0]$ it follows that $R(k, l, \vec{w}, \tau)$ must be constant. It may be possible to adapt the techniques of sections 2 and 3 to evaluate these constants, however in any given example the constants can always be evaluated by expanding the first few terms of (5.4). Given the explicit form of the shifts, $\vec{\mu}$, and the twist eigenvalues $e^{2\pi i \theta_j}$ corresponding to a Weyl twist (see for example [32] or I and appendix B of this paper), the knowledge that $R(k, l, \vec{w}, \tau)$ is constant implies many non-trivial theta function identities.

The simplest example is to consider the Coxeter element of D_2 . The twist eigenvalues are both -1 , and in the usual basis of vector weights $\{\pm \vec{e}_1, \pm \vec{e}_2\}$, the shift vector is $\vec{\mu} = \frac{1}{2} \vec{e}_1$. For simplicity set $\vec{w} \equiv 0$ in (5.4) and choose Λ to be the lattice of roots and vector weights, i.e. $Z \times Z$. Taking $\theta_1 = \frac{1}{2}$, one finds from (5.4) that for any choice of k and l ,

$$R(k, l, \tau) \equiv a \frac{\theta_2(0|\tau) \theta_3(0|\tau) \theta_4(0|\tau)}{\eta^3(\tau)} \quad (5.5)$$

for some constant a that depends on k and l . The constancy of this is the well known theta function identity. The fact that it reflects supersymmetry on a $D_1 \times D_2$ lattice was observed in [6].

A rather less trivial identity is given by considering the Coxeter element of A_2 . If $\vec{\alpha}$ and $\vec{\beta}$ are the simple roots of A_2 , the Coxeter shift vector is $\frac{1}{3}(\vec{\alpha} + \vec{\beta})$ while the twist eigenvalues are $e^{2\pi i/3}$. Taking $\vec{w} \equiv 0$, $k = 0$, $l = 1$ and $\theta_1 = \frac{2}{3}$ in (5.4), and choosing Λ to be the root lattice of A_2 , yields

$$\frac{\theta_1(\frac{2}{3}|\tau)}{\eta^3(\tau)} \times \left\{ \sum_{\vec{r} \in \Lambda_\tau(A_2)} q^{(\vec{r})^2} e^{\frac{2}{3}\pi i \vec{r} \cdot (\vec{\alpha} + \vec{\beta})} \right\} = \sqrt{3} \quad (5.6)$$

One may also derive further identities between the standard set of theta functions, $\theta_i(\nu|\tau)$, $i = 1, \dots, 4$, by taking $\mathcal{G} = D_{2M}$. For example, take Λ to be the root lattice plus vector conjugacy class of D_{2M} , set $\vec{w} \equiv 0$ and take $k = 0$. Let $\vec{e}_j$ denote the usual orthonormal basis for describing the weights and roots of D_n (see, for example, [33] or I), and let $\vec{\mu} = \mu_j \vec{e}_j$. One finds that the μ_j can be chosen so that $\mu_j = \frac{1}{2} - \mu_{2M+1-j}$, $j = 1, \dots, 2M$ (see appendix B, equation (B 3)) and so one obtains, for $l = 1, 2, \dots, N$

$$\prod_{j=1}^M \left\{ \frac{\theta_1(l\theta_j|\tau) \theta_3(l\mu_j|\tau) \theta_p(l\mu_j|\tau)}{\eta^3(\tau)} \right\} \equiv \text{constant} \quad (5.7)$$

where $p = 4$ for l odd and $p = 3$ for l even. This is the natural generalization of the classical theta function identity $\theta_2\theta_3\theta_4 = 2\eta^3$. For Coxeter elements of the Weyl group one has

$$\theta_j = \frac{(2j-1)}{2(2M-1)} \quad \mu_j = \frac{(j-1)}{2(2M-1)} \quad j = 1, 2, \dots, M \quad (5.8)$$

and for example, using the Coxeter element of D_4 , one obtains the identity $\theta_1(\frac{1}{6}|\tau) \theta_3(\frac{1}{6}|\tau) \theta_4(\frac{1}{6}|\tau) = \eta^3(\tau)$.

5.1 CONCLUSIONS

We have presented a systematic investigation of a class of $(1,1)$ lattice models, for all known bosonic supercurrents. This shows first of all that the supercurrents constructed in I can indeed all be used to build new chiral theories. Because of the way the supercurrent was constructed, we know that all these theories can also be realized as asymmetric orbifolds, but looking at them in a different way yields additional insight into their structure, and provides a conceptually simpler way of calculating their spectra.

Since lattice theories are the simplest among the known string theories with space-time fermions, and yet display all their interesting features, it is often possible to guess general results by first deriving them for lattice theories. For example, the lattice point of view enabled us to derive a simple formula for the number of generations for asymmetric shifts. Even though explicit use of lattices was made in the derivation, it seems plausible that generalizations to other constructions should be possible.

The price one pays for simplicity is that the number of possible theories is smaller. Indeed, it should be possible to enumerate the $(1,1)$ models (including multiple-shifted ones) completely, and the final set is perhaps not much bigger than the one listed in this paper. Other approaches, which are not limited to producing theories with gauge groups of maximal rank, are likely to yield longer lists. Because of its limited size, the class of $(1,1)$ models is very useful for comparing different kinds of string constructions, and studying their overlaps.

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APPENDIX A

Some properties of the Weyl group

Let G denote any semi-simple, simply-laced Lie group of rank l , and let $\mathcal{G}$ denote its Lie algebra. Choose a Cartan sub-algebra $\mathcal{H}$ and a corresponding torus T . We denote the Weyl group of $\mathcal{G}$ by $W(\mathcal{G})$. Given some representation of G , take $\tilde{W}(\mathcal{G})$ to be some lift of W to G . (See I for a more detailed discussion of this). Then for any element $\sigma \in W(\mathcal{G})$ there is $\tilde{\sigma} \in \tilde{W}(\mathcal{G})$ such that for any element $\vec{\lambda} \cdot \vec{H} \in \mathcal{H}$

$$\tilde{\sigma}(\vec{\lambda} \cdot \vec{H})\tilde{\sigma}^{-1} = \sigma(\vec{\lambda}) \cdot \vec{H} \quad (\text{A1})$$

Since $\tilde{\sigma} \in G$, there is an element $k \in G$ such that

$$k\tilde{\sigma}k^{-1} = \exp(2\pi i \vec{\sigma} \cdot \vec{H}) \equiv t \in T \quad (\text{A2})$$

The vector $\vec{\sigma}$ is called a shift vector corresponding to a Weyl element σ . This shift is obviously ambiguous up to the addition of weights and up to Weyl reflections, (these are in fact the only ambiguities).

In I we defined an element $\sigma \in W$ to be non-degenerate if $\det(1 - \sigma) \neq 0$. Further, we define a proper element of $W(\mathcal{G})$ to be any element $\sigma \in W(\mathcal{G})$ such that $\sigma \notin W(\mathcal{G}')$ for any regular, semi-simple sub-algebra $\mathcal{G}'$ of $\mathcal{G}$. All such elements (up to conjugation) have been classified in [34]. Given an element $\sigma \in W(\mathcal{G})$, define a semi-simple sub-algebra $\mathcal{G}' \subset \mathcal{G}$ to be minimal for σ if $\mathcal{G}'$ is regular in $\mathcal{G}$ and σ is a proper element of $W(\mathcal{G}')$. Note that if σ is non-degenerate then the Cartan sub-algebra, $\mathcal{H}$, of $\mathcal{G}$ is contained in any such $\mathcal{G}'$. Finally, for any Lie algebra $\mathcal{G}$, $\Lambda_r(\mathcal{G})$ and $\Lambda_w(\mathcal{G})$ will denote the corresponding root and weight lattices. Let $Z(G)$ denote the center of the group G , then we have

$$|Z(G)| = \left| \frac{\Lambda_w(\mathcal{G})}{\Lambda_r(\mathcal{G})} \right| \quad (\text{A3})$$

where the vertical bars, $| \cdot |$, denote the order. (Indeed (A 3) holds without the bars if it is viewed as an identity between abelian groups).

The first observation we make is that for a non-degenerate $\sigma \in W(\mathcal{G})$ we have

$$\det(1 - \sigma) = \left| \frac{\Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}')} \right| = |Z(G')| \quad (\text{A4})$$

for any sub-algebra $\mathcal{G}'$ of $\mathcal{G}$ that is minimal relative to σ .

To prove this let $\vec{w}'$ be a weight of $\mathcal{G}'$ and define

$$s \equiv \exp[2\pi i((\sigma^{-1} - 1)^{-1}(\vec{w}') \cdot \vec{H})] \in T \quad (\text{A5})$$

Since $\vec{\sigma} \in G'$ we may choose k and t in (A 2) to lie in G' as well. Define $g \in G'$ by

$$g = k s k^{-1} \quad (\text{A6})$$

and observe that

$$\begin{aligned} g t g^{-1} &= t (k \exp[2\pi i \vec{w}' \cdot \vec{H}] k^{-1}) \\ &= t \exp[2\pi i \vec{w}' \cdot \vec{H}] \end{aligned} \quad (\text{A7})$$

Thus we have defined a map from $\Lambda_w(\mathcal{G}')$ onto $Z(G') \equiv \frac{\Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}')}$. This map is trivial if and only if $\vec{w}'$ is a root of $\mathcal{G}'$ or, from (A 7), if and only if g commutes with t . Suppose that g commutes with t . Let $Z(s)$ be the centralizer of s in G' . The centralizer, $Z(s)$, is a Lie subgroup of G' , moreover it contains $\vec{\sigma}$ since $g t g^{-1} = t$ is equivalent to $s \vec{\sigma} s^{-1} = \vec{\sigma}$. Therefore, by minimality of G' we must have $Z(s) \supseteq G'$ and hence $Z(s) = G'$. It follows that s lies in the center of G' and hence $(\sigma^{-1} - 1)^{-1}(\vec{w}')$ is a weight of $\mathcal{G}'$. The converse is obvious, if $(\sigma^{-1} - 1)^{-1}(\vec{w}')$ is a weight of $\mathcal{G}'$ then $g = s$ and $g t g^{-1} = s t s^{-1} = t$, and so $\vec{w}'$ must be a root of $\mathcal{G}'$. Thus we have shown that the kernel of the map described above is precisely $(\sigma^{-1} - 1)(\Lambda_w(\mathcal{G}')) \equiv \Lambda_r(\mathcal{G}')$. Hence

$$\left| \frac{\Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}')} \right| = \det(\sigma^{-1} - 1) = \det(1 - \sigma) \quad (\text{A8})$$

as required.

We have, in fact, established considerably more than (A 4). The identity (A 7) strongly suggests that g induces a Weyl rotation of $\vec{\sigma}$ to $\vec{\sigma} + \vec{w}'$. Indeed it was shown in I that there is an element $f \in G'$ that commutes with gtg^{-1} and for which $(fg)T(fg)^{-1} = T$. Thus fg induces a Weyl rotation, and therefore the action of g on $\vec{\sigma}$ is that of a Weyl rotation. Thus we have shown that for any weight $\vec{w}'$ of $\mathcal{G}'$ there is an element γ' of $W(\mathcal{G}')$ such that

$$\vec{\sigma} + \vec{w}' = \gamma'(\vec{\sigma}) + \vec{\alpha}' \quad (\text{A9})$$

for some root $\vec{\alpha}'$ of $\mathcal{G}'$. Moreover we know that a lift, $\tilde{\gamma}'$, of γ' is given by $\tilde{\gamma}' = (fk)s(fk)^{-1}$, and thus the shift corresponding to γ' is $(\sigma^{-1} - 1)^{-1}(\vec{w}')^*$. In particular it follows that for each conjugacy class of $\frac{\Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}')}$ there is at least one weight such that

$$(\vec{\sigma} + \vec{w}')^2 = (\vec{\sigma})^2 \quad (\text{A10})$$

Suppose that this identity holds for both $\vec{w}'$ and $\vec{w}' + \vec{\beta}'$ for some root vector $\vec{\beta}'$ on $\Lambda_r(\mathcal{G}')$, then we have

$$-\frac{1}{2}(\vec{\beta}')^2 = (\vec{\sigma} + \vec{w}') \cdot \vec{\beta}' = \gamma'(\vec{\sigma}) \cdot \vec{\beta}' = \vec{\sigma} \cdot (\gamma'^{-1}(\vec{\beta}')) \quad (\text{A11})$$

Observe that $\gamma'^{-1}(\vec{\beta}')$ is also a root vector on $\Lambda_r(\mathcal{G}')$. If one can show that for all non-zero root vectors $\vec{r}$ on $\Lambda_r(\mathcal{G}')$, and for a Weyl shift $\vec{\sigma}$, of minimal length, that

$$|\vec{\sigma} \cdot \vec{r}| < \frac{1}{2}(\vec{r})^2 \quad (\text{A12})$$

then (A 11) implies that $\vec{\beta}' \equiv 0$ and we have the following result.

Let $\sigma \in W(\mathcal{G})$ be non-degenerate, and let $\mathcal{G}'$ be a minimal sub-algebra for σ . Let $\vec{\sigma}$ be a shift of minimal length corresponding to σ , then there is precisely one weight in each conjugacy class of $\frac{\Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}')}$ such that (A 10) is satisfied, and the number of such weights is equal to $\det(1 - \sigma)$.

* It is amusing to note that given a non-degenerate Weyl element σ , we have defined a natural map of $Z(\mathcal{G}')$ into $W(\mathcal{G}')$.

All that remains to is to establish (A 12). Write $\vec{r} = \sum_{\vec{\alpha}} a_{\alpha} \vec{\alpha}$ where a_{α} are some integer coefficients and $\vec{\alpha}$ are root vectors such that $(\vec{\alpha})^2 = 2$. Then

$$\frac{1}{2}(\vec{r})^2 = \sum (a_{\alpha})^2 + \left(\sum_{\alpha \neq \beta} a_{\alpha} a_{\beta} \vec{\alpha} \cdot \vec{\beta} \right) \quad (\text{A13})$$

We can choose the a_{α} in such a way that all terms in the second sum in (A 13) are positive (for if one were negative the pair of roots involved can always be replaced by some linear combinations so that the corresponding term in (A 13) is positive). It follows that we can arrange that $\frac{1}{2}(\vec{r})^2 \geq \sum (a_{\alpha})^2$ and hence

$$\begin{aligned} |\vec{\sigma} \cdot \vec{r}| &= \left| \sum_{\alpha} a_{\alpha} \vec{\alpha} \cdot \vec{\sigma} \right| \\ &\leq \left(\frac{1}{2}(\vec{r})^2 \right) \times \max_{\{\vec{\alpha}: (\vec{\alpha})^2=2\}} |\vec{\alpha} \cdot \vec{\sigma}| \end{aligned} \quad (\text{A14})$$

By inspection of all the minimal shifts of all proper elements of all Lie groups one finds that $|\vec{\sigma} \cdot \vec{\alpha}| < 1$ for all roots $\vec{\alpha}$ with $(\vec{\alpha})^2 = 2$ [†]. The result follows.

As a final remark we observe that with $\sigma \in W(\mathcal{G})$ and $\mathcal{G}'$ a minimal sub-algebra for σ , we have, by duality

$$\frac{\Lambda_r(\mathcal{G})}{\Lambda_r(\mathcal{G}')} \equiv \frac{\Lambda_w(\mathcal{G}')}{\Lambda_w(\mathcal{G})} \quad (\text{A15})$$

hence

$$\begin{aligned} \left| \frac{\Lambda_r(\mathcal{G})}{\Lambda_r(\mathcal{G}')} \right|^2 &= \left| \frac{\Lambda_r(\mathcal{G}) \times \Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}') \times \Lambda_w(\mathcal{G})} \right| \\ &\equiv \left| \frac{\Lambda_w(\mathcal{G})}{\Lambda_r(\mathcal{G})} \right|^{-1} \times \left| \frac{\Lambda_w(\mathcal{G}')}{\Lambda_r(\mathcal{G}')} \right| \\ &\equiv \frac{\det(1 - \sigma)}{|Z(G)|} \end{aligned} \quad (\text{A16})$$

and so

$$\left| \frac{\Lambda_r(\mathcal{G})}{\Lambda_r(\mathcal{G}')} \right| = \left| \frac{\Lambda_w(\mathcal{G}')}{\Lambda_w(\mathcal{G})} \right| = \sqrt{\frac{\det(1 - \sigma)}{|Z(G)|}} \quad (\text{A17})$$

[†] It would be nice to have a more direct and simple proof of this fact.

$$\left| \frac{\Lambda_w(\mathcal{G})}{\Lambda_r(\mathcal{G}')} \right| = \sqrt{\det(1 - \sigma) |Z(G)|} \quad (\text{A18})$$

APPENDIX B

In I we gave a list of shift vectors for all the non-degenerate Weyl elements of simple, simply-laced Lie algebras of rank less than or equal to 8. In this appendix we wish to complete this list by removing the restriction on the rank. The only non-degenerate Weyl elements in A_n are the Coxeter elements, whose equivalent shifts were given in I for all n . Moreover, I also contains the shift vectors of the non-degenerate Weyl elements of the exceptional Lie algebras, thus it only remains to consider the D_n series for $n > 8$. There is, of course, no unique choice for the shift vector corresponding to a particular non-degenerate Weyl twist, however, once one knows one such vector all others can be obtained by a Weyl rotation of the known vector^{*}. Therefore we will only give one of the possible shift vectors for the Weyl element.

Let $\{e_i\}, i = 1, \dots, n$ denote the standard weight basis for D_n , where the roots of D_n are $\{\pm e_i \pm e_j\}$ (see for example [33]). The conjugacy classes of non-degenerate Weyl elements of D_n are labelled (see [34], or for a short review see I) as $D_n(a_{k-1}), k = 1, \dots, [\frac{n}{2}]$ (where $[x]$ denotes the greatest integer less than or equal to x) and the action of a typical element, g , of this class on the weight basis is:

$$\begin{aligned} e_1 &\rightarrow e_2 \rightarrow \dots \rightarrow e_k \rightarrow -e_1 \rightarrow -e_2 \rightarrow \dots \rightarrow -e_k \rightarrow e_1. \\ e_{k+1} &\rightarrow e_{k+2} \rightarrow \dots \rightarrow e_n \rightarrow -e_{k+1} \rightarrow -e_{k+2} \rightarrow \dots \rightarrow -e_n \rightarrow e_{k+1}. \end{aligned} \quad (\text{B1})$$

Note that for $k = 1$ this describes the action of a Coxeter element, and thus $D_n(a_0)$ is the Coxeter conjugacy class. The easiest way to determine the corresponding shift vector is to observe that since the order of a non-degenerate Weyl element of D_n does *not* double on the vector representation, the eigenvalues of $\tilde{g}$, a lift of the Weyl element g to the Lie group,

^{*} The addition of a weight vector is also allowed, but for non-degenerate elements this is equivalent to a Weyl rotation of the shift.

can be read off from (B 1) directly. Since (B 1) represents a simple permutation of the vector weight spaces, the eigenvalues of $\tilde{g}$ are:

$$e^{\pi i p/k}, \quad e^{\pi i q/(n-k)} \quad p = 0, \dots, 2k-1; \quad q = 0, \dots, 2(n-k)-1. \quad (\text{B2})$$

Since $\exp(2\pi i \vec{\sigma} \cdot \vec{H})$ must have the same eigenvalues, it is elementary to see that one way to achieve this, and hence one possible choice for the shift vector is:

$$\begin{aligned} \vec{\sigma} = & \frac{1}{2k} [k e_1 + (k-1)e_2 + \dots + 2e_{k-1} + e_k] \\ & + \frac{1}{2(n-k)} [(n-k-1)e_{k+1} + (n-k-2)e_{k+2} + \dots + e_{n-1}] \end{aligned} \quad (\text{B3})$$

For $k=1$ this, of course, reproduces the Coxeter shift derived in **I**, and for $n \leq 8$ it also reproduces the other non-degenerate shifts given in **I**.

It is easily checked that

$$(\vec{\sigma})^2 = \frac{n}{24} \left[2 + \frac{1}{k(n-k)} \right] \quad (\text{B4})$$

Moreover, let $e^{2\pi i \theta_j}$, $j = 1, 2, \dots, n$ denote the eigenvalues of the original Weyl twist, g , in $D_n(a_{k-1})$ acting on the Cartan sub-algebra of D_n . It is shown in [34] that these eigenvalues are

$$e^{2\pi i(p-\frac{1}{2})/k}, \quad e^{2\pi i(q-\frac{1}{2})/(n-k)} \quad p = 1, 2, \dots, k; \quad q = 1, 2, \dots, (n-k).$$

One can then explicitly check that if the θ_j are chosen so that $0 \leq \theta_j < 1$ then the strict form of level matching is satisfied:

$$\sum_{j=1}^n \theta_j(1 - \theta_j) = \frac{n}{24} \left[2 + \frac{1}{k(n-k)} \right] = (\vec{\sigma})^2 \quad (\text{B5})$$

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Table Captions

— *Table 1*

List of all non-tachyonic spectra obtained with single shifts. Column 1 gives the type (A,B or C) defined in the text, as well as the number of chiral generations (subscript). The superscript is only a dummy label. The definition of N_i is given in the text. Column 8 indicates the lattice obtained by left-right interchange; “S” indicates that the lattice is left-right symmetric. The last column gives the additional gauge groups, not including E_6 (D_5) or the $U(1)$ factors.

— *Table 2a – 2h*

Spectra obtained for all combinations of Weyl twists on EN-lattices. The unshifted lattice is indicated in the upper left corner. The first column indicates the twists applied to the right (fermionic) sector and the first row indicates the twist applied to the left (bosonic) sector. If a left (right) shift has two inequivalent realizations, there are two columns (rows) associated with it. They represent the primary and secondary shift vector respectively. The spectra are labelled as in table 1.

— *Table 3*

Spectra obtained for the lattices that admit only one single shift.

Table 1

Spectrum	N_1	N_2	N_G	N_3	N_4	N_5	Refl.	Gauge-group
A_{36}^1	36	0	6	270	—	—	S	A_2
A_{36}^2	36	0	24	324	—	—	S	$(A_2)^4$
A_{24}^1	24	0	0	192	—	—	S	—
A_{24}^2	25	1	0	180	—	—	S	—
A_{24}^3	25	1	2	208	—	—	S	A_1
A_{24}^4	25	1	8	236	—	—	S	$(A_1)^4$
A_{24}^5	27	3	0	192	—	—	S	—
A_{24}^6	27	3	12	270	—	—	S	$(A_1)^6$
A_{24}^7	29	5	0	202	—	—	S	—
A_{24}^8	31	7	4	254	—	—	S	$(A_1)^2$
A_{12}^1	16	4	8	236	—	—	S	$(A_1)^4$
A_0^1	13	13	24	238	—	—	A_0^2	$(A_2)^4$
A_0^2	13	13	26	322	—	—	A_0^1	$D_4 A_1$
B_{48}^1	48	0	12	54	12	—	C_{48}^1	$(A_1)^6$
B_{48}^2	48	0	12	172	24	—	C_{48}^2	$(A_1)^6$
B_{48}^3	48	0	20	164	30	—	C_{48}^3	$(A_1)^4 A_3$
B_{48}^4	48	0	36	108	18	—	C_{48}^4	$(A_3)^3$
B_{36}^1	37	1	24	120	12	—	C_{36}^1	$(A_2)^4$
B_{12}^1	17	5	8	264	28	—	C_{12}^1	$(A_1)^4$
B_0^1	8	8	0	206	26	—	C_0^1	—
B_0^2	10	10	2	234	26	—	C_0^2	A_1
C_{48}^4	48	0	12	324	18	432	B_{48}^4	$(A_1)^6$
C_{36}^1	37	1	8	312	12	368	B_{36}^1	$(A_1)^4$

Table 2a

E_6	E_6		$E_6(a_1)$		$E_6(a_2)$		$A_5 A_1$	$(A_2)^3$	
E_6	A_{24}^2	B_0^1	B_{36}^1	A_{36}^2	A_{12}^1	B_{12}^1	A_{24}^4	B_{36}^1	A_{36}^2
	C_0^1	D_{24}^3	B_{36}^1	A_{36}^2	A_{12}^1	B_{12}^1	A_{24}^4	B_{36}^1	A_{36}^2
$E_6(a_1)$	C_{36}^1	C_{36}^1	D_{48}^1	D_{24}^1	C_{36}^1	D_0^1	C_{36}^1	D_{72}^1	A_{36}^2
	A_{36}^2	A_{36}^2	D_{24}^1	D_{24}^1	A_0^2	A_{36}^2	A_{36}^2	A_{36}^2	A_{36}^2
$E_6(a_2)$	A_{12}^1	A_{12}^1	B_{36}^1	A_0^1	A_{24}^4	B_{12}^1	A_{12}^1	B_{36}^1	A_0^1
	C_{12}^1	C_{12}^1	D_0^2	A_{36}^2	C_{12}^1	D_{24}^2	C_{12}^1	D_0^2	A_{36}^2
$A_5 A_1$	A_{24}^4	A_{24}^4	B_{36}^1	A_{36}^2	A_{12}^1	B_{12}^1	A_{24}^4	B_{36}^1	A_{36}^2
$(A_2)^3$	C_{36}^1	C_{36}^1	D_{72}^1	A_{36}^2	C_{36}^1	D_0^1	C_{36}^1	D_{72}^1	A_{36}^2
	A_{36}^2	A_{36}^2	A_{36}^2	A_{36}^2	A_0^2	A_{36}^2	A_{36}^2	A_{36}^2	A_{36}^2

Table 2b

D_6	D_6	$D_6(a_1)$		$D_6(a_2)$	$D_4 D_2$	$D_3 D_3$	$(D_2)^3$
D_6	D_{48}^2	C_{48}^4	C_{48}^4	D_{96}^{11}	D_{96}^{11}	C_{48}^4	D_{96}^{11}
$D_6(a_1)$	B_{48}^4	A_{24}^3	B_0^2	B_{48}^4	B_{48}^4	A_{24}^6	B_{48}^4
	B_{48}^4	C_0^2	D_{24}^4	B_{48}^4	B_{48}^4	A_{24}^6	B_{48}^4
$D_6(a_2)$	D_{96}^{11}	C_{48}^4	C_{48}^4	D_{48}^4	D_0^4	C_{48}^4	D_{96}^{11}
$D_4 D_2$	D_{96}^{11}	C_{48}^4	C_{48}^4	D_0^3	D_{96}^{12}	C_{48}^4	D_{96}^{11}
$D_3 D_3$	B_{48}^4	A_{24}^6	A_{24}^6	B_{48}^4	B_{48}^4	A_{24}^6	B_{48}^4
$(D_2)^3$	D_{96}^{11}	C_{48}^4	C_{48}^4	D_{96}^{11}	D_{96}^{11}	C_{48}^4	D_{96}^{11}

Table 2c

$D_4 D_2$	$D_4 D_2$	$D_4(a_1)D_2$	$(D_2)^3$
$D_4 D_2$	D_{96}^5	C_{48}^3	D_{96}^6
$D_4(a_1)D_2$	B_{48}^3	A_{24}^8	B_{48}^3
$(D_2)^3$	D_{96}^6	C_{48}^3	D_{96}^6

Table 2d

$D_5 A_1$	$D_5 A_1$	$D_3 D_2 A_1$	$D_5(a_1)A_1$
$D_5 A_1$	A_{24}^5	B_{48}^1	B_{48}^1
$D_3 D_2 A_1$	C_{48}^1	D_{96}^7	D_{96}^7
$D_5(a_1)A_1$	C_{48}^1	D_{96}^7	D_{96}^8

Table 2e

$D_4 A_2$	$D_4 A_2$	$D_2 D_2 A_2$	$D_4(a_1)A_2$
$D_4 A_2$	A_{24}^7	B_{48}^2	B_{48}^2
$D_2 D_2 A_2$	C_{48}^2	D_{96}^9	D_{96}^9
$D_4(a_1)A_1$	C_{48}^2	D_{96}^9	D_{96}^{10}

Table 2f

A_6	A_6	
A_6	A_{24}^1	A_{24}^1
	A_{24}^1	D_{48}^3

Table 2g

$A_4 A_2$	$A_4 A_2$	
$A_4 A_2$	D_{96}^4	D_{96}^4
	D_{96}^4	D_{192}^2

Table 2h

$(A_2)^3$	$(A_2)^3$	
A_2^3	D_{72}^2	A_{36}^1
	A_{36}^1	A_{36}^1

Table 3

Lattice	Spectrum
$A_5 A_1$	A_{24}^2
$A_3 A_3$	A_{24}^3
$A_4 D_2$	D_{144}^2
$A_3 A_2 A_1$	D_{192}^1
$A_3 D_2 A_1$	D_{96}^3
$(A_2)^2 D_2$	D_{144}^1
$A_2 (D_2)^2$	D_{96}^2
$(D_2)^3$	D_{96}^1
$((D_2)^3)^*$	D_{96}^1
(*) non-EN	