CERN-TH.5076/88

OPEN AND CLOSED p-ADIC STRINGS AND QUADRATIC EXTENSIONS OF NUMBER FIELDS

I.Ya. Aref'eva^{*)}
CERN - Geneva

B.G. Dragovic
Institute of Physics, P.O. Box 57
11001 Beograd, Yugoslavia

and

I.V. Volovich
Steklov Mathematical Institute, Vavilov Str. 42
Moscow, GSP-1, 117966, U.S.S.R.

ABSTRACT

In order to elaborate the hypothesis of three types of gravitons being related to three types of closed p-adic strings, the corresponding scattering amplitudes are constructed. The graviton scattering amplitude in the form of convolution of two characters on the quadratic extension of a locally compact field can be constructed only for the closed superstring II. Relations between closed and open p-adic string amplitudes change substantially in comparison to the ordinary case. Moreover, the p-adic amplitude for open strings is defined in many ways, depending on the particular ordering on the field of the p-adic numbers.

^{*)} Permanent address: Steklov Mathematical Institute, Vavilov Str. 42, Moscow, GSP-1, 117966, U.S.S.R.

1. - MOTIVATIONS

It is well known that we traditionally use ordinary (real or complex) numbers in theoretical physics. The use of p -adic numbers has been proposed in Ref. [1]. There are various physical and mathematical reasons to use p -adic numbers in modern theoretical physics [2-4]. In particular, p -adic numbers find their applications in string theory.

As we know, there are two basic types of bosonic strings - open and closed strings. The open string corresponds to the segment in the field R of real numbers (or more precisely it is an image of segment in R^D), while the closed string is related to a circle. Hence in the p -adic case, according to Ref. [1], an open string should be related to the field Q_p of p -adic numbers and a closed string to the p -adic circle. An ordinary circle can be defined in the plane by the equation $z\bar{z} = r$. Such a circle is related to the field C of complex numbers and this field is a quadratic extension of R , i.e., $C = R(\sqrt{-1})$. In an analogous way, the p -adic circle is related to the quadratic extension of the p -adic field Q_p (for an introduction to p -adic mathematics see Ref. [2], and also references quoted therein on the corresponding mathematical textbooks). In contrast to the real case, where one has only one quadratic extension, in the p -adic case there are three quadratic extensions. Hence, there should exist at least three types of closed p -adic strings [1]. This is a very important fact. The point is in the connection which exists in an ordinary case between the theory of the gravitational field (graviton) and closed strings. Since there are three types of closed strings in the p -adic case, it has been predicted in Ref. [1] that the graviton may have an additional quantum number, say p -adic colour. From this consideration it follows that there may be three types of gravitons. Let us note that this prediction is quite general and does not depend upon the dynamics itself but rather on the general algebraic properties of p -adic case numbers.

Nevertheless, let us mention the two possible phenomenological approaches to the theory of a few different gravitons. From one point of view it is possible that there is one basic graviton $g_{\mu\nu}(x)$ related to the metric of space-time and several additional particles of spin two with corresponding fields $\phi_{\mu\nu}^i(x)$, $i = 1, 2, \dots, N$. As a related total Lagrangian, we can take

$$L = R(g) + \sum_{i=1}^N g^{\mu\nu} R_{\mu\nu}(\phi^i) + L_m(g),$$

where $L_m(g)$ corresponds to the interaction of matter with $g_{\mu\nu}(x)$, and $R_{\mu\nu}(\phi^i)$ is a Ricci tensor constructed over the tensor field $\phi_{\mu\nu}^i(x)$.

However, another point of view is also possible. For instance, one can hope that the desired theory is related to the gravity theory with torsion. Namely it is known that the theory with torsion contains in its spectrum not only the graviton but also additional particles of spin two (see the corresponding references in [5] and discussion of physical aspects in [6]). Standard problems of non-renormalizability of the theory with torsion as well as the problems with ghosts in the previous Lagrangian could be overcome by taking into account all possible fields in the theory (cf. with fields of higher spins in the usual strings).

From the string theory point of view, in order to find a phenomenological Lagrangian we have to consider a limit $\alpha' \rightarrow 0$ in the theory of closed strings [7]. We will discuss this task in the theory of p-adic strings.

As we already pointed out, an idea on the p-adic colour of the graviton is closely connected with algebraic properties of p-adic number fields, i.e., the existence of not one but three quadratic extensions. Note that $\mathbb{Q}_p$ has non-trivial algebraic extensions, not only of quadratic degrees but also of cubic and higher degrees. Recently Gervais [8] proposed a very interesting idea that such p-adic extensions could be related to the membrane theory [9].

Let us now discuss the question of the construction of a dynamical theory for open and closed p-adic strings. In Ref. [1] it was proposed to consider p-adic generalization of the Veneziano string amplitude in two ways, according to two equivalent representations

$$A(a, b) = \int_0^1 |x|^{a-1} |1-x|^{b-1} dx = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \quad (1)$$

The first way corresponds to an interpretation of the amplitude $A(a, b)$ as a convolution of two characters and the second one to the p-adic interpolation of the gamma function. Using the first approach a complex-valued string amplitude over a finite Galois field has been constructed [1] (see also Ref. [10]).

In this paper we will mainly consider the method of the convolution of characters. Consideration of string amplitudes as a convolution of characters is a very general concept applicable to characters on number fields, groups and algebras [4]. Here we will deal with string amplitudes of the following form

$$A(\gamma_a, \gamma_b) = \int_K \gamma_a(x) \gamma_b(1-x) dx, \quad (2)$$

where K is a field F , i.e., $K = F$, $\gamma_a(x)$ is a multiplicative character on K , and dx is a measure on K . If K is a locally compact field then one can take dx as the Haar measure but it is not necessary. For instance, if F is a finite Galois field, then for dx one can take a discrete measure and (2) becomes the Jacobi sum [1].

Note that the range of integration in (2) is over the entire field F , and hence this p -adic generalization is rather one of the Virasoro-Shapiro amplitude

$$A = \int_C |z|^{a-1} |1-z|^{b-1} dz, \quad (3)$$

than of the Veneziano amplitude (1), where the integration is over the unit segment on the real axis. The equation (3) is just a particular case of (2) for $K = \mathbb{C}$ and $\gamma_a(z) = |z|^{a-1}$. The ordinary Veneziano amplitude can be rewritten in the following way

$$A = \int_R |x|^{a-1} |1-x|^{b-1} \theta_{[0,1]}(x) dx, \quad (4)$$

where $\theta(x)$ is the characteristic function of the segment $[0,1]$. In particular, it can be written in terms of the Heaviside function $\theta_{[0,1]}(x) = \theta(x)\theta(1-x)$. Hence, in order to have a generalization of the expression (4) on an arbitrary field F one should have on F an analogue of the Heaviside function or the function $\text{sign } x$. This demand is very general and is connected with an introduction of some partial ordering on a field F . As is known, $\mathbb{Q}_p$ is a non-ordered field but it can be made partially ordered (discussion of this problem in connection with the arrow of time is presented in Ref. [2]). Recall that already in the theory of ordinary strings there is a qualitative difference between open and closed strings. This difference is connected with the impossibility of making the order of points on a complex plane (Riemann sphere) and the possibility of doing so on the real axis.

In the present paper we propose a generalization of the amplitude (4), in the case of an arbitrary locally compact disconnected field F , in the following form

$$A_{F, \tau}^{\text{open}}(\gamma_a, \gamma_b) = \int_F |x|^{a-1} |1-x|^{b-1} \theta_{\tau[0,1]} dx, \quad (5)$$

where $\theta_{\tau[0,1]}(x)$ is a p -adic generalization of the characteristic function of the segment $[0,1]$ on F related to a quadratic extension $F(\sqrt{\tau})$ (for more details see Section 2). In particular one can take the function $\theta_{\tau[0,1]}(x)$ in the form $\theta_{\tau}(x)\theta_{\tau}(1-x)$ where $\theta_{\tau}(x)$ is a p -adic analogue of the Heaviside function. We will also discuss some other ways of setting the characteristic function on a segment $[0,1]$. So, in order to have an analogue of the open string on F it is necessary that the field F has a non-trivial quadratic extension $F(\sqrt{\tau})$. The field $\mathbb{C}$ of complex numbers does not have a non-trivial extension, i.e., $\mathbb{C}(\sqrt{\tau}) = \mathbb{C}$, and as a consequence one cannot introduce an ordering. The field $\mathbb{R}$ of real numbers has a unique non-trivial extension $\mathbb{R}(\sqrt{-1}) = \mathbb{C}$, and for that reason an open string is related to $\mathbb{R}$. In such a way, already at the algebraic level one can understand a general fact that the theory involving only open string does not exist (cf., with usual statement that the theory of open strings is not a closed theory, see

Ref. [7]). Such a theory exists only when it is possible to construct a quadratic extension on the field F .

Recall that in the ordinary case there are two types of relations between amplitudes of the open and the closed strings. The first relation gives a connection on the tree level as follows

$$A_{tree}^{closed}(s, t, u) = \sin\left(\frac{\pi t}{8}\right) A_{tree}^{open}\left(\frac{s}{4}, \frac{t}{4}\right) A_{tree}^{open}\left(\frac{t}{4}, \frac{u}{4}\right), \quad (6)$$

where s, t, u are the Mandelstam variables. From unitarity follows the second connection, which says that the non-planar one-loop diagrams for the open strings have closed strings' poles.

One of the aims of this paper is also to learn what is left of the relation (6) in the corresponding p -adic case (a p -adic analogue of the second relation will be studied in a separate publication). In order to bring this consideration to the more concrete cases, we have to say something more about the objects contained in the formulae (2) and (5). For this reason the following ingredients must be specified:

- i) the object K , and an appropriate measure on K ;
- ii) the characters $\gamma_a(x)$, with their parametrizations in terms of the momenta of scattered particles, i.e., $a = a(\{k_i\})$ should be defined.

For K one can take an arbitrary mathematical structure which is convenient for an introduction of the multiplicative characters and the Haar measure, and hence [4]:

- a) K is an arbitrary field F , in particular, $F = R, C, Q_p, Q_p(\sqrt{\tau})$;
- b) K is a group;
- c) K is an algebra.

Parametrizing characters, i.e., defining $a = a(\{k_i\})$, we may think that:

- α) k_i belong to the usual space R^D (or M^D),
- β) k_i are elements of a more complex (e.g., p -adic) number fields,
- γ) k_i are elements of the space which is dual to K .

These α)- γ) possibilities are related to the different structures of the space-time [1-4], i.e., there are profound potentialities inherent in the generalized string amplitude (2). Namely, starting to construct the theory from this amplitude one can deal with various mathematical structures of the space-time itself.

The most modest point of view is to leave the space-time structure as usual, i.e., $k_i \in R^D(M^D)$, and consider the various structures a)-c) of K. However, we think that the most interesting physical possibilities are just related to the change of the known space-time structure and in particular, to an appearance of the non-Archimedean geometry at the Planck scale. Nevertheless it is now rather instructive to analyze possibilities which follow from the expression (2) maintaining $k_i \in R^D(M^D)$. For a discussion of this possibility see Refs. [11,12].

2. - p-ADIC HEAVISIDE FUNCTION

Let F in Eq. (5) be a non-discrete totally disconnected and locally compact field (e.g., $F = Q_p$).

In order to construct the string amplitude we need to know the corresponding Heaviside function. As it has been already noticed in Section 1, we will solve this problem using quadratic extensions of the field F. In order to introduce a quadratic extension of the field F, recall that one can attach to F a multiplicative group F^* ($F^* = F - \{0\}$). Also there exists the Haar measure on F and consequently a norm |a| on F^* can be defined by an equality $d(ax) = |a|dx$. This multiplicative group F^* is isomorphic to the direct product $Z \times Z_q \times A$, where Z is an infinite cyclic group, Z_q is a cyclic group of the order of q-1 ($q = p^n$, p = a prime number) and a group A consists of elements $x \in F$ such that $|x-1| < 1$. Let p be an odd prime, i.e., $p \neq 2$. Quadratic extensions of the field F will be connected with generating elements of Z and Z_q . When $n = 2$, the algebraic extension is the quadratic one. The field F, with a multiplicative group F^* isomorphic to the direct product $Z \times Z_q \times A$, has the three quadratic extensions: $F(\sqrt{p})$, $F(\sqrt{\epsilon})$ and $F(\sqrt{p\epsilon})$, where p and ϵ are generating elements of Z and Z_q , respectively [13].

Let a quadratic extension be $K = F(\sqrt{\tau})$ where $\tau \in F$ and $\sqrt{\tau} \notin F$. Any element z of K is of the form $z = x + \sqrt{\tau}y$ with $x, y \in F$. Define $\bar{z} = x - \sqrt{\tau}y$, $S(z) = z + \bar{z}$ and $N(z) = z\bar{z}$. For the Haar measure on K we take $dz = dx dy$. Define the sign function of an element w of K with respect to τ : $\text{sign}_{\tau} w = +1$ if w can be expressed in the form $w = N(z) = x^2 - \tau y^2$ for a certain $z \in K$, and $\text{sign}_{\tau} w = -1$ otherwise. The function $\text{sign}_{\tau} w$ defined in the above way is a multiplicative character on K. Define also the generalized Heaviside function in the form

$$\theta_{\tau}(w) = \frac{1 + \text{Sign}_{\tau} w}{2} \quad (7)$$

which is an analogue of the ordinary one.

3. - GENERALIZED s-CHANNEL AMPLITUDES

In this section we will consider the amplitude (5) with the characteristic function in one of the following forms:

$$\theta_{\tau[0,1]}(x) = \theta_{\tau}(x) \theta_{\tau}(1-x), \quad (8.1)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2} (1 + \text{Sign}_{\tau} x \cdot \text{Sign}_{\tau}(1-x)), \quad (8.2)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2} (\text{Sign}_{\tau} x + \text{Sign}_{\tau}(1-x)), \quad (8.3)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2} (\text{Sign}_{\tau} x - \text{Sign}_{\tau}(-1) \cdot \text{Sign}_{\tau}(1-x)), \quad \tau = \varepsilon, \quad (8.4)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2} (1 - \text{Sign}_{\tau}(-1) \cdot \text{Sign}_{\tau} x \cdot \text{Sign}_{\tau}(1-x)). \quad (8.5)$$

Equations (8.1)-(8.4) were considered in Ref. [14] and (8.5) in Ref. [15]. The corresponding amplitudes (5) can be calculated with the help of the general formula [13]

$$\mathcal{B}(\pi_1, \pi_2) = \frac{\Gamma(\pi_1) \Gamma(\pi_2)}{\Gamma(\pi_1 + \pi_2)}, \quad (9)$$

which connects the beta function

$$\mathcal{B}(\pi_1, \pi_2) = \int_F \pi_1(x) |x|^{-1} \pi_2(1-x) |1-x|^{-1} dx, \quad (10)$$

where $\pi(x)$ is a multiplicative character with the gamma function defined by an additive character χ

$$\Gamma(\pi) = \int_F \chi(x) \pi(x) |x|^{-1} dx. \quad (11)$$

We have

$$\mathcal{A}_{F, \varepsilon}^{(1)}(a, b) = \frac{1}{4} \left[\frac{\Gamma(\pi_a) \Gamma(\pi_b)}{\Gamma(\pi_{a+b})} + \frac{\Gamma(\tilde{\pi}_a) \Gamma(\pi_b)}{\Gamma(\tilde{\pi}_{a+b})} + \frac{\Gamma(\pi_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\tilde{\pi}_{a+b})} + \frac{\Gamma(\tilde{\pi}_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\pi_{a+b})} \right], \quad (12.1)$$

$$\mathcal{A}_{F, \varepsilon}^{(2)}(a, b) = \frac{1}{2} \left[\frac{\Gamma(\pi_a) \Gamma(\pi_b)}{\Gamma(\pi_{a+b})} + \frac{\Gamma(\tilde{\pi}_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\tilde{\pi}_{a+b})} \right], \quad (12.2)$$

$$\mathcal{A}_{F, \varepsilon}^{(3)}(a, b) = \frac{1}{2} \left[\frac{\Gamma(\tilde{\pi}_a) \Gamma(\pi_b)}{\Gamma(\tilde{\pi}_{a+b})} + \frac{\Gamma(\pi_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\pi_{a+b})} \right], \quad (12.3)$$

$$\mathcal{A}_{F,\varepsilon}^{(4)}(a,b) = \frac{1}{2} \left[\frac{\Gamma(\tilde{\pi}_a) \Gamma(\pi_b)}{\Gamma(\tilde{\pi}_{a+b})} - \frac{\Gamma(\pi_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\tilde{\pi}_{a+b})} \right], \quad (12.4)$$

$$\mathcal{A}_{F,\varepsilon}^{(5)}(a,b) = \frac{1}{2} \left[\frac{\Gamma(\pi_a) \Gamma(\pi_b)}{\Gamma(\pi_{a+b})} - \frac{\Gamma(\tilde{\pi}_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\tilde{\pi}_{a+b})} \right], \quad (12.5)$$

where π_a and $\tilde{\pi}_a$ correspond to the multiplicative characters $\pi_a(x) = (x)^a$ and $\tilde{\pi}_a(x) = |x|^a \text{sign}_\tau x$. The corresponding gamma functions are

$$\Gamma(\pi_a) = \frac{1 - q^{a-1}}{1 - q^{-a}}, \quad (13.1)$$

$$\Gamma(\tilde{\pi}_a) = \begin{cases} \pm \sqrt{\text{Sign}_\tau(-1)} q^{a-1}, & \text{for } \tau = p, \varepsilon p \\ \frac{1 + q^{a-1}}{1 + q^{-a}}, & \text{for } \tau = \varepsilon, \end{cases} \quad (13.2)$$

where q is the order of the residue field, (e.g., $q = p$ for $F = \mathbb{Q}_p$). Using expressions (13.1) and (13.2) the "s-channel amplitudes" (12) for $\tau = \varepsilon$ can be rewritten in the following forms

$$\begin{aligned} \mathcal{A}_{q,\varepsilon}^{(1)}(a,b,c) &= \frac{1}{(1-q^{-2a})(1-q^{-2b})(1-q^{-2c})} \left(1 - \frac{1}{q}\right) \left(1 - \frac{1}{q} + \frac{1}{q^2}\right) + \\ &+ \frac{1}{q} (q^{-2a} + q^{-2b} + q^{-2c}) - \frac{1}{q^2} (q^{2a} + q^{2b} + q^{2c}), \end{aligned} \quad (14.1)$$

$$\mathcal{A}_{q,\varepsilon}^{(2)}(a,b,c) = \frac{1 - q^{c-1}}{1 - q^{-c}} \frac{(1 - \frac{1}{q})^2 + q^{c-1} + q^{-c-1} - q^{a+b-1} - q^{-a+b-1}}{(1 - q^{-2a})(1 - q^{-2b})}, \quad (14.2)$$

$$\mathcal{A}_{q,\varepsilon}^{(3)}(a,b,c) = \frac{1 + q^{-a-b}}{1 + q^{a+b-1}} \frac{(1 - \frac{1}{q})^2 - q^{-a-b} - q^{a+b-2} + q^{-a+b-1} + q^{a-b-1}}{(1 - q^{-2a})(1 - q^{-2b})}, \quad (14.3)$$

$$A_{p,\varepsilon}^{(4)}(a,b,c) = \left(1 - \frac{1}{q}\right) \frac{1 + q^{c-1}}{1 + q^{-c}} \frac{q^{-b} - q^{b-1} - q^{-a} + q^{a-1}}{(1 - q^{-2a})(1 - q^{-2b})}, \quad (14.4)$$

$$A_{p,\varepsilon}^{(5)}(a,b,c) = \left(1 - \frac{1}{q}\right) \frac{1 - q^{c-1}}{1 - q^{-c}} \frac{q^{-b} - q^{b-1} + q^{-a} - q^{a-1}}{(1 - q^{-2a})(1 - q^{-2b})}, \quad (14.5)$$

where $a = -\alpha(x) = -1 - \frac{1}{2}s$, $b = -\alpha(t)$, $c = -\alpha(u)$, and

$$a + b + c = 1 \quad (15)$$

In the particular case $F = \mathbb{Q}_p$ we have to replace q by p in formula (14).

Note some properties of the obtained string amplitudes (15). The amplitude (14.1) has one very surprising property. Namely, as is evident from (14.1), this amplitude is a completely crossing symmetric one. As a consequence of symmetry properties, it appears that the amplitude in the s -channel contains a u -pole, i.e., a pole at $c = 0$. Moreover the amplitude (14.1) contains a set of complex poles which do not correspond to poles of the "total amplitude" $A(a,b) = B_p(\pi_a, \pi_e)$. The amplitudes (14.2)-(14.5) are not totally crossing symmetric. The amplitude (14.2), like (14.1), contains the u -pole but on the contrary amplitudes (14.3)-(14.5) do not contain it.

In such a way there are four generalizations of the Veneziano string amplitude which deserve further attention. These amplitudes are: $A_{p,\varepsilon}^{(1)}(a,b,c)$ which exhibit total crossing symmetry, and amplitudes $A_{p,\varepsilon}^{(i)}(a,b,c)$, $i = 3, 4, 5$, which are not totally crossing symmetric but do not contain undesired u -poles.

4. - FORMULAE BETWEEN CLOSED AND OPEN STRING AMPLITUDES

Consider now the string amplitudes, constructed over the p -adic fields $\mathbb{Q}_p$ and their quadratic extensions $\mathbb{Q}_p(\sqrt{\tau})$, from the point of view of the product formulae (6) which relates amplitudes of closed and open strings in a very simple form. Let us first pay attention to the case $\tau = \varepsilon$. In this case the closed string amplitude defined on the quadratically extended field $K = \mathbb{Q}_p(\sqrt{\varepsilon})$ is presented in Ref. [11] and has the form

$$A_{Q_p(\sqrt{\epsilon})}^{\text{closed}}(a, b, c) = \int_{Q_p(\sqrt{\epsilon})} |x|^{a-1} |1-x|^{b-1} dx = \frac{1-q^{a-1}}{1-q^{-a}} \cdot \frac{1-q^{b-1}}{1-q^{-b}} \cdot \frac{1-q^{c-1}}{1-q^{-c}} \quad (16)$$

where $q = p^2$. Evidently there are no such formulae as simple as (7) for the above constructed open string amplitudes. However, there exists a formula in the following form

$$A_{Q_p(\sqrt{\epsilon})}^{\text{closed}}(a, b, c) = A_{Q_p}^{\text{open, total}}(a, b, c) \tilde{A}_{Q_p}(a, b, c), \quad (17)$$

where

$$A_{Q_p}^{\text{open, total}}(a, b, c) = \int_{Q_p} |x|^{a-1} |1-x|^{b-1} dx = \frac{1-p^{a-1}}{1-p^{-a}} \cdot \frac{1-p^{b-1}}{1-p^{-b}} \cdot \frac{1-p^{c-1}}{1-p^{-c}} \quad (18)$$

is a p-adic analogue of the totally crossing symmetric Veneziano amplitude (see for example Refs. [12,16] and about N-point p-adic amplitudes see Refs. [15,17,18]. The amplitude $\tilde{A}_{Q_p}(a, b, c)$ has no integral representation and can be defined by the formula

$$\tilde{A}_{Q_p}(a, b, c) = \frac{\Gamma(\tilde{\pi}_a) \Gamma(\tilde{\pi}_b)}{\Gamma(\tilde{\pi}_{a+b})}, \quad (19)$$

where $\Gamma(\tilde{\pi}_a)$ is given by the expression (13.2) for $\tau = \epsilon$.

Notice that any amplitude originating from the multiplicative characters $\pi_a(x)$ and $\tilde{\pi}_a(x)$ through the formula (13) necessarily contains only an even number of the gamma functions $\Gamma(\tilde{\pi})$. This circumstance leads to an obstacle in obtaining new relations of desired simplicity like that in (6). However the more complex formulae may take place. So we obtain the following interesting relation

$$\left(A_{Q_p(\sqrt{\epsilon})}^{\text{closed}}(a, b, c) \right)^2 = B_p(\pi_a, \pi_b) B_p(\tilde{\pi}_a, \tilde{\pi}_b) B_p(\tilde{\pi}_b, \tilde{\pi}_c) B_p(\tilde{\pi}_c, \tilde{\pi}_a), \quad (20)$$

where the index p in B_p denotes that the beta function is defined on the field Q_p . If we want to take as an open string amplitude one of those constructed in the previous section, for instance $A_{p,\epsilon}^{(3)}$ or $A_{p,\epsilon}^{(4)}$, then it follows that no simple

relation with the closed string amplitude can be obtained. This fact can be interpreted as the absence of the "gluing formulae" for one separate prime number p , at least for the strings corresponding to the quadratic extension with $\tau = \varepsilon$.

Some simpler situations are found in the cases of quadratic extensions with $\tau = p$ and $\tau = p\varepsilon$. In these cases the amplitudes for closed strings are given by formula (16) with the replacement $q = p$. Here ($\tau = p, p\varepsilon$), the simplest gluing formula, has the following form

$$A_{Q_p(\sqrt{\varepsilon})}^{\text{closed}}(a, b, c) = p B(\tilde{\pi}_a, \tilde{\pi}_b) B(\tilde{\pi}_b, \tilde{\pi}_c) B_p(\tilde{\pi}_c, \pi_a), \quad (21)$$

i.e., the closed string amplitude looks like being composed of the three open amplitudes. The desired product formulae become no more simple when some linear combinations of amplitudes $A^{(i)}$ ($i = 1, \dots, 4$) are included.

5. - p-ADIC STRING AMPLITUDES FOR GRAVITONS

Recall that the basic motivation for the present paper is our wish to construct the simplest extension of the Veneziano amplitude in order to have a possibility to investigate in a simple framework the prediction of several types of gravitons. We started to consider already in the first section the connection between p-adic string amplitudes and possible types of gravitons. Now we come back again to this problem. Firstly let us notice the presence of only the tachyons' poles (and complex poles) in the above constructed amplitudes [e.g., see Eq. (14)]. The poles corresponding to massless particles (massless vector bosons or gravitons) are not contained in these individual amplitudes.

Let us now consider a p-adic generalization of the N-point tree amplitude for vector particles in the bosonic case. It can be proposed in the following form

$$A(\zeta_1 k_1, \dots, \zeta_n k_n) = g^{n-2} \int_{(Q_p)^{n-3}} \theta_{[0, y_{n-1}, \dots, y_{3,1}]}(y) F(\zeta, k, y). \quad (22)$$

$$\prod_{3 \leq i < j \leq n-1} |y_i - y_j|_p^{k_i k_j} \prod_{3 \leq i \leq n-1} (|y_i|_p^{k_n k_i} |1 - y_i|^{k_2 k_i} dy_i),$$

where $\theta_{[0, y_{n-1}, \dots, y_{3,1}]}(y)$ is a p-adic generalization of the characteristic function of the simplex $0 \leq y_{n-1} \leq \dots \leq y_4 \leq y_3 \leq 1$ [for example, for $n = 4$ it is the one of (12)] and $F(\zeta, k, y)$ is the part of $\exp \sum_{i=j} \{ \frac{1}{2} (\zeta_i \zeta_j) / (y_i - y_j)^2 - k_i k_j / y_i - y_j \}$

that is multilinear in all the polarization vectors ζ_i . The answer depends on the choice of the p-adic characteristic function. Taking the characteristic function as in Eqs. (8.3)-(8.5), the four-particle amplitude can be written as a superposition of the amplitudes $A_{p,T}^{(i)}(a,b,c)$ where a and b take different values, but always $c = 1-a-b$,

$$\begin{aligned} A_4(\zeta_1 k_1, \zeta_2 k_2, \zeta_3 k_3, \zeta_4 k_4) = g^2 & \left[(\zeta_1 \zeta_2)(\zeta_3 \zeta_4) A_{p,T}^{(i)}\left(-\frac{s}{2}-1, -\frac{t}{2}+1, 1-\frac{u}{2}\right) + \right. \\ & + (\zeta_1 \zeta_3)(\zeta_2 \zeta_4) A_{p,T}^{(i)}\left(-\frac{s}{2}+1, -\frac{t}{2}+1, -\frac{u}{2}-1\right) + (\zeta_1 \zeta_4)(\zeta_2 \zeta_3) A_{p,T}^{(i)}\left(-\frac{s}{2}+1, -\frac{t}{2}-1, 1-\frac{u}{2}\right) + \\ & + (k_1 \zeta_2)(k_2 \zeta_3)(k_3 \zeta_4)(k_4 \zeta_1) A_{p,T}^{(i)}\left(-\frac{s}{2}, -\frac{t}{2}, 1+\frac{u}{2}\right) + \dots + \\ & \left. + (k_2 \zeta_1)(k_3 \zeta_4)(\zeta_2 \zeta_3) A_{p,T}^{(i)}\left(-\frac{s}{2}+1, -\frac{t}{2}-1, 1-\frac{u}{2}\right) + \dots \right]. \end{aligned} \quad (23)$$

Here we collect all terms containing the different combinations of $(\zeta_i \zeta_j)(\zeta_k \zeta_m)$ and we present only one term containing the second power of k_1 , as well as one term containing the fourth power of k_1 . Analyzing Eq. (23), one can conclude that the amplitudes $A_4^{(i)}(\zeta_1 k_1 \zeta_2 k_2 \zeta_3 k_3 \zeta_4 k_4)$, $d = 3, 4, 5$ contain real poles at points $s = -2, 0, 2$ and $t = -2, 0, 2$ but they do not contain u-poles.

Now it seems rather natural to try to get a p-adic amplitude for the graviton, following the way of obtaining the amplitude for tachyons in the p-adic closed string. The last amplitude has been obtained starting from the usual amplitude by the changing of the usual absolute values norm on the p-adic norm. However, the general structure of the amplitudes of the excited states is the following

$$\int d\bar{z} |z|^{K_3 K_4/2} |1-z|^{K_2 K_3/2} (\bar{z})^{n_1} (1-\bar{z})^{n_2} (\bar{z})^{n_3} (1-\bar{z})^{n_4} \quad (24)$$

and, generally speaking, $n_1 \neq n_3$, $n_2 \neq n_4$.

In particular, taking the scattering amplitude for four gravitons in the bosonic string in the form

$$A_4^{\mu\mu', \nu\nu', \gamma\gamma', \delta\delta'}(k_1 k_2 k_3 k_4) = \kappa^2 \sin \frac{\pi t}{8} A_4^{\mu\nu\gamma\delta}\left(\frac{s}{4}, \frac{t}{4}\right) A_4^{\mu'\nu'\gamma'\delta'}\left(\frac{t}{4}, \frac{s}{4}\right) \quad (25)$$

and substituting here the well-known expression for $A_4^{\mu\nu\gamma\delta}$ [7] we find that the resulting expression cannot be written as a combination of the Gel'fand-Graev beta functions $B_\infty(a,b)$.

In other words, the amplitude (25) cannot be represented as a combination of the expressions (24) taken under restrictions $n_1 = n_3, n_2 = n_2$. It is interesting to note that the scattering amplitude for massless particles in superstring II can be written as the combination of the Gel'fand-Graev beta functions $B_\infty(a,b)$. This fact provides the possibility to write down the scattering amplitude for the massless particle for the p-adic closed superstring theory. The answer [14,19] is the following:

$$\begin{aligned} A_K^{\text{closed, super}}(s, t, u) = & (1,1) B_K(-\frac{s}{8}+1, -\frac{t}{8}) + (2,2) B_K(-\frac{s}{8}, -\frac{t}{8}+1) + \\ & + (3,3) B_K(-\frac{s}{8}, -\frac{t}{8}) + (1,2) [B_K(-\frac{s}{8}, -\frac{t}{8}) - B_K(-\frac{s}{8}, -\frac{t}{8}+1) - B_K(-\frac{s}{8}+1, -\frac{t}{8})] + \\ & + (2,3) [B_K(-\frac{s}{8}+1, -\frac{t}{8}) - B_K(-\frac{s}{8}, -\frac{t}{8}+1) - B_K(-\frac{s}{8}, -\frac{t}{8})] + \\ & + (3,1) [B_K(-\frac{s}{8}, -\frac{t}{8}+1) - B_K(-\frac{s}{8}+1, -\frac{t}{8}) - B_K(-\frac{s}{8}, -\frac{t}{8})], \end{aligned} \quad (26)$$

(i,j) are kinematical factors which depend on spins and polarizations. For $K = Q_p(\sqrt{\tau})$ B_K is given by (16) with $q = p^2, \tau = \varepsilon$; $q = p, \tau = p, \varepsilon p$. The right-hand side of (26) has poles at $s, t, u = 0, 8$ for each of $\tau = \varepsilon, p, \varepsilon p$. It has the poles in all channels at the following real point $s, t, u = 0$ and $s, t, u = 8$.

Our future aim is to construct an action describing the scattering of the different p-adic gravitons. In order to solve this problem we have to know how to construct the tree diagrams which correspond to the scattering of gravitons with different p-adic colour. The basic diagram which can be realized in a rather simplified approach is presented in Fig. 1, where shaded and non-shaded surfaces correspond to closed strings of different types, i.e., different τ . In order to calculate the corresponding amplitude we have to construct a W_{τ_1, τ_2} operator which glues two types of closed strings (cf., with the W-operator which connects an open and a closed string as is indicated in Fig. 2). A construction of the operator W_{τ_1, τ_2} demands the introduction of a mapping of a " $Q_p(\sqrt{\tau_1})$ upper half plane" into " $Q_p(\sqrt{\tau_2})$ upper half plane". We will consider this construction elsewhere.

6. - CONCLUDING REMARKS

Although we have obtained several formulae connecting amplitudes of closed and open p-adic strings defined as a convolution of multiplicative characters, it is worth noticing that there is probably no simple product formulae in terms of the

corresponding beta functions. This conclusion can be used to support our conjecture that the p-adic gamma functions are more fundamental objects in the construction of the p-adic string theory than the p-adic beta functions. The simple gluing formula (19) is an evident example for an advantage of the gamma functions.

ACKNOWLEDGEMENTS

One of the authors (I.Ya.A.) would like to thank L. Alvarez-Gaumé for discussions and the Theoretical Physics Division of CERN for the warm hospitality extended to her.

REFERENCES

- [1] I.V. Volovich, Teor. Mat. Fiz. 71 (1987) 337; Class. Quantum Grav. 4 (1987) L83.
- [2] I.V. Volovich, "Number theory as the ultimate physical theory", preprint CERN-TH.4781/87 (July 1987);
I.Ya. Aref'eva and I.V. Volovich, "Strings, gravity and p-adic space-time", Seminar on Quantum Gravity, Moscow, May 1987 (to be published by World Scientific, Singapore, 1988).
- [3] I.Ya. Aref'eva, B.G. Dragovic and I.V. Volovich, Phys. Lett. B200 (1988) 512.
- [4] I.V. Volovich, "Harmonic analysis and p-adic strings", (October 1987), to be published in Lett. Math. Phys.
- [5] M.O. Katanayev and I.V. Volovich, Phys. Lett. B175 (1986) 413.
- [6] I.V. Volovich and M.I. Katanayev, Teor. Mat. Fiz. 66 (1986) 79.
- [7] M.B. Green, J.H. Schwarz and E. Witten, "Superstring theory", (Cambridge Univ. Press, Cambridge, 1987).
- [8] J.-L. Gervais, Phys. Lett. B201 (1988) 189.
- [9] M.J. Duff, Class. Quantum Grav. 5 (1988) 189.
- [10] B. Grossman, Phys. Lett. B197 (1987) 101.
- [11] P.G.O. Freund and M. Olson, Phys. Lett. B199 (1987) 186.
- [12] P.G.O. Freund and E. Witten, Phys. Lett. B199 (1987) 191.
- [13] I.M. Gel'fand, M.I. Graev and I.I. Piatetskii-Shapiro, "Representation theory and automorphic functions", (Nauka, Moscow, 1966).
- [14] I.Ya. Aref'eva, B.G. Dragovic and I.V. Volovich, Beograd preprint IF-14/88 (April 1988).
- [15] L. Brekke, P.G.O. Freund, M. Olson and E. Witten, Nucl. Phys. B302 (1988) 365.
- [16] I.Ya. Aref'eva, B.G. Dragovic and I.V. Volovich, "On the adelic string amplitudes", preprint IF-13/88, to be published in Phys. Lett. B.
- [17] P.H. Frampton and Y. Okada, Phys. Rev. D37 (1988) 3077; Phys. Rev. D37 (1988) 3077.
- [18] E. Marinari and G. Parisi, Phys. Lett. B203 (1988) 52.
- [19] I.Ya. Aref'eva, B.G. Dragovic and I.V. Volovich, "p-adic superstring", CERN preprint TH.5089/88 (1988).

FIGURE CAPTIONS

Fig. 1 : Basic tree diagram for gravitons of different p-adic colour.

Fig. 2 : A tree diagram with both closed and open strings as external particles.

Fig. 1

Fig. 2