CERN-TH.5067/88

HAWKING RADIATION AND THE VIRASORO ALGEBRA

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A B S T R A C T

The Hawking radiation in two-dimensional moving mirror models of black hole formation is shown to be a direct consequence of the central charge in the Virasoro algebra. Hence mirror trajectories related by the action of an anomaly free $SL(2, \mathbb{R})$ subgroup lead to the same energy-momentum tensor expectation value. The same invariance is found in the energy spectrum of the created particles. The relation to the Frolov-Vilkovisky effective action approach is discussed.

CERN-TH.5067/88

June 1988

Soon after Hawking's [1,2] famous discovery that black holes evaporate particles with a thermal distribution at a temperature $T = 1/8\pi M$ it was realized that this effect is not mainly due to the intense gravitational field near a black hole, but is rather a consequence of the global structure of the space-time manifold. In particular, the existence of a future horizon turned out to be crucial. Motivated by this observation, field theory systems were constructed in flat space [3-7] which reproduce the salient features of Hawking's result but avoid additional complications due to space-time curvature. In their simplest version these so-called moving mirror models consist of a free, massless scalar field ϕ defined on a two-dimensional semi-infinite Minkowski space which is bounded by a "reflecting mirror" following a timelike trajectory $x = z(t)$. (In two dimensions the mirror actually degenerates to a point.) Roughly speaking, this two-dimensional space models the radial world of a spherically collapsing system forming a black hole [4]. The field operator $\phi(t, x)$ is a solution of the wave equation $\square \phi = 0$ which obeys Dirichlet boundary conditions $\phi[t, z(t)] = 0$, i.e. it vanishes on the trajectory of the mirror. Because of this time-dependent boundary condition the system does not possess a timelike Killing vector field and hence non-trivial particle production effects are to be expected. The quantum state of the field is assumed to be the "in"-vacuum, a state defined by the requirement that there is no particle flux through past null infinity $\mathcal{I}^-$. For a trajectory which is static for $t < 0$, say, this means that in the remote past there are no particles present. A trajectory accelerating for $t > 0$ can lead to a flux of particles leaving the mirror and propagating to future null infinity $\mathcal{I}^+$. The formal reason for this particle creation is a non-trivial Bogoliubov transformation relating the creation and annihilation operators associated with field modes behaving like plane waves on $\mathcal{I}^-$ and $\mathcal{I}^+$, respectively. There are certain trajectories which are known to lead to a thermal particle flux [3,6,7]. They give rise to a future horizon.

To investigate the properties of the radiation from moving mirrors it is important to know the expectation value of the energy-momentum tensor $T_{\mu\nu}$. Using light-cone co-ordinates $u = t - x$ and $v = t + x$ its non-vanishing components are $T_{uu} = (\partial_u \phi)^2$ and $T_{vv} = (\partial_v \phi)^2$. A particle flux towards future null infinity created by the mirror would be indicated by a non-zero expectation value of T_{uu} in a state (we shall denote it by $|\text{vac}\rangle$) which has vanishing expectation value of T_{vv} on $\mathcal{I}^-$. A calculation of $\langle \text{vac} | T_{uu} | \text{vac} \rangle$ was first done by Fulling and Davies [3]. They encountered a divergent sum over field modes which they regularized by point splitting methods. The free, massless, real scalar field ϕ was quantized by expanding the field operator as

$$\phi(u, v) = \int_0^\infty d\omega [a_\omega \phi_\omega(u, v) + a_\omega^\dagger \phi_\omega^*(u, v)] \quad (1)$$

in terms of a complete set of mode functions given by

$$\phi_\omega(u, v) = (4\pi\omega)^{-1/2} [e^{-i\omega v} - e^{-i\omega p(u)}], \quad \omega > 0. \quad (2)$$

Here p is the "ray tracing" function [3] belonging to the trajectory $x = z(t)$. In terms of light cone co-ordinates the location of the mirror is given by $(u, v = p(u))$. As required the ϕ_ω 's vanish along this line. The operators a_ω and $a_\omega^\dagger$ obey the canonical commutation relations $[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega')$. Since ϕ_ω behaves like an ingoing plane wave on $\mathcal{H}^-$ (i.e. for $u \rightarrow -\infty$) the "in"-vacuum is defined by

$$a_\omega |vac\rangle = 0 \quad (3)$$

For this state Fulling and Davies obtain the renormalized expectation value [3]

$$\langle vac | T_{uu}(u) | vac \rangle = \frac{1}{12\pi} [p'(u)]^{\frac{1}{2}} \partial_u^2 [p'(u)]^{-1/2}, \quad (4)$$

and a vanishing expectation value for T_{vv} . An alternative derivation [2] of Eq. (4) uses the fact that by a conformal co-ordinate transformation any accelerated trajectory $x = z(t)$ can be mapped onto a straight line. This results in a time-independent boundary condition, but also in a non-trivial space-time metric conformally related to the Minkowski metric. In two dimensions the effective action and hence $\langle T_{\mu\nu} \rangle$ can be evaluated in closed form. Within the framework of dimensional regularization^{*}, for instance, one finds [2,8], for a metric of the form $ds^2 = C(u, v) du dv$:

$$\langle T_{\mu\nu} \rangle = \Theta_{\mu\nu} - \frac{1}{48\pi} R g_{\mu\nu} \quad (5a)$$

^{*}) A point splitting derivation has been given in Ref. [9].

where R is the curvature scalar and

$$\begin{aligned}\Theta_{uu} &= -\frac{1}{12\pi} C^{1/2} \partial_u^2 C^{-1/2} \\ \Theta_{vv} &= -\frac{1}{12\pi} C^{1/2} \partial_v^2 C^{-1/2} \\ \Theta_{uv} &= \Theta_{vu} = 0.\end{aligned}\tag{5b}$$

[If the space-time under consideration is conformal only to a part of Minkowski space, the RHS of Eq. (5a) contains additional terms.] Applying (5) to the (flat) space-time which results from the transformations making the boundary condition time-independent again leads to the result (4). This second derivation makes it obvious that a non-vanishing $\langle T_{uu} \rangle$ must be related to a quantum mechanical breaking of conformal symmetry. At the classical level the theory is invariant under conformal transformations and hence a moving mirror and a mirror at rest should be equivalent. However, contrary to the curved space-time discussions of the Hawking effect, in the moving mirror model there is no trace anomaly, which is the most prominent symptom of conformal symmetry breaking [10]. In this note we give a simple proof of (4) which shows that the radiation from the mirror is due to another manifestation of the conformal anomaly, viz. the Schwinger term (or central charge) in the Virasoro algebra of the energy momentum tensor [11-13]. It is known that in a certain sense the central charge can be considered a remnant of the trace anomaly when the gravitational field is switched off^{*)}. The resulting non-canonical transformation law of T_{uu} under conformal transformations will lead to the inequivalence between the accelerating mirror and the mirror at rest. The relation between the central charge and the trace anomaly is paralleled by the flat space (i.e. moving mirror) and curved space models of the two-dimensional Hawking effect. As was pointed out by Christensen and Fulling [10] in the framework of two dimensional^{**)} gravitational collapse models using a curved spacetime metric, Hawking radiation and trace anomaly are equivalent to each other. We shall prove the corresponding statement for their flat space-time analogues, namely that the radiation from moving mirrors is equivalent to a non-zero central charge. As is well known in the context of string theory [16], for instance, the central charge is not "felt" by the subalgebra of the Virasoro generators L_{-1} , L_0 and L_{+1} generating Möbius transformations. It turns out that this implies that the T_{uu} -expectation values for all mirror trajectories with ray tracing functions related by a Möbius transformation are the same.

^{*)} For a recent discussion, see [14,15].

^{**)} In a more realistic four-dimensional setting the trace anomaly is not sufficient to determine the Hawking flux, of course.

The relation between $\langle T_{uu} \rangle$ and the central charge is as follows. Given the expansion (1) one has $T_{uu} = (\partial_u \phi)^2$ with

$$\partial_u \phi(u) = i p'(u) \int_0^\infty d\omega \left(\frac{\omega}{4\pi} \right)^{1/2} \left[a_\omega e^{-i\omega p(u)} - a_\omega^\dagger e^{+i\omega p(u)} \right]. \quad (6)$$

To compute $\langle \text{vac} | T_{uu}(u) | \text{vac} \rangle$ we introduce a new operator $T_{uu}^0 \equiv (\partial_u \phi_0)^2$ where

$$\partial_u \phi_0(u) = i \int_0^\infty d\omega \left(\frac{\omega}{4\pi} \right)^{1/2} \left[a_\omega e^{-i\omega u} - a_\omega^\dagger e^{+i\omega u} \right].$$

Obviously T_{uu}^0 is the energy momentum tensor for a trajectory with $p(u) = u$, i.e. for a mirror which remains at $x = 0$ for all t . We know that there is no radiation from such a mirror. Our aim is to perform a conformal co-ordinate transformation $u \rightarrow p^{-1}(u)$ reducing T_{uu} to its free counterpart T_{uu}^0 . The energy momentum tensor obeys a Virasoro algebra with a central extension [11-14]:

$$\begin{aligned} [T_{uu}(u), T_{uu}(u')] &= -i T_{uu}'(u) \delta(u-u') \\ &\quad - 2i T_{uu}(u) \delta'(u-u') - \frac{i}{24\pi} \delta'''(u-u'). \end{aligned} \quad (7)$$

Constructing unitary operators

$$U = \exp \left[-i \int du \epsilon(u) T_{uu}(u) \right] \quad (8)$$

implementing infinitesimal transformations $u \rightarrow u - \epsilon(u)$ in the Hilbert space, one consequently has

$$U T_{uu}(u) U^\dagger = T_{uu}(u) + \delta T_{uu}(u) \quad (9)$$

with

$$\delta T_{uu}(u) = \varepsilon(u) T'_{uu}(u) + 2\varepsilon'(u) T_{uu}(u) + \frac{1}{24\pi} \varepsilon'''(u). \quad (10)$$

The first two terms on the RHS of Eq. (10) correspond to the classical transformation law, the third term results from the anomaly. Iterating transformations like (10) one finds [11] for finite transformations $u \rightarrow \tilde{u}(u)$:

$$U T_{uu}(\tilde{u}) U^\dagger = \left(\frac{du}{d\tilde{u}} \right)^2 T_{uu}(u(\tilde{u})) + \frac{1}{24\pi} \{u; \tilde{u}\}. \quad (11)$$

Here

$$\{f; x\} \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2 \quad (12)$$

is the Schwarzian derivative. Equation (11) clearly shows that T_{uu} does not transform like a classical tensor. This is the origin of the conformal inequivalence of a mirror at rest and an accelerating one. Now we may write

$$\begin{aligned} \langle \text{vac} | T_{uu}(u) | \text{vac} \rangle &= [p'(u)]^2 \langle \text{vac} | T_{uu}^0(p(u)) | \text{vac} \rangle \\ &= [p'(u)]^2 \langle 0 | U T_{uu}^0(p(u)) U^\dagger | 0 \rangle, \end{aligned} \quad (13)$$

where U is the operator implementing the co-ordinate change from u to $p^{-1}(u)$ and $|0\rangle \equiv U|\text{vac}\rangle$ is the transformed vacuum state. From (11) one has

$$U T_{uu}^0(p(u)) U^\dagger = [p'(u)]^{-2} T_{uu}^0(u) + \frac{1}{24\pi} \{u; p\}. \quad (14)$$

Inserting this into (13) and using the identity

$$\{y; x\} = -\left(\frac{dy}{dx}\right)^2 \{x; y\}$$

one finds

$$\langle \text{vac} | T_{uu}(u) | \text{vac} \rangle = \langle 0 | T_{uu}^0(u) | 0 \rangle - \frac{1}{24\pi} \{p; u\} \quad (15)$$

The first term on the RHS of Eq. (15) is the expectation value of the free energy momentum tensor, i.e. the one for $z(t) \equiv 0$, in the corresponding vacuum state. It therefore vanishes. [The state $|0\rangle$ is indeed the conventional vacuum for time independent boundary conditions. Since T_{vv} is unaffected by the u -transformation one has $\langle \text{vac} | T_{vv} | \text{vac} \rangle = 0$ as well as $\langle 0 | T_{vv} | 0 \rangle = 0$, i.e. zero flux through $\mathcal{J}^-$ also for $z(t) \equiv 0$. A standard calculation [2] then shows that $\langle 0 | T_{uu}^0 | 0 \rangle = 0$.] Thus equation (15) reads

$$\begin{aligned} \langle \text{vac} | T_{uu}(u) | \text{vac} \rangle &= -\frac{1}{24\pi} \{p; u\} \\ &= -\frac{1}{24\pi} \left[\frac{p'''(u)}{p'(u)} - \frac{3}{2} \left(\frac{p''(u)}{p'(u)} \right)^2 \right] \end{aligned} \quad (16)$$

This is identical to the results (4) obtained by point-splitting regularization. Obviously the non-zero flux to $\mathcal{J}^+$ produced by the mirror is entirely due to the Schwinger term in the Virasoro algebra^{*)} from which it follows without having to perform any renormalization. Since the central charge itself also can be calculated without encountering any divergent mode sums (cf., the discussion in Refs. [16,17], for instance), the present derivation is finite at each step. An immediate corollary of Eq. (16) is that the flux vanishes if the conformal transformation used is generated by the anomaly-free subalgebra of the Virasoro

*) The Schwarzian derivatives on the RHS of eq. (5b) have a similar origin.

Virasoro algebra, which is unaffected by the presence of the central extension [12,13,16]. This subalgebra generates the Möbius transformations

$$u \longrightarrow \frac{au+b}{cu+d} \equiv Mu \quad (17)$$

For the present application the coefficients a , b , c and d are assumed to be real and normalized according to $ad-bc = 1$. Then the group of Möbius transformations is isomorphic to $SL(2,R)$. Hence the trajectories with $\langle T_{uu} \rangle = 0$ are of the form

$$p(u) = \frac{Au+B}{Cu+D} \quad (18)$$

The Schwarzian derivative on the RHS of Eq. (16) vanishes if and only if $p(u)$ has this structure. As was already realized by Fulling and Davies [3], Eq. (18) describes trajectories with constant acceleration. Furthermore, because of the $SL(2,R)$ invariance of the Schwarzian derivative,

$$\left\{ \frac{ay+b}{cy+d} ; x \right\} = \{ y ; x \} ,$$

two trajectories related by a $SL(2,R)$ transformation lead to the same particle flux, i.e. the trajectories $p(u)$ and

$$\tilde{p}(u) = \frac{ap(u)+b}{cp(u)+d} \equiv Mp(u) \quad (19)$$

have the same T_{uu} -expectation value. The function $p(u)$ can be transformed into $\tilde{p}(u)$ without using Virasoro generators affected by the central charge. In the following we shall mainly use the inverse ray tracing function $u = f(v)$ with $f[p(u)] = u$. Under the transformation (19), it becomes

$$\tilde{f}(v) = f(M^{-1}v)$$

One can check that if the original trajectory is timelike, i.e. $p'(u) > 0$ and $f'(u) > 0$, then the new trajectory is timelike as well. However, not each $SL(2, R)$ transformed trajectory is physically meaningful. Assume the original trajectory is defined by the map $f: [v_{\min}, v_{\max}] \rightarrow [u_{\min}, u_{\max}]$. For $\tilde{f}(v)$ to be defined, one must require $M^{-1}v \in [v_{\min}, v_{\max}]$ for all v in the domain of $\tilde{f}$. Writing $M^{-1}v \equiv (\alpha v + \beta)/(\gamma v + \delta)$, M^{-1} has a pole at $-\delta/\gamma$ which divides the domain of M^{-1} into two disjoint pieces, viz. $v < -\delta/\gamma$ and $v > -\delta/\gamma$. If the requirement $M^{-1}v \in [v_{\min}, v_{\max}]$ is met for values of v both with $v < -\delta/\gamma$ and $v > -\delta/\gamma$ also the domain of $\tilde{f}$ consists of two disjoint pieces. Geometrically this means that the original trajectory is mapped onto a curve consisting of two branches which usually no longer can be considered the boundary of spacetime. On the other hand, if only one branch of M^{-1} fits into the interval $[v_{\min}, v_{\max}]$, the new trajectory has a well-defined interpretation.

As an example let us look at the trajectory [6,7]

$$f(v) = -\kappa^{-1} \ln \left[\frac{B-v}{B} \right], \quad \kappa, B > 0.$$

It is known to lead to a thermal particle spectrum. The domain of f is $(-\infty, B)$. The trajectory has a v -asymptote at $v = B$. If the $SL(2, R)$ transformations are restricted to those with $\alpha/\gamma > B$, the domain of $\tilde{f}$ is $(-\delta/\gamma, MB)$. Then the new trajectory is perfectly acceptable. In general it has two v -asymptotes, namely for $v = -\delta/\gamma$ and $v = MB$. they are approached for $u \rightarrow -\infty$ and $u \rightarrow +\infty$, respectively. In the limit $\alpha/\gamma \rightarrow B$ the second asymptote is shifted to infinity. [For $f(v) = -\kappa^{-1} \ln[-v]$ this corresponds to the simple inversion $M^{-1}v = -1/v$.]

We saw that the flux $\langle T_{uu} \rangle = 0$ is a consequence of the Schwinger term in the energy-momentum tensor algebra. On the other hand it is known [14,15] that this Schwinger term is related to the trace anomaly in two-dimensional curved spacetime:

$$\langle T^\mu_\mu \rangle = -\frac{1}{24\pi} R. \quad (20)$$

As was first discussed by Christensen and Fulling [10], knowledge of eq.(20) is sufficient to determine the Hawking flux created by the two-dimensional metric describing a collapsing body. A related approach is due to Frolov and Vilkovisky [18]. It is based on the effective action $W[g_{\mu\nu}]$. In two dimensions W can be computed exactly, e.g. by functionally integrating the trace anomaly equation:

$$W[g_{\mu\nu}] = \frac{1}{96\pi} \int d^2x \sqrt{-g} R \square_g^{-1} R. \quad (21)$$

Let us briefly discuss the relation between the Frolov-Vilkovisky approach on one side and the moving mirror models and the Virasoro anomaly on the other. Quite generally, computing the effective action and the mean field using conventional Minkowski space Feynman rules, one finds that they are complex. The reason is that one actually calculates the matrix element $\langle \text{out} | F(g_{\mu\nu}) | \text{in} \rangle$ rather than the desired expectation value $\langle \text{in} | F(g_{\mu\nu}) | \text{in} \rangle$. Since the diagrammatic techniques yielding the latter expectation value directly are not easy to implement, Frolov and Vilkovisky propose the following heuristic procedure for computing $\langle \text{in} | T^{\mu\nu} | \text{in} \rangle$ (applicable in any number of space-time dimensions): (i) compute the Euclidean effective action W , (ii) calculate $\langle T^{\mu\nu} \rangle = -2g^{-\frac{1}{2}} \delta W / \delta g_{\mu\nu}$, and (iii) replace all Euclidean Green's functions $\square_g^{-1}$ appearing in $\langle T^{\mu\nu} \rangle$ by the retarded Minkowski space propagator $\hat{G}_{\text{ret}}$.

We shall now prove that at least in two dimensions the results of this procedure and of the conventional field theory methods coincide. Taking the functional derivative of (21) and substituting $\square_g^{-1} \rightarrow \hat{G}_{\text{ret}}$, one obtains

$$\begin{aligned} \langle T^{\mu\nu} \rangle &= \frac{1}{48\pi} \left[2 \nabla^\mu \nabla^\nu (\hat{G}_{\text{ret}} R) \right. \\ &\quad \left. + \nabla^\mu (\hat{G}_{\text{ret}} R) \nabla^\nu (\hat{G}_{\text{ret}} R) \right] \end{aligned}$$

$$-g^{\mu\nu} \left\{ 2R + \frac{1}{2} \nabla_\rho (\hat{G}_{\text{ret}} R) \nabla^\rho (\hat{G}_{\text{ret}} R) \right\}]$$

(22)

To isolate the Hawking effect we consider an asymptotically flat metric $g_{\mu\nu}(x)$ and introduce two sets (u_+, v_+) and (u_-, v_-) of null co-ordinates which become Minkowskian on future null infinity and past null infinity, respectively [18]:

$$ds^2 = 2 \psi_+(u_+, v_+) du_+ dv_+ = 2 \psi_-(u_-, v_-) du_- dv_- . \quad (23)$$

We require $\phi_+(\gamma^+) = \phi_-(\gamma^-) = 1$. The co-ordinates are related by two equations $u_+ = u_+(u_-)$ and $v_+ = v_+(v_-)$. The conformal factors obey

$$\psi_- = \frac{du_+}{du_-} \frac{dv_+}{dv_-} \psi_+ . \quad (24)$$

Because in two dimensions $R = \square_g \ln \phi$, $\ln \phi$ is a solution of an inhomogeneous d'Alembert equation. For $\square_g^{-1} = \hat{G}_{\text{ret}}$ one obtains $\ln \phi_-$ since for this boundary condition the solution necessarily vanishes on γ^- . Inserting $\hat{G}_{\text{ret}} R = \ln \phi_-$ into (22) one finds for the flux reaching γ^+ :

$$\langle T^{\mu\nu} \rangle l_\mu l_\nu \Big|_{\gamma^+} = \frac{1}{48\pi} \left[2 \frac{d^2 \ln \psi_-}{du_+^2} + \left(\frac{d \ln \psi_-}{du_+} \right)^2 \right] \Big|_{\gamma^+} . \quad (25)$$

Here $l^\mu \equiv \nabla^\mu v_+$ and we assume that γ^+ is a $v_- = \text{const}$ line, so that the flux through γ^+ is given by $\langle T^{\mu\nu} \rangle l_\mu l_\nu$. (This corresponds to $\langle T_{uu} \rangle$ above.) Because of the retarded boundary condition the flux through γ^- vanishes. The Hawking effect,

if present, is entirely due to the inequivalence of the two null co-ordinate systems. Since v_+ is constant on $\mathcal{Y}^+$, Eq. (24) implies $\ln \phi_- = \ln |du_+/du_-| + \text{const.}$ Inserting this into (25) yields the final result (the primes denote derivatives with respect to u_+):

$$\begin{aligned} \langle T^{\mu\nu} \rangle \ell_\mu \ell_\nu \Big|_{\mathcal{Y}^+} &= -\frac{1}{24\pi} \left[\frac{u_-'''}{u_-'} - \frac{3}{2} \left(\frac{u_-''}{u_-'} \right)^2 \right] \\ &= -\frac{1}{24\pi} \{ u_-; u_+ \} \end{aligned} \quad (26)$$

On the RHS of Eq. (26) we observe a Schwarzian derivative. It implies that there will be no Hawking radiation if u_- and u_+ are related by $u_- = (au_+ + b)/(cu_+ + d)$ and that the same flux is obtained for two functions $u_+(u_-)$ and $\tilde{u}_+(u_-)$ related by a Möbius transformation. (This is not to mean that the transformations necessarily are physically meaningful.) We can rederive the result (26) by a simple argument based on Eq. (11) applied to the co-ordinate transformation $u_+ \rightarrow u_+(u_-)$. First the field is quantized using the (u_-, v_-) system leading to the correct vacuum definition with vanishing flux through $\mathcal{Y}^-$. Then $\langle T^{\mu\nu}(\mathcal{Y}^+) \rangle$ is computed by expressing the energy-momentum tensor in terms of the (u_+, v_+) -co-ordinates via Eq. (11). [Note that on $\mathcal{Y}^+$ space-time is flat, i.e. the asymptotic form of the outgoing modes in terms of u_- is $\exp[-i\omega u_+(u_-)]$. As in the moving mirror discussion this becomes replaced by free out-modes $\exp[-i\omega u_+]$.] The naive term vanishes because the (u_+, v_+) -system is Minkowskian on $\mathcal{Y}^+$, whereas the anomaly exactly reproduces (26). Thus the answers of the Fradkin-Vilkovisky method based on the effective action or the trace anomaly and of conventional (conformal) field theory employing the anomalous Virasoro algebra are in complete agreement. In both derivations the two null co-ordinate systems become inequivalent at the quantum level. Comparing eqs. (26) and (16) we observe that the flux obtained in the curved space-time model could be simulated by a moving mirror model with $u_+ \equiv u$, $p(u) \equiv u_-(u)$ and $v \equiv v_+ = v_-$. This is not an accident since with these identifications the solutions of the wave equation $\square_g \phi = 0$ reduce on $\mathcal{Y}^-$ to a sum of the same exponentials as in (2), viz. $\exp[-i\omega v]$ and $\exp[-i\omega p(u)]$; the co-ordinates v_+ and v_- may be identified because v_+ is a constant on $\mathcal{Y}^+$. [Cf. the argument which led to (26).]

Until now we analyzed the Hawking radiation from moving mirrors in terms of

the energy momentum tensor $\langle T_{uu} \rangle$ which was found to be invariant under $SL(2, R)$ transformations of $p(u)$. Now we ask whether this invariance also manifests itself in the particle number spectrum $n(\omega)$. There is no a priori reason to expect such an invariance since it is well known [3,6] that there are trajectories where the mirror radiates zero (or even negative) energy but nevertheless produces particles. To define what is meant by a "particle" on $\mathcal{H}^+$ we introduce the "out"-modes

$$\chi_{\omega}(u, v) = (4\pi\omega)^{-1/2} [e^{-i\omega f(v)} - e^{-i\omega u}] , \quad \omega > 0 . \quad (27)$$

They reduce to plane waves on $\mathcal{H}^+$. Expanding the field operator in terms of the χ_{ω} 's the resulting creation and annihilation operators are related to a_{ω} and a_{ω}^+ by a Bogoliubov transformation with the coefficients

$$\alpha_{\omega\omega'} = (\phi_{\omega}, \chi_{\omega'}) , \quad \beta_{\omega\omega'} = -(\phi_{\omega}, \chi_{\omega'}^*) ,$$

where $(.,.)$ denotes the Klein-Gordon inner product. They are given by^{*)}

$$\left. \begin{array}{l} \alpha_{\omega\omega'} \\ \beta_{\omega\omega'} \end{array} \right\} = \pm \frac{1}{2\pi} \sqrt{\frac{\omega}{\omega'}} \int_{v_{\min}}^{v_{\max}} dv \exp[-i\omega v \pm i\omega' f(v)] \quad (28)$$

The number of particles created in the mode ω is determined by the β -coefficients:

$$n(\omega) = \int_0^{\infty} d\omega' |\beta_{\omega'\omega}|^2 . \quad (29)$$

Inserting (28) into (29) and performing the ω' -integration yields

*) Up to irrelevant surface terms these equations are exact [6]. The integral formulae of Ref. [3] are approximations of eq. (28).

$$n(\omega) = -\frac{1}{(2\pi)^2 \omega} \int_{v_{\min}}^{v_{\max}} \frac{dv_1 dv_2}{(v_1 - v_2 - i\varepsilon)^2} \exp[-i\omega\{f(v_1) - f(v_2)\}] \quad (30)$$

It is easy to see that this expression is $SL(2, R)$ invariant. Under a Möbius transformation $f(v)$ in the exponential of Eq. (30) is replaced by $\tilde{f}(v) = f(M^{-1}v)$. Introducing new integration variables $v'_{1,2} = M^{-1}v_{1,2}$ and noting that the measure $dv_1 dv_2 / (v_1 - v_2)^2$ is invariant under a simultaneous Möbius transformation of v_1 and v_2 , one indeed finds that $n(\omega)$ is $SL(2, R)$ invariant: $n(\omega) = \tilde{n}(\omega)$. As we already mentioned, this is not an obvious consequence of $SL(2, R)$ transformations being unaffected by the central charge, because in general the particle number distribution is unrelated to the (integrated) energy momentum tensor. As was discussed in detail by Walker [6], the equation

$$\int_{u_{\min}}^{u_{\max}} du \langle T_{uu} \rangle = \int_0^{\infty} d\omega \omega n(\omega) \quad , \quad (31)$$

stating that the total energy released by the mirror can be computed by adding the energies of the individual particles, does not hold in general. It is true only for trajectories without u -asymptotes. In this case $u_{\min} = -\infty$ and $u_{\max} = +\infty$.

To summarize, we have shown that in the language of conformal field theory the Hawking radiation from (two-dimensional) moving mirrors has a simple interpretation in terms of the central charge of the Virasoro algebra. It can be obtained without encountering divergent mode sums. Classically the accelerating mirror and a mirror at rest are conformally equivalent, at the quantum mechanical level the anomalous transformation law of T_{uu} distinguishes between the two situations. The moving mirror models may be considered the flat space limit of models describing gravitational collapse by a curved background metric in a very similar way as the central charge is a flat space remnant of the trace anomaly in

curved space [14,15]. This fact is illustrated by the equivalence of the moving mirror and the Frolov-Vilkovisky approach based on the effective action and hence on the trace anomaly. Since the Virasoro algebra has an anomaly free $SL(2,R)$ subgroup, mirror trajectories related by a Möbius transformation lead to the same energy momentum tensor. This invariance is shared by the particle spectrum. In this context we should mention a crucial difference between the radiation from the moving mirrors and the Unruh effect [19], i.e. the phenomenon that an observer with a constant acceleration a perceives the Minkowski vacuum as a thermal bath of particles at a temperature $T = \frac{a}{2\pi}$. The transformation from Minkowski co-ordinates to the Rindler co-ordinates natural for the accelerated observer is a conformal co-ordinate transformation also, but it belongs to the anomaly-free subgroup [12]. Hence the effect is completely independent of the conformal anomaly, but rather is related to the fact that Rindler space is conformal only to a part of Minkowski space, viz., the (RHS, say) Rindler wedge.

I would like to thank K. Isler, C. Schmid, C.A. Trugenberger and K. Zalewski for stimulating discussions.

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