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**CURVATURES, GAUSS-BONNET AND CHERN-SIMONS MULTIPLETS
IN OLD-MINIMAL $N = 1$ SUPERGRAVITY**

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ABSTRACT. The complete form of the three curvature multiplets of old-minimal supergravity are given. The full super--Gauss-Bonnet theorem is derived and the Chern-Simons multiplets constructed. The latter construction allows a coupling to antisymmetric tensor fields which is the supersymmetric generalization of the Green-Schwarz mechanism occurring in superstring theories.

1. Introduction.

Gravity theories with higher curvature terms¹ have recently received much attention in connection with the point-field limit of string theories. In particular, the Green-Schwarz² mechanism needed for anomaly cancellations in higher-dimensional theories imply specific couplings of Chern-Simons forms to antisymmetric (second-rank) tensor fields. In superstrings these properties must have a supersymmetric extension which lead to a generalization of supergravity theories including higher order superinvariants.³ The local supersymmetric completion of higher curvature theories require a complete knowledge of the supercurvature multiplets⁴ as well as the super--Chern-Simons ones.^{5,6} The superspace form of these multiplets⁷ have been found recently but an explicit component determination has not yet been completed. Partial results including the full linearized expansion and the full nonlinear bosonic terms have been given in Refs. 3 and 4. Here we complete those results and give the complete proof of the super--Gauss-Bonnet theorem for old-minimal Poincaré supergravity. The full Chern-Simons multiplet is also obtained and its coupling to antisymmetric tensor fields via the Green-Schwarz mechanism described.

In dealing with supergravity theories with higher curvature terms, an important role is played by the so-called auxiliary fields of the gravity multiplets.⁸ Different auxiliary field formulations correspond to different "compensators" of conformal supergravity. In canonical supergravity, when only terms linear in the curvatures are present, it is known that all these formulations are equivalent⁹ in the sense that they can be transformed into each other via duality transformations or by adding extra multiplets.

In the minimal formulations, the auxiliary fields are dependents of a chiral and a linear multiplet respectively.

For simple supergravity, without higher curvature terms, the two formulations are dual of each other. In fact, a local transformation can be performed which transforms a chiral into a linear multiplet and vice versa.

When higher curvature terms are present, as is the case in superstring theories, this is no longer true, due to the occurrence of derivatives on the compensators.¹⁰ An explicit example of a Lagrangian quadratic in the curvatures was given which revealed this inequivalence.

In the present paper we confine our investigation to Poincaré supergravity in the so-called old minimal formulation. Results for the new minimal formulation are treated elsewhere.¹¹

Our results are relevant in connection with a superspace formulation of background fields in superstrings with $N = 1$ space-time supersymmetry in $D = 4$ dimensions. Heterotic strings seem to have a natural formulation in the new-minimal formulation.¹² This does not mean that an old minimal formulation is impossible.¹³ In fact, even in the presence of higher derivative terms, the two formulations may be transformed, in general, into one another provided a sufficient number of new degrees of freedom is introduced. The question is which formulation is more economical in terms of tensor fields. In view of the different possible formulations of superstring theories, we believe that a complete knowledge of the structure of both formulations is useful.

The plan of the paper is as follows. In Section 2, we review some basics of the Poincaré calculus with old-minimal formulation, especially some properties of the supercovariant derivatives. In Section 3, the curvature multiplets are explicitly displayed. In Section 4, the complete proof of the super--Gauss-Bonnet theorem is given. In Section 5, we give the Lorentz Chern-Simons and Gauss-Bonnet multiplets and their couplings to an antisymmetric tensor.

2. Elements of Tensor Calculus.

We will follow the conventions of Ref. 6. The transformation rules of the supergravity multiplet are given by:¹⁴

$$\begin{aligned}
 \delta e^a_{\mu} &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_{\mu} \\
 \delta \psi_{\mu} &= \left(D_{\mu} + \frac{i}{2} \gamma_5 A_{\mu} \right) \epsilon - \frac{1}{2} \gamma_{\mu} \eta \epsilon \\
 \delta S &= - \frac{1}{2} \bar{\epsilon} \gamma \cdot z \\
 \delta P &= \frac{1}{2} \bar{\epsilon} \gamma_5 \gamma \cdot z \\
 \delta A_a &= \frac{1}{2} \bar{\epsilon} \gamma_5 z_a
 \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
 \eta &= - \frac{1}{3} (S - i \gamma_5 P - i A \gamma_5) \\
 z_a &= \frac{3}{2} \left(R^{(\text{cov})}_a - \frac{1}{3} \gamma_a \gamma \cdot R^{(\text{cov})} \right)
 \end{aligned} \tag{2.2}$$

and $R^{(\text{cov})}_a$ is the supercovariant Rarita-Schwinger field strength:

$$\begin{aligned}
 R^{(\text{cov})a} &= \frac{1}{2} \Gamma^{abc} \psi^{(\text{cov})}_{bc} = \frac{1}{2} \epsilon^{abcd} \gamma_5 \gamma_d \psi^{(\text{cov})}_{bc} , \\
 \psi^{(\text{cov})}_{bc} &= e^{\mu}_b e^{\nu}_c \psi^{(\text{cov})}_{\mu\nu} , \quad \psi^{(\text{cov})}_{\mu\nu} = 2 \left(D_{[\mu} + \frac{i}{2} A_{[\mu} \gamma_5 - \frac{1}{2} \gamma_{[\mu} \eta \right] } \psi_{\nu]} .
 \end{aligned} \tag{2.3}$$

The tangent space covariant derivative is:

$$D_\mu \psi_\nu = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \right) \psi_\nu \quad (2.4)$$

as usual. The spin connection $\omega_\mu^{ab} = \omega_\mu^{ab}(e, \psi)$ is given by the usual expression¹⁴ and its transformation rule is:^{4,14,15}

$$\begin{aligned} \delta \omega_{\mu ab} &= -\frac{1}{4} (\bar{\epsilon} \gamma_\mu \psi_{ab}^{(cov)} - 2 \bar{\epsilon} \gamma_{[a} \psi_{b]\mu}^{(cov)}) + \frac{1}{4} \bar{\epsilon} \{\Gamma_{ab}, \eta\} \psi_\mu = \\ &= \frac{1}{2} \bar{\epsilon} (\Gamma_{[ab} \zeta_{\mu]} - \gamma_\mu \psi_{ab}^{(cov)}) + \frac{1}{4} \bar{\epsilon} \{\Gamma_{ab}, \eta\} \psi_\mu . \end{aligned} \quad (2.5)$$

The Lagrangian for the supergravity multiplet which is invariant under the rules (2.1) is

$$\mathcal{L}_{sg} = -\frac{1}{2} e R - \frac{1}{2} e \bar{\psi}_\mu R^\mu - \frac{1}{3} e (S^2 + P^2 - A_a A^a) \quad (2.6)$$

where R is the scalar curvature and R^μ is the Rarita-Schwinger field strength:

$$R = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = R_{\mu\lambda}^{ab} e_a^\lambda e_{b\nu},$$

$$R^\mu = \frac{1}{2} \Gamma^{\mu\nu\rho} \psi_{\nu\rho}, \quad \psi_{\nu\rho} = 2D_{[\nu} \psi_{\rho]}.$$

Now let us turn to the matter multiplets. A scalar multiplet $[A, B, \chi', F, G]$ consists of a scalar (A) and a pseudoscalar (B) physical fields, a Majorana fermion χ' and a scalar (F) and a pseudoscalar (G) auxiliary fields with the transformation rules:

$$\delta A = \frac{1}{2} \bar{\epsilon} \chi'$$

$$\delta B = -\frac{1}{2} \bar{\epsilon} \gamma_5 \chi'$$

$$\delta \chi' = \frac{1}{2} \hat{D} (A - i \gamma_5 B) \epsilon + \frac{1}{2} (F + i \gamma_5 G) \epsilon \quad (2.7)$$

$$\delta F = \frac{1}{2} \bar{\epsilon} \left(\hat{D} - \frac{i}{2} \kappa \gamma_5 \right) \chi' + \frac{1}{2} \bar{\epsilon} \eta \chi'$$

$$\delta G = \frac{i}{2} \bar{\epsilon} \gamma_5 \left(\hat{D} - \frac{i}{2} \kappa \gamma_5 \right) \chi' - \frac{1}{2} \bar{\epsilon} \eta \gamma_5 \chi' .$$

The super-Poincaré covariant derivative $\hat{D}_\mu$ on any field with only tangent space indices is defined as

$$\hat{D}_\mu \phi \dots = D_\mu \phi \dots - \delta(\psi_\mu) \phi \dots , \quad (2.8)$$

the first term being the tangent space covariant derivative and the second is the subtraction of a supersymmetry transformation with parameter ψ_μ . The chiral multiplet $[z, \chi_L, h]$ is the complex combination:

$$z = i(A + iB) , \quad \chi_L = \frac{i}{2} (I + \gamma_5) \chi' , \quad h = i(F + iG) \quad (2.9)$$

while the complex conjugate combination is an antichiral multiplet.

In the derivation of the components and transformation rules of the Ricci multiplet in Section 3, we will need the supersymmetry transformations of the supercovariant derivatives of these fields and they are:

$$\delta \hat{D}_a A = \frac{1}{2} \bar{\epsilon} \left[\hat{D}_a - \frac{i}{2} A_a \gamma_5 - \frac{1}{2} \eta \gamma_a \right] \chi'$$

$$\delta \hat{D}_a B = -\frac{1}{2} \bar{\epsilon} \gamma_5 \left(\hat{D}_a - \frac{i}{2} A_a \gamma_5 \right) \chi' + \frac{1}{4} \bar{\epsilon} \eta \gamma_a \gamma_5 \chi'$$

$$\begin{aligned}
\delta \hat{D}_a F &= \frac{1}{2} \bar{\epsilon} \left(\hat{D}_a - \frac{1}{2} A_a \gamma_5 - \frac{1}{2} \eta \gamma_a \right) \left(\hat{D} - \frac{1}{2} \kappa \gamma_5 + \eta \right) \chi' \\
\delta \hat{D}_a G &= \frac{1}{2} \bar{\epsilon} \left(\hat{D}_a - \frac{1}{2} A_a \gamma_5 - \frac{1}{2} \eta \gamma_a \right) \left[\gamma_5 \left(\hat{D} - \frac{1}{2} \kappa \gamma_5 \right) - \eta \gamma_5 \right] \chi' \\
\delta \hat{D}_a \chi' &= \frac{1}{2} \left\{ \hat{D}_a [\hat{D}(A - i\gamma_5 B) + F + i\gamma_5 G] \right\} \epsilon \\
&\quad - \frac{1}{4} [\hat{D}(A - i\gamma_5 B) + F + i\gamma_5 G] (iA_a \gamma_5 - \gamma_a \eta) \epsilon \\
&\quad + \frac{1}{8} \bar{\epsilon} (\Gamma_{[cd]a}^{Z} - \gamma_a \psi_{cd}^{(\text{cov})}) \Gamma^{cd} \chi' .
\end{aligned} \tag{2.10}$$

A locally supersymmetric Lagrangian for a chiral multiplet $[z, \chi_L, h]$ is given by:

$$e^{-1} \mathcal{L} = h + \bar{\psi}_R \cdot \gamma \chi_L + \frac{1}{2} \bar{\psi}_{\mu R} \Gamma^{\mu\nu} \psi_{\nu R} Z + u z \tag{2.11}$$

with

$$u = S - iP . \tag{2.12}$$

Density (2.11) is complex, thus usually one takes the real part, but we will use it as it stands since the imaginary part is important in providing the supersymmetric generalization of the Chern-Simons form. The latter in turn plays a crucial role in the Green-Schwartz mechanism, when formula (2.11) is applied to the square of the Weyl multiplet.¹⁶ This will become transparent in Section 5.

The second basic multiplet is the vector multiplet $[C, Z, H, K, B_a, \Lambda, D]$ where Z and Λ are Majorana fermions; C, H, K and D are scalar or pseudoscalar fields

and B_a is a vector one. Its transformation laws are:^{4,17}

$$\delta C = \frac{i}{2} \bar{\epsilon} \gamma_5 Z$$

$$\delta Z = \frac{1}{2} (H i \gamma_5 - K - B + \hat{D} C i \gamma_5) \epsilon$$

$$\delta H = \frac{i}{2} \bar{\epsilon} \gamma_5 \left(\hat{D} - \frac{i}{2} K \gamma_5 \right) Z + \frac{i}{2} \bar{\epsilon} \gamma_5 \Lambda - \frac{i}{2} \bar{\epsilon} \eta \gamma_5 Z$$

$$\delta K = -\frac{1}{2} \bar{\epsilon} \left(\hat{D} - \frac{i}{2} K \gamma_5 \right) Z - \frac{1}{2} \bar{\epsilon} \Lambda - \frac{1}{2} \bar{\epsilon} \eta Z \quad (2.13)$$

$$\delta B_a = -\frac{1}{2} \bar{\epsilon} \left(\hat{D}_a - \frac{i}{2} A_a \gamma_5 \right) Z - \frac{1}{2} \bar{\epsilon} \gamma_a \Lambda + \frac{1}{4} \bar{\epsilon} \eta \gamma_a Z$$

$$\delta \Lambda = \frac{1}{4} [\Gamma^{ab} Y_{ab} + 2 i \gamma_5 D] \epsilon$$

$$\delta D = \frac{i}{2} \bar{\epsilon} \gamma_5 \left(\hat{D} + \frac{i}{2} K \gamma_5 \right) \Lambda$$

where

$$Y_{ab} = 2 \hat{D}_{[a} B_{b]} - \frac{1}{2} \bar{Z} \psi_{ab}^{(cov)} \quad (2.14)$$

Let us also give the following transformation rules of covariant derivatives:

$$\delta \hat{D}_a C = \frac{i}{2} \bar{\epsilon} \gamma_5 \left(\hat{D}_a - \frac{i}{2} A_a \gamma_5 \right) Z - \frac{i}{4} \bar{\epsilon} \eta \gamma_a \gamma_5 Z$$

$$\delta \hat{D}_a Z = \frac{1}{2} [\hat{D}_a (H i \gamma_5 - K - B + \hat{D} C i \gamma_5)] \epsilon +$$

$$\begin{aligned}
& - \frac{1}{4} (H i \gamma_5 - K - B + \hat{D} C i \gamma_5) (i A_a \gamma_5 - \gamma_a \eta) \epsilon + \\
& + \frac{1}{8} \epsilon (\Gamma_{[cd]a} Z_a - \gamma_a \psi_{cd}^{(cov)}) \Gamma^{cd} Z
\end{aligned} \tag{2.15}$$

where we left the last term in $\delta \hat{D}_a Z$ "unfierz" for simplicity.

A locally supersymmetric Lagrangian for the vector multiplet is provided by the D-density:^{4,17}

$$\begin{aligned}
e^{-1} \mathcal{L}_D &= D - \frac{1}{2} \bar{\psi} \cdot \gamma \gamma_5 \Lambda - \frac{2}{3} (HS-KP) + \frac{2}{3} A^a B_a \\
&- \frac{1}{3} \bar{Z} \gamma_5 \gamma \cdot R + \frac{1}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho B_\sigma - \frac{1}{8} e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \bar{Z} \psi_\sigma \\
&- \frac{2}{3} C e^{-1} \mathcal{L}_{sg} .
\end{aligned} \tag{2.16}$$

A real function $\phi(\Sigma_i, \Sigma^{*j})$ of a set of chiral multiplets $\Sigma_i = [z_i, \chi_{iL}, h_i]$ and their complex conjugates will be a vector multiplet with components:

$$\begin{aligned}
\phi(\Sigma_i, \Sigma^{*j}) &= \left[\phi(z_i, z^{*j}), -2i(\phi^j \chi_{jL} + \phi_j \chi_R^j), \right. \\
&- \phi^j h_j - \phi_j h^{*j} + \phi^{jk-} \chi_{jL} \chi_{kL} + \phi_{jk} \chi_R^j \chi_R^k, \\
&i\phi^j h_j - i\phi_j h^{*j} - i\phi^{jk-} \chi_{jL} \chi_{kL} + i\phi_{jk} \bar{\chi}_R^j \chi_R^k, \\
&+ i\phi^j \hat{D}_\mu z_j - i\phi_j \hat{D}_\mu z^{*j} + 2i\phi_k^j \bar{\chi}_{jL} \gamma_\mu \chi_R^k, \\
&\left. - 2i\phi_k^j \left\{ (h_j + \hat{D} z_j) \chi_R^k - (h^{*k} + \hat{D} z^{*j}) \chi_{jL} \right\} + \right.
\end{aligned}$$

$$\begin{aligned}
& + 2i\phi^{jk}{}_{\ell} \bar{\chi}_{jL} \chi_{kL} \chi_R^{\ell} - 2i\phi^{\ell}{}_{jk} \bar{\chi}_R^j \chi_R^k \chi_{\ell L} , \\
& 2\phi^j{}_k \left\{ h_j h^{*k} - \hat{D}_{\mu} z_j \hat{D}^{\mu} z^{*k} - (\bar{\chi}_{jL} \hat{D} \chi_R^k + \bar{\chi}_R^k \hat{D} \chi_{jL}) \right\} \\
& - 2\phi^{jk}{}_{\ell} (h^{*\ell} \bar{\chi}_{jL} \chi_{kL} + \bar{\chi}_R^{\ell} (\hat{D} z_j) \chi_{kL}) \\
& - 2\phi^{\ell}{}_{jk} (h_{\ell} \bar{\chi}_R^j \chi_R^k + \bar{\chi}_{\ell L} (\hat{D} z^{*j}) \chi_R^k) \\
& + 2\phi^{ij}{}_{k\ell} \bar{\chi}_{iL} \chi_{jL} \bar{\chi}_R^k \chi_R^{\ell} \Big] . \tag{2.17}
\end{aligned}$$

One can obtain a chiral multiplet from a vector multiplet by chiral projection. This operation is used to define the so-called linear multiplet, which is the last basic type of multiplet in supergravity. We will postpone these matters until Section 5, however, so that we have a chance to introduce the scalar curvature multiplet which plays a significant role in all this.

3. The Curvature Multiplets.

There are three basic curvature multiplets: the scalar curvature multiplet R ,¹⁸ the Weyl multiplet W_{ab} ¹⁶ and the Ricci multiplet W_a .⁴ The first two are scalar multiplets, which we will present in their chiral form and the last one is a vector multiplet.

Let us start with the scalar curvature multiplet R , whose components are

$$R = [S, P, -\gamma \cdot z, B, -\hat{D}_a A^a] \quad (3.1a)$$

or in chiral form:

$$R = [u^*, -i\gamma \cdot z_R, B - i\hat{D}_a A^a] \quad (3.1b)$$

where u is given in (2.12) and

$$B = \eta^{ab} B_{ab} = \frac{1}{2} R^{(cov)} - \frac{2}{3} \left(|u|^2 + \frac{1}{2} A_a A^a \right). \quad (3.2)$$

The symbol B_{ab} will be one of the components of the Ricci multiplet. The supercovariantized scalar curvature is:

$$R^{(cov)} = R - \bar{\psi}^a \gamma^b \psi_{ab}^{(cov)} + \frac{1}{2} \bar{\psi}_a \Gamma^{ab} \eta \psi_b. \quad (3.3)$$

The product of R with its complex conjugate $\bar{R}$ is a vector multiplet whose components we can obtain by applying the formula (2.17) to (3.1):

$$R\bar{R} = \left[|u|^2, 2i\gamma_S(S - i\gamma_S P) \gamma \cdot z, -2SB + 2P\hat{D}_a A^a, \right.$$

$$\begin{aligned}
 & 2PB + 2S\hat{D}_a A^a, \quad iu\hat{D}_b u^* - i\bar{z}\cdot\gamma\gamma_5\gamma_b\gamma\cdot z, \\
 & - 2i\gamma_5\left\{ B + i\gamma_5\hat{D}_a A^a - \hat{D}(S-i\gamma_5 P)\right\} \gamma\cdot z, \\
 & 2\left\{ B^2 + (\hat{D}_a A^a)^2 - \hat{D}_b u\hat{D}^b u^* + \bar{z}\cdot\gamma\left(\hat{D} + \frac{1}{2}\gamma_5 A\right)\gamma\cdot z\right\}. \quad (3.4)
 \end{aligned}$$

Next we turn to the Weyl multiplet W_{ab}^α , which has two external vector indices a, b (it is antisymmetric in them) and one external spinor index. Its components are

$$W_{ab} = \left[R_{abL}(Q), \quad \frac{1}{4}(1+\gamma_5)(\Gamma^{cd}W_{abcd} - 3iR^{cd}(A)T_{cdab})C^{-1}, \quad Z_{abL} \right] \quad (3.5)$$

where C^{-1} is the inverse of the charge conjugation matrix and

$$Z_{ab} = 2T_{abcd}R^{(cov)cd}(S) - \frac{1}{2}(S-i\gamma_5 P)R_{ab}(Q). \quad (3.6)$$

$R_{ab}(Q)$ and $R_{ab}^{(cov)}(S)$ are the curvatures of conformal supergravity (they correspond to what have been usually called $\bar{R}_{ab}(Q)$ and $\bar{R}_{ab}^{(cov)}(S)$). The covariant S-curvature $R_{ab}^{(cov)}(S)$ is:

$$R_{ab}^{(cov)}(S) = R_{ab}(S) - \frac{1}{2}\bar{\psi}_{[a}\gamma^c(\gamma_5 R_{b]c}(A) - \tilde{R}_{b]c}(A)), \quad (3.7)$$

while $R_{ab}(A)$ is the conformal A-curvature. The Weyl tensor W_{abcd} is the traceless part of the covariant M-curvature $R_{ab}^{(cov)}(M_{cd})$. Since that removes the dependence on f_μ^a and b_μ , W_{abcd} is actually the traceless part of:

$$\hat{R}_{\mu\nu}^{(\text{cov})ab} = \hat{R}_{\mu\nu}^{ab} - \bar{\psi}_{[\mu} \gamma_{\nu]} R^{ab}(Q) - \frac{2}{3} e^{[a} \bar{\psi}_{[\mu} \psi_{\nu]} z^{b]} ,$$

$$\hat{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - \frac{1}{3} \bar{\psi}_{[\mu} \Gamma^{ab} z_{\nu]} + \frac{1}{6} \bar{\psi}_{\mu} \Gamma^{ab} (S - i\gamma_5 P) \psi_{\nu}$$

$$+ \frac{1}{6} \epsilon^{abcd} A_c \bar{\psi}_{\mu} \gamma_d \psi_{\nu} , \quad (3.8)$$

and $R_{\mu\nu}^{ab}$ is the Riemann tensor for $\omega_{\mu}^{ab}(e, \psi)$.[†]

Finally, the matrix projector T_{abcd} is given by:^{4,16}

$$T_{ab}^{cd} = \frac{1}{3} \left(\delta_{[a}^c \delta_{b]}^d - \frac{1}{2} \Gamma_{ab}^{cd} - \delta_{[a}^{[c} \Gamma_{b]}^d \right) , \quad (3.9)$$

$$(\Gamma_{abcd} = \epsilon_{abcd} \gamma_5)$$

which has the following properties:

$$T_{abcd} \gamma^d = \gamma^a T_{abcd} = 0$$

$$T_{abcd} \gamma_5 = \gamma_5 T_{abcd} = -\frac{1}{2} \epsilon_{abef} T^{ef}_{cd}$$

$$T_{abef} T^{ef}_{cd} = T_{abcd} , \quad T_{abef} T^{ef}_{cd} = \frac{1}{3} T_{abcd} ,$$

$$CT_{abcd} C^{-1} = T_{abcd}^T \quad (3.10)$$

$$(CT_{abcd})_{\alpha\beta} = -(CT_{cdab})_{\beta\alpha}$$

[†]There is a misprint in the last term of $\hat{R}_{\mu\nu}^{ab}$ in Eq. (4.10) of Ref. 4.

where C is the charge conjugation matrix.

The Weyl multiplet satisfies $W_{ab} = T_{abcd} W^{cd}$ (which implies $\gamma^b W_{ab} = 0$ among other things) and is independent of the conformal dilation field b_μ . As a matter of fact, one has:

$$R_{ab}(Q) = -\psi_{ab}^{(cov)} + \frac{2}{3} \gamma_{[a} z_{b]} . \quad (3.11)$$

Clearly (3.11) does not depend on b_μ , nor does $R_{ab}(A)$ or W_{abcd} (see Eq. (3.8)). For more details, we refer the reader to Appendix A.

The square of the Weyl multiplet $\bar{W}_{ab} W^{ab}$ is also a chiral multiplet with no external indices and its components are:^{6,16}

$$\begin{aligned} \bar{W}_{ab} W^{ab} = & \left[\bar{R}_{ab}(Q) R_L^{ab}(Q) , \frac{1}{2} (\Gamma^{cd} W_{abcd} - 3i R_{ab}(A)) R_L^{ab}(Q) , \right. \\ & 4\bar{R}_L^{ab}(Q) R_{ab}^{(cov)}(S) + \frac{1}{2} W_{abcd} W^{abcd} - \frac{1}{4} \epsilon^{abcd} W_{\mu\nu ab} W^{\mu\nu cd} \\ & \left. - \frac{3}{4} R_{ab}(A) R^{ab}(A) + \frac{3}{4} R_{ab}(A) \tilde{R}^{ab}(A) - u \bar{R}_{ab}(Q) R_L^{ab}(Q) \right] . \quad (3.12) \end{aligned}$$

If we put $u = 0$ in the last component, we get the conformal $\bar{W}_{ab} W^{ab}$ multiplet.

The third basic curvature multiplet in old-minimal Poincaré supergravity is the Ricci multiplet $W_a = [C_a, Z_a, H_a, K_a, B_{ab}, \lambda_a, d_a]$ given by

$$C_a = A_a$$

$$Z_a = z_a$$

$$H_a = \hat{D}_a P + A_a S$$

$$K_a = \hat{D}_a S - A_a P$$

$$B_{ab} = \frac{3}{2} \left(\hat{R}_{ab} - \frac{1}{6} \eta_{ab} \hat{R} \right) - \frac{3}{4} \bar{\psi}^c \gamma_a R_{bc}^{(Q)} + \frac{i}{2} \tilde{F}_{ab}$$

$$- \frac{1}{6} (|u|^2 + A_c A^c) \eta_{ab} + \frac{1}{3} A_a A_b + \frac{1}{2} \bar{\psi}_a z_b$$

$$\lambda_a = 2\gamma^b \left(\hat{D}_{[a} - \frac{i}{2} \gamma_5 A_{[a} \right) z_{b]} + \frac{1}{2} (S + i\gamma_5 P) z_a$$

$$+ \frac{i}{2} \gamma_5 \mathcal{K} z_a + \frac{i}{6} \gamma_5 \mathcal{K} \Gamma_{ac} z^c \quad (3.13)$$

$$d_a = \hat{D}_a \hat{D}_b A^b - \hat{D}_a A - \frac{2}{3} B_{ac} A^c - \frac{1}{3} B A_a$$

$$- \frac{1}{6} A_a |u|^2 - \frac{1}{12} \bar{z}^b \gamma_5 \Gamma_{abc} z^c + \frac{i}{6} \bar{z}_a \gamma_5 \gamma \cdot z$$

$$- \frac{i}{6} \bar{z}^c \gamma_5 \gamma_a z_c + i \bar{z}^c \gamma_5 \psi_{ac}^{(cov)}$$

where

$$\hat{R}_{\mu\nu} = \hat{R}_{\mu\lambda}{}^{ab} e_a{}^\lambda e_{bv} ,$$

$$\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu} ,$$

(3.14)

$$\hat{F}_{ab} = 2\hat{D}_{[a} A_{b]} ,$$

$$\hat{D}_a A_a = \hat{D}_b \hat{D}^b A_a ,$$

and $\hat{R}_{\mu\lambda}{}^{ab}$ has been given in (3.8).

With the results of Appendix B it is easy to see that the antisymmetric part of B_{ab} is simply

$$B_{[ab]} = \frac{1}{2} \tilde{F}_{ab} \quad (3.15)$$

and that $\hat{F}_{ab}$ is related to the conformal A-curvature:

$$\hat{F}_{ab} = \frac{3}{2} R_{ab}(A) . \quad (3.16)$$

The transformation rule of A_a has been given in (2.1) which indeed has the form dictated by (2.13). As already pointed out in Ref. 6, this is not true for the remaining components because of the external index. One gets:

$$\begin{aligned} \delta Z_a &= \frac{1}{2} (H_a i\gamma_5 - K_a - B_{ab} \gamma^b + i\hat{D}A_a \gamma_5) \epsilon \\ &\quad - \frac{1}{6} \Gamma_{ab} \gamma_5 (S - i\gamma_5 P) A^b \epsilon , \\ \delta H_a &= \frac{1}{2} \bar{\epsilon} \gamma_5 \left(\hat{D} - \frac{1}{2} K \gamma_5 \right) z_a + \frac{1}{2} \bar{\epsilon} \gamma_5 \lambda_a - \frac{1}{2} \bar{\epsilon} \eta \gamma_5 z_a \\ &\quad - \frac{1}{12} \bar{\epsilon} (S - i\gamma_5 P) \gamma_5 \Gamma_{ac} z^c , \\ \delta K_a &= - \frac{1}{2} \bar{\epsilon} \left(\hat{D} - \frac{1}{2} K \gamma_5 \right) z_a - \frac{1}{2} \bar{\epsilon} \lambda_a - \frac{1}{2} \bar{\epsilon} \eta z_a \\ &\quad - \frac{1}{12} \bar{\epsilon} (S - i\gamma_5 P) \Gamma_{ac} z^c , \\ \delta B_{ab} &= - \frac{1}{2} \bar{\epsilon} \left(\hat{D}_b - \frac{1}{2} A_b \gamma_5 \right) z_a - \frac{1}{2} \bar{\epsilon} \gamma_b \lambda_a + \frac{1}{4} \bar{\epsilon} \eta \gamma_b z_a + \end{aligned} \quad (3.17)$$

$$\begin{aligned}
& + \frac{1}{12} \bar{\epsilon} (S - i\gamma_5 P) \Gamma_{ca} \gamma_b z^c + \frac{i}{6} \bar{\epsilon} \gamma_5 (A_b z_a - A \cdot z \eta_{ab}) \\
& - \frac{i}{2} \bar{\epsilon} \gamma_b \gamma_5 A^c \psi_{ac}^{(cov)} - \frac{i}{6} \bar{\epsilon} \gamma_5 A^c \Gamma_{ca} z_b, \\
\delta \lambda_a = & \frac{1}{2} \Gamma^{bc} \left[\hat{\Gamma}_b B_{ac} - \frac{1}{4} \bar{z}_a \psi_{bc}^{(cov)} - \frac{1}{6} \bar{z}^d \Gamma_{adc} z_b - \frac{1}{12} \bar{z}_a \Gamma_{bcd} z^d + \frac{1}{12} \bar{z}^d \Gamma_{abc} z^d \right. \\
& - \frac{1}{12} \bar{z}_b \gamma_a z_c - \frac{1}{6} (S - i\gamma_5 P) \hat{\Gamma}_c (S + i\gamma_5 P) - \frac{i}{2} \gamma_5 (\hat{R}_{bcad}^{(cov)} A^d + \frac{1}{3} |u|^2 \eta_{ab} A_c) \left. \right] \epsilon \\
& + \frac{1}{6} A^c \hat{F}_{ac} \epsilon + \frac{i}{2} \gamma_5^d a \epsilon \\
& + \frac{1}{2} \left\{ \frac{1}{3} \left[\tilde{F}_{ab} - \hat{F}_{ab} \gamma_5 - \gamma_5 (A_b \delta_a^c - \eta_{ab} A^c) \hat{\Gamma}_c - \frac{2i}{3} (A_a A_b - A^c A_c \eta_{ab}) \right. \right. \\
& + \frac{i}{3} (\delta_a^c \delta_b^d - \eta_{ab} \eta^{cd}) (B_{cd} + i\gamma_5 \hat{\Gamma}_d A_c) \left. \right] (S + i\gamma_5 P) \\
& + \frac{1}{2} \bar{z}^c \gamma_b \psi_{ac}^{(cov)} - \frac{1}{2} \bar{z}^c \gamma_5 \gamma_b \psi_{ac}^{(cov)} \gamma_5 - \frac{1}{12} \bar{z}^c \Gamma_{cb} z_a + \frac{1}{12} \bar{z}^c \Gamma_{cb} \gamma_5 z_a \gamma_5 \\
& + \frac{1}{12} \left[(\bar{z}_a z_b - \bar{z}_a \gamma_5 z_b \gamma_5) - \eta_{ab} (\bar{z}^c z_c - \bar{z}^c \gamma_5 z_c \gamma_5) \right] \left. \right\} \gamma^b \epsilon, \\
\delta d_a = & \frac{i}{2} \bar{\epsilon} \gamma_5 \left(\hat{D} + \frac{i}{2} K \gamma_5 \right) \lambda_a - \frac{i}{6} \bar{\epsilon} (S - i\gamma_5 P) \gamma_5 \Gamma_{ab} \lambda^b - \frac{1}{6} \bar{\epsilon} A^c \Gamma_{cab} \lambda^b \\
& + \frac{i}{6} \bar{\epsilon} \gamma_5 (S - i\gamma_5 P) \gamma_{[b} \left(\hat{D}^b - \frac{i}{2} A^b \gamma_5 \right) z_{a]} + \frac{1}{3} \bar{\epsilon} A^b \Gamma_{c[a} \left(\hat{D}_{b]} - \frac{i}{2} A_{b]} \gamma_5 \right) z^c \\
& + \frac{2}{3} \bar{\epsilon} A^b \left(\hat{D}_{[a} - \frac{i}{2} \gamma_5 A_{[a} \right) z_{b]} - \frac{i}{6} \bar{\epsilon} \gamma_5 (H_{[a} i\gamma_5 - K_{[a}) \gamma_{b]} z^b \\
& + \frac{i}{6} \bar{\epsilon} (B_{(ac)} - B \eta_{ac}) \gamma_5 z^c + \frac{i}{12} \bar{\epsilon} (B_{ac} - i\gamma_5 \hat{\Gamma}_c A_a) \gamma_5 \Gamma^{cd} z_d \\
& - \frac{i}{12} \bar{\epsilon} (B_{bc} + i\gamma_5 \hat{\Gamma}_b A_c) \gamma_5 \Gamma_a^c z^b - \frac{i}{12} \bar{\epsilon} (B - i\hat{\Gamma}_c A^c \gamma_5) \Gamma_{ab} \gamma_5 z^b +
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{4} \bar{\epsilon} \gamma^b \gamma_a \hat{F}_{bc} z^c - \frac{1}{3} \bar{\epsilon} \hat{F}_{b[a} \Gamma^{bc} z_{c]} - \frac{1}{24} |u|^2 \bar{\epsilon} \gamma_5 \gamma_a \gamma \cdot z \\
& + \frac{1}{72} \bar{\epsilon} (S - i\gamma_5 P) \Gamma_{acd} A^c z^d + \frac{1}{24} \bar{\epsilon} (S - i\gamma_5 P) A z_a \\
& + \frac{1}{36} \bar{\epsilon} (S - i\gamma_5 P) A_{[a} \gamma_{b]} z^b + \frac{i}{6} \bar{\epsilon} \gamma_5 A^b A_{[a} z_{b]} \\
& + \frac{1}{9} \bar{\epsilon} \gamma_5 A_{[a} \Gamma_{b]c} A^b z^c - \frac{1}{36} \bar{\epsilon} \gamma_5 \Gamma_{ad} A^d A \cdot z + \frac{i}{6} \bar{\epsilon} A \gamma_5 A^c \psi_{ac}^{(cov)} \\
& + \frac{i}{2} \bar{\epsilon} (B^{cd} \gamma_d - i \hat{D} A^c \gamma_5) \gamma_5 \psi_{ac}^{(cov)},
\end{aligned}$$

where $B_{(ac)} = \frac{1}{2} (B_{ac} + B_{ca})$ is the symmetric part of B_{ac} .

The derivation of δz_a and δB_{ab} has been explained in Ref. 4.[†] The derivation of δH_a and δK_a is simple after using the rules (2.10) as applied to the multiplet in (3.1a). In order to obtain $\delta \lambda_a$ one needs the γ -trace of the antisymmetric part of $\delta \hat{D}_a z_b$ (the more general transformation is given in Appendix C).

$$\begin{aligned}
\gamma^b \delta \hat{D}_{[a} z_{b]} &= \frac{1}{8} \left[\frac{1}{2} \hat{D}_a |u|^2 + \frac{1}{3} \hat{D}_a (A_c A^c) + \frac{4}{3} A_a \hat{D}_c A^c - A^c \hat{D}_c A_a - \frac{5}{6} \bar{z}_a \gamma \cdot z \right] \epsilon \\
&+ \frac{i}{4} \gamma_5 \left[\hat{D}_a \hat{D}_b A^b - \hat{D} A_a + \frac{1}{2} (P \hat{D}_a S - S \hat{D}_a P) - \frac{2}{3} A_a |u|^2 - \frac{5}{6} B_{ac} A^c + \frac{1}{3} B A_a \right. \\
&- \frac{i}{8} \bar{z}^b \gamma_5 \Gamma_{abc} z^c + \frac{7i}{12} \bar{z}_a \gamma_5 \gamma \cdot z - \frac{5i}{24} \bar{z}^c \gamma_5 \gamma_a z_c + i \bar{z}^c \gamma_5 \psi_{ac}^{(cov)} \left. \right] \epsilon \\
&+ \frac{1}{12} \gamma^b \left\{ i (\hat{F}_{ab} \gamma_5 + \tilde{F}_{ab}) + 2 A_a A_b + B_{ab} - i \hat{D}_b A_a + \right.
\end{aligned}$$

[†]The result for δB_{ab} is incorrectly reported, however.

$$\begin{aligned}
& + \frac{1}{2} (B - i\gamma_5 \hat{D}_c A^c) \eta_{ab} \left\{ (S - i\gamma_5 P) + A_b (H_a + i\gamma_5 K_a) - 2A_a (H_b + i\gamma_5 K_b) \right. \\
& - \frac{1}{2} \eta_{ab} A^c (H_c + i\gamma_5 K_c) - \frac{1}{2} \epsilon_{abcd} A^c \hat{D}^d (S - i\gamma_5 P) + \frac{3}{2} \bar{z}^c \gamma_b \psi_{ac}^{(cov)} \\
& + \frac{3}{2} \bar{z}^c \gamma_5 \gamma_b \psi_{ac}^{(cov)} \gamma_5 + \frac{1}{2} \bar{z}_b \Gamma_{ad} z^d - \frac{1}{2} \bar{z}^d \Gamma_{db} z_a - \frac{3}{8} \bar{z}_c \Gamma^{cd} z_d \eta_{ab} + \frac{5}{4} \bar{z}_b z_a \\
& - \frac{1}{8} \bar{z}^c z_c \eta_{ab} + \frac{1}{4} \bar{z}_b \gamma_5 \Gamma_{ad} z^d \gamma_5 - \frac{1}{4} \bar{z}^d \Gamma_{db} \gamma_5 z_a \gamma_5 - \frac{1}{8} \bar{z}_c \gamma_5 \Gamma^{cd} z_d \gamma_5 \eta_{ab} \\
& \left. + \frac{3}{4} \bar{z}_b \gamma_5 z_a \gamma_5 - \frac{3}{8} \bar{z}^c \gamma_5 z_c \gamma_5 \eta_{ab} \right] \epsilon \\
& + \frac{1}{4} \Gamma^{bc} \left[\hat{D}_b B_{ac} - \frac{1}{2} \gamma_5 \hat{R}_{bcad}^{(cov)} A^d - \frac{1}{4} \bar{z}_a \psi_{bc}^{(cov)} - \frac{1}{6} (S - i\gamma_5 P) \eta_{ab} \hat{D}_c (S + i\gamma_5 P) \right. \\
& + \frac{1}{6} \gamma_5 \left\{ (B_{ab} - i\gamma_5 \hat{D}_b A_a) - (B - i\gamma_5 \hat{D}_c A^c) \eta_{ab} \right\} A_c \\
& - \frac{1}{6} \bar{z}^d \Gamma_{dca} z_b - \frac{1}{8} \bar{z}^d \Gamma_{dbc} z_a + \frac{1}{16} \bar{z}_d \Gamma_{abc} z^d - \frac{1}{12} \bar{z} \cdot \gamma z_c \eta_{ab} \\
& \left. - \frac{1}{12} \bar{z}_b \gamma_c z_a - \frac{1}{24} \bar{z}_b \gamma_a z_c \right] \epsilon , \tag{3.18}
\end{aligned}$$

where use has been made of the Bianchi identity:

$$\begin{aligned}
\hat{D}_a B - \hat{D}^b B_{ab} &= -\frac{1}{2} \hat{D}_a \left(|u|^2 + \frac{1}{3} A_c A^c \right) - \frac{1}{3} \hat{D}_b (A_a A^b) \\
&+ \bar{z}^c \psi_{ac}^{(cov)} + \frac{1}{6} \bar{z}_a \gamma \cdot z \tag{3.19}
\end{aligned}$$

and of

$$\frac{1}{4} \bar{z}^c \psi_{ac}^{(cov)} + \frac{1}{8} \epsilon_{abcd} \bar{z}^d \gamma_5 \psi^{(cov)bc} = \frac{1}{12} \bar{z}_a \gamma \cdot z . \tag{3.20}$$

In (3.18) $\hat{R}_{ab}^{(cov)cd}$ arises because of the commutator of two supercovariant derivatives, which is given by a Lorentz transformation with parameter $-R_{ab}^{(cov)cd}$ plus a supersymmetry transformation with parameter $-\psi_{ab}^{(cov)}$:

$$[\hat{D}_a, \hat{D}_b] = -\delta_L(\hat{R}_{ab}^{(cov)cd}) - \delta_{sup}(\psi_{ab}^{(cov)}) \quad (3.21)$$

on any field with only tangent space indices. In particular

$$\begin{aligned} [\hat{D}_a, \hat{D}_b] A^c &= \hat{R}_{ab}^{(cov)cd} A_d - \frac{1}{2} \bar{z}^c \gamma_5 \psi_{ab}^{(cov)} , \\ [\hat{D}_a, \hat{D}_b] z^c &= \frac{1}{4} \hat{R}_{ab}^{(cov)de} \Gamma_{de} z^c + \hat{R}_{ab}^{(cov)cd} z_d \\ &\quad - \frac{1}{2} \left[H^c i \gamma_5 - K^c - B^{cd} \gamma_d + i \hat{D} A^c \gamma_5 - \frac{1}{3} \Gamma^{cd} \gamma_5 (S - i \gamma_5 P) A_d \right] \psi_{ab}^{(cov)} . \end{aligned} \quad (3.22)$$

The second commutator in (3.22) is used in the derivation of δd_a together with the Bianchi identities for $\hat{R}_{abcd}^{(cov)}$ of Appendix B.

Finally, to obtain the transformation of d_a , one needs as additional ingredients $\delta \hat{A}_a$ and $\delta \psi_{ab}^{(cov)}$:

$$\begin{aligned} \delta \hat{A}_a &= \frac{1}{2} \bar{\epsilon} \left(\hat{D}_c - \frac{1}{2} \gamma_5 A_c - \frac{1}{2} \eta \gamma_c \right) \left[\gamma_5 \left(\hat{D}^c - \frac{1}{2} A^c \gamma_5 \right) - \frac{1}{2} \eta \gamma^c \gamma_5 \right] z_a \\ &\quad - \frac{1}{2} \bar{\epsilon} A^d \left(\hat{D} - \frac{1}{2} \gamma_5 A - 2\eta \right) \psi_{ad}^{(cov)} + \frac{1}{2} \bar{\epsilon} \Gamma_{[ca} z_{d]} \hat{F}^{cd} - \frac{1}{4} \bar{\epsilon} \eta \gamma^c \Gamma_{[ca} z_{d]} A^d \\ &\quad - \frac{1}{3} \bar{\epsilon} \gamma^d \gamma_{[d} z_{c]} \hat{D}^c A_a - \bar{\epsilon} \gamma_c \psi_{ad}^{(cov)} \hat{D}^c A^d + \frac{1}{2} \bar{\epsilon} A^d \Gamma_{[ca} \hat{D}^c z_{d]} , \end{aligned} \quad (3.23)$$

$$\begin{aligned}
\delta\psi_{ab}^{(\text{cov})} = & \left[\frac{1}{4} \hat{R}_{ab}^{(\text{cov})cd} + \frac{i}{3} \delta^c_{[a} \hat{D}_{b]} A^d \gamma_5 + \frac{1}{18} (|u|^2 + A_e A^e) \delta_a^c \delta_b^d \right. \\
& - \frac{1}{9} \delta^c_{[a} A_{b]} A^d \left. \right] \Gamma_{cd} \epsilon + \frac{i}{3} \gamma_5 \hat{F}_{ab} \epsilon \\
& + \frac{1}{3} \left[H_{[a} i \gamma_5 + K_{[a} - \frac{i}{3} (S + i \gamma_5 P) \gamma_5 A_{[a} \right] \gamma_{b]} \epsilon \\
& + \frac{i}{9} \epsilon_{abcd} A^c \gamma^d (S - i \gamma_5 P) \epsilon .
\end{aligned} \tag{3.24}$$

In order to obtain the final form for δd_a , one must also use the Bianchi identities for $\hat{R}_{abcd}^{(\text{cov})}$ and $\psi_{ab}^{(\text{cov})}$ which we provide in Appendix B.

The square of the Ricci multiplet $W_a W^a$ has to be derived from its first component $C^{\text{new}} = A_a A^a$, since the multiplication rule of vector multiplets without external indices does not apply here, as anticipated in Ref. 4. Using their nomenclature, we get:[†]

$$W_a W^a = [C^{\text{new}}, Z^{\text{new}}, H^{\text{new}}, K^{\text{new}}, B_b^{\text{new}}, \Lambda^{\text{new}}, D^{\text{new}}] \tag{3.25}$$

with

$$C^{\text{new}} = A_a A^a$$

$$Z^{\text{new}} = 2A_a Z^a$$

$$H^{\text{new}} = 2A^a H_a - \frac{1}{2} \bar{z}^a z_a$$

[†]The component D^{new} has not been correctly reported in Ref. 4.

$$K^{\text{new}} = 2A^a K_a + \frac{i}{2} \bar{z}^a \gamma_5 z_a$$

$$B_b^{\text{new}} = 2A^a B_{ab} + \frac{i}{2} \bar{z}^a \gamma_5 \gamma_b z_a$$

$$\Lambda^{\text{new}} = 2A^a \lambda_a + (H_a - i\gamma_5 K_a - iB_{ab} \gamma^b \gamma_5 - \hat{D}A_a) z^a$$

$$\begin{aligned} D^{\text{new}} = & H_a H^a + K_a K^a - B_{ab} B^{ab} - (\hat{D}_b A_a)(\hat{D}^b A^a) + 2A^a(\hat{D}_a \hat{D}_b A^b - \hat{D}A_a) \\ & - \frac{4}{3} B_{ab} A^a A^b - \frac{2}{3} B A_a A^a - \frac{1}{3} |u|^2 A_a A^a + \bar{z}^a \left(\hat{D} - \frac{i}{2} \kappa \gamma_5 \right) z_a \\ & - 2\bar{z}^a \left(\hat{D}_a + \frac{i}{2} A_a \gamma_5 \right) \gamma \cdot z - \bar{z}^a (S + i\gamma_5 P) z_a - \frac{1}{3} A_a \bar{z}_b \gamma_5 \Gamma^{abc} z_c \\ & - \frac{4}{3} i \bar{z}^a \gamma_5 \kappa z_a + \frac{i}{3} \bar{z} \cdot A \gamma_5 \gamma \cdot z + 2i \bar{z}^b \gamma_5 \psi_{ab}^{(\text{cov})} A^a. \end{aligned} \quad (3.26)$$

We stress that these components are derived directly. As it turns out, all the components except D^{new} can be expressed in terms of the components of W_a according to the naive multiplication rule. For D^{new} we have an extra term though:

$$\begin{aligned} D^{\text{new}} = & 2A^a d_a - 2\bar{z}^a \lambda_a + H_a H^a + K_a K^a - B_{ab} B^{ab} - (\hat{D}_b A_a) \hat{D}^b A^a \\ & - \bar{z}^a \left(\hat{D} - \frac{i}{2} \kappa \gamma_5 \right) z_a + \frac{i}{6} \bar{z}^a \gamma_5 A^c \Gamma_{cab} z^b. \end{aligned} \quad (3.27)$$

4. The Super Gauss-Bonnet Theorem.

Applying the density formula (2.11) to the multiplet (3.12) we obtain after some manipulations:[†]

$$\begin{aligned}
 e^{-1} \mathcal{L}(\bar{W}_{ab} W^{ab}) &= \frac{1}{2} \hat{R}_{\mu\nu ab} \hat{R}^{ab\mu\nu} - \hat{R}_{ab} \hat{R}^{ba} + \frac{1}{6} \hat{R}^2 - \frac{3}{4} R_{\mu\nu}(A) \tilde{R}^{\mu\nu}(A) \\
 &- \frac{1}{2} e^{-1} \epsilon^{\mu\nu\rho\sigma} R_{\nu\rho ab} \bar{\phi}_\mu \gamma_5 \Gamma^{ab} \psi_\sigma - i e^{-1} \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho} \bar{\phi}_\mu \psi_\sigma \\
 &+ \bar{\psi}_\mu \hat{R}_{\nu\rho} \gamma^{\rho} R^{\mu\nu}(Q) + 4 e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\phi}_\sigma \gamma_\rho \gamma_5 \left[D_\nu - \frac{1}{2} \gamma_5 A_\nu \right] \phi_\mu \\
 &- \frac{1}{2} \bar{R}_{ab}(Q) \gamma_\mu \psi_\nu \bar{R}^{\mu\nu}(Q) \gamma^a \psi^b - \frac{3}{4} \bar{R}_{bd}(Q) \gamma_\nu \psi^d \bar{R}^{av}(Q) \gamma^b \psi_a \\
 &+ \frac{1}{4} \bar{\psi}_\rho \Gamma^{\rho\sigma} \psi_\sigma \bar{R}_{\mu\nu}(Q) R^{\mu\nu}(Q) - \frac{1}{4} \bar{\psi}_\rho \Gamma^{\rho\sigma} \gamma_5 \psi_\sigma \bar{R}_{\mu\nu}(Q) \gamma_5 R^{\mu\nu}(Q) \\
 &+ \frac{1}{2} e^{-1} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left[-\Omega_{\nu\rho\sigma}^{\text{LOR}} + \frac{2}{3} \Omega_{\nu\rho\sigma}(A) - 8 \bar{\phi}_\nu (1 + \gamma_5) \mathfrak{D}_\rho \psi_\sigma + 4 \bar{\phi}_\nu \gamma_\rho \psi_\sigma \right] . \quad (4.1)
 \end{aligned}$$

The symbols $\hat{R}_{\mu\nu ab}$, $\hat{R}_{ab}$ and $\hat{R}$ have been defined in (3.8) and (3.14) and

$$\Omega_{\nu\rho\sigma}^{\text{LOR}} = \omega_{[\nu}{}^{ab} R_{\rho\sigma]ab} + \frac{2}{3} \omega_{\nu ab} \omega_\rho{}^{bc} \omega_{\sigma c}{}^a$$

$$\Omega_{\nu\rho\sigma}(A) = A_{[\nu} F_{\rho\sigma]} , \quad F_{\rho\sigma} = 2\partial_{[\rho} A_{\sigma]}$$

$$\phi_\nu = \frac{1}{3} \gamma^\rho \left(S_{\nu\rho} + \frac{1}{2} \gamma_5 \tilde{S}_{\nu\rho} \right)$$

[†]The imaginary part of this Lagrangian as given here has already been given in Ref. 6. The factor 4 in the last term $4\phi_\nu \gamma_\rho \psi_\sigma$ is missing though in Eq. (7.6) there, which propagates to Eqs. (7.16), (7.21) and (7.24). Eq. (7.6) also needs an overall factor 1/2.

$$S_{\nu\rho} = -2\mathfrak{D}_{[\nu}\psi_{\rho]}$$

$$\mathfrak{D}_{\mu}\psi_{\nu} = \left(\partial_{\mu} + \frac{1}{4} \omega_{\mu}^{ab} \Gamma_{ab} + \frac{i}{2} \gamma_5 A_{\mu} \right) \psi_{\nu} . \quad (4.2)$$

The spin connection ω_{μ}^{ab} contains ψ_{μ} torsion but no b_{μ} terms. The object ϕ_{μ} is obtained from the φ_{μ} field of conformal supergravity by putting the dilation field b_{μ} equal to zero and is related to z_a in (2.2)

$$\phi_{\mu} = \varphi_{\mu} (b_{\nu} = 0) = \frac{1}{3} z_{\mu} + \eta \psi_{\mu} \quad (4.3)$$

If we apply now the D-density formula (2.16) to the multiplet $R\bar{R}$ in (3.4), we get the following Lagrangian after the necessary algebra:

$$\begin{aligned} \frac{1}{2} e^{-1} \mathcal{L}(R\bar{R}) = & \frac{1}{4} \hat{R}^2 + \bar{B} \bar{\psi}^a z_a - \frac{1}{3} B (|u|^2 + 2A_a A^a) + \frac{1}{6} (3|u|^2 + 2A_c A^c) \bar{\psi}^a z_a \\ & - \frac{1}{9} \left[|u|^4 + 4|u|^2 A_c A^c + (A_c A^c)^2 \right] + (\hat{D}_a A^a)^2 - \hat{D}_{\mu} u \hat{D}^{\mu} u^* + \frac{1}{3} A^{\mu} (u \hat{D}_{\mu} u^*) \\ & + 2\bar{z}^a D_a \gamma \cdot z - \bar{z}^a \not{D} z_a + e^{-1} \partial_{\mu} \left[e \bar{z} \cdot \gamma z^{\mu} - \frac{3}{2} e \bar{\psi}_{\nu} \Gamma^{\mu\nu\rho} (S+i\gamma_5 P) \phi_{\rho} \right. \\ & \left. - \frac{1}{2} e \bar{z}_a \Gamma^{apc} A \gamma_5 \psi_c \right] + \frac{2}{3} \bar{z}_a (S+i\gamma_5 P) R^a - \frac{1}{3} \bar{z}_a (S+i\gamma_5 P) \Gamma^{ab} R_b \\ & + 3ie^{-1} \epsilon^{\mu\nu\rho\sigma} A_{\mu} \bar{\psi}_{\nu} D_{\rho} \phi_{\sigma} + 3i\bar{\psi} \cdot A \gamma_5 \Gamma^{bc} D_b \phi_c - 6i\bar{\psi}_a \gamma_5 A^b \Gamma^{ac} D_{[b} \phi_{c]} \\ & + \bar{\psi} \cdot \gamma \hat{D}_a (S+i\gamma_5 P) z^a - \bar{\psi}_a \hat{D} (S+i\gamma_5 P) z^a + 9\bar{\phi}_{\mu} \Gamma^{\mu\nu\rho} D_{\nu} \phi_{\rho} - \frac{1}{2} \bar{\psi} \cdot A \gamma^b \psi^d D_b A_d \\ & + \frac{1}{2} \bar{\psi} \cdot \gamma A^b \psi^d D_b A_d + \frac{1}{2} \bar{\psi} \cdot A \gamma \cdot \psi D_b A^b + \frac{1}{4} A_a A^a \bar{\psi}_b R^b - \frac{1}{2} \bar{\psi} \cdot A A_d \Gamma^{bcd} D_b \psi_c + \end{aligned}$$

$$\begin{aligned}
& + \bar{\psi}_a A_d \Gamma^{acd} A^b D_{[b} \psi_{c]} + \frac{i}{6} A_a \bar{z} \gamma_S \Gamma^{abc} z_c - \frac{i}{3} \bar{z} \cdot A \gamma_S \gamma \cdot z + \frac{i}{6} \bar{z}^a \gamma_S A z_a \\
& + \frac{i}{24} |u|^2 \bar{\psi}_a \Gamma^{abc} \gamma_S A_b \psi_c + \frac{1}{8} e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \bar{z}^a \gamma_S \gamma_\sigma z_a + \frac{i}{8} \bar{\psi}_c \gamma_S A \Gamma^{cab} \psi_a \bar{\psi}_b \gamma \cdot z \\
& - \frac{i}{8} \bar{\psi}_c A \Gamma^{cab} \psi_a \bar{\psi}_b \gamma_S \gamma \cdot z - \frac{1}{4} (\bar{\psi}_a z_a)^2 + \frac{1}{4} \bar{\psi}_a \gamma_c \psi_b \bar{z}^a \gamma^c z^b - \frac{3i}{8} \bar{\psi}_b \gamma_d A \cdot \psi \bar{\psi}_c \gamma_S \Gamma^{bd} z^c \\
& + \frac{3i}{16} \bar{\psi}_a A \psi_b \bar{\psi}_c \gamma_S \Gamma^{ab} z^c - \frac{5i}{8} \bar{\psi} \cdot \gamma A \cdot \psi \bar{\psi}_a \gamma_S z^a + \frac{i}{16} \epsilon^{acbe} A_e \bar{\psi}_a \gamma_c \psi_b \bar{\psi}_d z^d \\
& - \frac{i}{8} \bar{\psi} \cdot \gamma \psi_b \bar{\psi}_c \gamma_S \Gamma^{bd} A_d z^c - \frac{i}{32} \bar{\psi}_a \Gamma_{\mu\nu} \psi_b \bar{\psi}_c A \Gamma^{\mu\nu} \Gamma^{ab} \gamma_S z^c \\
& + \frac{i}{8} \bar{\psi}_a \Gamma^{\mu\nu} A \cdot \psi \bar{\psi}_d \gamma^a \Gamma_{\mu\nu} \gamma_S z^d + \frac{i}{4} \epsilon^{abce} \bar{\psi}_b \gamma_e \psi_c \bar{z} \cdot A \psi_a - \frac{i}{8} \epsilon^{abcd} \bar{\psi}_a \gamma_b \psi_c \bar{z} \cdot \gamma A \psi_d \\
& + \frac{i}{8} \bar{\psi}_a \gamma_b \psi_c \bar{\psi}^b (S - i \gamma_S P) \Gamma^{ac} \gamma_S A \cdot \psi - \frac{i}{4} \bar{\psi}_a \gamma_b A \cdot \psi \bar{\psi}^b (S - i \gamma_S P) \Gamma^{ad} \gamma_S \psi_d . \quad (4.4)
\end{aligned}$$

The Lagrangian for the Ricci square multiplet (3.25) is obtained by using again (2.15) with the components (3.26) and the result can be cast in the form:

$$\begin{aligned}
e^{-1} \mathcal{L}(W_a \bar{W}^a) &= \hat{D}_\mu u \hat{D}^\mu u^* - \frac{1}{3} A^a (u \hat{D}_a u^*) + \frac{1}{9} [|u|^4 + 4|u|^2 A_c A^c + (A_c A^c)^2] \\
&+ \frac{1}{3} B(|u|^2 + 2A_c A^c) - \frac{1}{6} (3|u|^2 + 2A_c A^c) \bar{\psi}^a z_a - \frac{9}{4} \left(\hat{R}_{ab} \hat{R}^{ba} - \frac{2}{9} \hat{R}^2 \right) \\
&+ \frac{9}{4} \hat{R}_{ab} \bar{\psi} \cdot \gamma R^{ab}(Q) + \frac{9}{4} \hat{R}^{ab} \bar{\psi}^c \gamma_a R_{bc}(Q) - \frac{9}{64} \left\{ \frac{1}{2} \bar{\psi}^c \psi_c \bar{R}^{bd}(Q) R_{bd}(Q) \right. \\
&- \frac{1}{2} \bar{\psi}^c \gamma_S \psi_c \bar{R}^{bd}(Q) \gamma_S R_{bd}(Q) - 2 \bar{\psi}^c \gamma^a \gamma_S \psi_d \bar{R}^{bd}(Q) \gamma_a \gamma_S R_{bc}(Q) \\
&\left. - \frac{1}{2} \bar{\psi}_c \Gamma^{cd} \psi_d \bar{R}^{ab}(Q) R_{ab}(Q) + \frac{1}{2} \bar{\psi}_c \gamma_S \Gamma^{cd} \psi_d \bar{R}^{ab}(Q) \gamma_S R_{ab}(Q) \right\} +
\end{aligned}$$

$$\begin{aligned}
& - \frac{3i}{2} \bar{\psi}_a \phi_b \tilde{F}^{ab} + \frac{9}{16} (\bar{\psi}_a \Gamma^{ab} \psi_b \phi_c \Gamma^{cd} \phi_d - 4 \bar{\psi}_a \Gamma^{ac} \psi_b \phi_c \Gamma^{bd} \phi_d \\
& + \bar{\psi}_a \Gamma^{cd} \psi_b \phi_c \Gamma^{ab} \phi_d) - \frac{1}{4} A^a A_a \bar{\psi}_b R^b - 3i \phi^b \tilde{\psi}_{ab} A^a - \frac{9i}{2} A_a \phi_b \gamma_5 \Gamma^{abc} \phi_c \\
& + \frac{3}{4} \hat{F}_{ab} \hat{F}^{ab} - (\hat{D}_a A^a)^2 + e^{-1} \partial_\mu [e e_a^\mu D_b (A^b A^a)] - i e^{-1} \partial_\mu (e \bar{\psi} \cdot A \gamma_5 z^\mu) \\
& - e^{-1} \partial_\mu [e \hat{D}^\mu (A^a A_a)] + \frac{1}{2} \bar{\psi} \cdot \gamma A \cdot \psi D_b A^b - \frac{1}{2} \bar{\psi} \cdot \gamma \psi_a A^b D_b A^a \\
& + \frac{1}{2} \bar{\psi} \cdot A \gamma^b \psi_c D_b A^c + \frac{1}{4} (\bar{\psi}^a z_a)^2 - \frac{1}{4} \bar{\psi}_a \gamma_c \psi_b \bar{z}^a \gamma^c z^b - 3i \bar{\psi} \cdot A \gamma_5 \Gamma^{bc} D_b \phi_c \\
& + 6i \bar{\psi}_b \gamma_5 \Gamma^{bc} A^a D_{[a} \phi_{c]} + \frac{1}{2} \bar{\psi} \cdot A \Gamma^{bcd} A_d D_b \psi_c - \bar{\psi}_b \Gamma^{bcd} A^a A_d D_{[a} \psi_{c]} + \bar{z}^a \not{D} z_a \\
& - 2 \bar{z}^a D_a \gamma \cdot z - \bar{z}^a [\hat{D}(S - i \gamma_5 P) + B] \psi_a - \bar{\psi} \cdot \gamma \hat{D}_a (S + i \gamma_5 P) z^a - \frac{1}{6} \bar{z}^a \gamma_5 \not{A} z_a \\
& + \frac{1}{3} \bar{z} \cdot A \gamma_5 \gamma \cdot z - \frac{1}{6} A_a \bar{z}_b \gamma_5 \Gamma^{abc} z_c - \frac{1}{24} |u|^2 A_a \bar{\psi}_b \gamma_5 \Gamma^{abc} \psi_c \\
& - \frac{2}{3} \bar{z}^a (S + i \gamma_5 P) \left(R_a - \frac{1}{2} \Gamma_{ab} R^b \right) + \frac{1}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \left(\frac{1}{2} \bar{z}^a \gamma_5 \gamma_\sigma z_a - \bar{z} \cdot A \psi_\sigma \right) \\
& - \frac{5i}{8} \bar{\psi} \cdot A \gamma \cdot \psi \bar{\psi}_a \gamma_5 z^a - \frac{3i}{8} \bar{\psi} \cdot A \gamma_d \psi_b \bar{\psi}_a \gamma_5 \Gamma^{bd} z^a + \frac{i}{8} \bar{\psi} \cdot A \Gamma_{cd} \psi_b \bar{\psi}_a \gamma^b \Gamma^{cd} \gamma_5 z^a \\
& - \frac{i}{8} \bar{\psi}_c \gamma_5 \not{A} \Gamma^{cab} \psi_a \bar{\psi}_b \gamma \cdot z + \frac{i}{8} \bar{\psi}_c \not{A} \Gamma^{cab} \psi_a \bar{\psi}_b \gamma_5 \gamma \cdot z + \frac{i}{8} \epsilon^{bdac} \bar{\psi}_b \gamma_d \psi_a \bar{z} \cdot \gamma \not{A} \psi_c \\
& - \frac{3i}{16} \bar{\psi}_c \not{A} \psi_a \bar{\psi}_b \gamma_5 \Gamma^{ca} z^b - \frac{1}{16} \epsilon^{cdae} A_e \bar{\psi}_c \gamma_d \psi_a \bar{\psi}_b z^b + \frac{i}{32} \bar{\psi}_c \Gamma^{\mu\nu} \psi_a \bar{\psi}_b \not{A} \Gamma_{\mu\nu} \Gamma^{ca} \gamma_5 z^b \\
& + \frac{i}{8} \bar{\psi}_c \gamma \cdot \psi \bar{\psi}_a \gamma_5 A_d \Gamma^{dc} z^a + \frac{i}{8} \bar{\psi} \cdot A \gamma_5 \Gamma^{bc} (S - i \gamma_5 P) \psi_d \bar{\psi}_b \gamma^d \psi_c + \\
& - \frac{1}{4} \bar{\psi}_b \gamma_5 \Gamma^{ba} (S - i \gamma_5 P) \psi_c \bar{\psi} \cdot A \gamma^c \psi_a .
\end{aligned} \tag{4.5}$$

A considerable simplification occurs when one considers the combination

$\mathcal{L}(\bar{W}_{ab} W^{ab}) + \frac{1}{2} \mathcal{L}(\bar{R}R)$, which gives the Lagrangian:

$$\begin{aligned}
 e^{-1} [\mathcal{L}(W_a W^a) + \frac{1}{2} \mathcal{L}(\bar{R}R)] = & -\frac{9}{4} \left(\hat{R}_{ab} \hat{R}^{ba} - \frac{1}{3} \hat{R}^2 \right) + \frac{9}{4} \hat{R}_{ab} \bar{\psi} \cdot \gamma R^{ab}(Q) \\
 & + \frac{9}{4} \hat{R}^{ab} \bar{\psi}^c \gamma_a R_{bc}(Q) - \frac{9}{64} \left\{ \frac{1}{2} \bar{\psi}^c \psi_c \bar{R}^{bd}(Q) R_{bd}(Q) - \frac{1}{2} \bar{\psi}^c \gamma_5 \psi_c \bar{R}^{bd}(Q) \gamma_5 R_{bd}(Q) \right. \\
 & - 2 \bar{\psi}^c \gamma_e \gamma_5 \psi_d \bar{R}^{bd}(Q) \gamma_e \gamma_5 R_{bc}(Q) - \frac{1}{2} \bar{\psi}^c \Gamma^{cd} \psi_d \bar{R}^{ab}(Q) R_{ab}(Q) \\
 & \left. + \frac{1}{2} \bar{\psi}^c \gamma_5 \Gamma^{cd} \psi_d \bar{R}^{ab}(Q) \gamma_5 R_{ab}(Q) \right\} - \frac{3i}{2} \bar{\psi}_a \phi_b \tilde{F}^{ab} - 3i \phi^b \tilde{\psi}_{ab} A^a \\
 & + 3ie^{-1} \epsilon^{\mu\nu\rho\sigma} A_\mu \bar{\psi}_\nu D_\rho \phi_\sigma + 9\phi_\mu \Gamma^{\mu\nu\rho} \left(D_\nu - \frac{1}{2} A_\nu \gamma_5 \right) \phi_\rho + \frac{3}{4} \hat{F}_{ab} \hat{F}^{ab} \\
 & + e^{-1} \partial_\mu [e e_a^\mu D_b (A^b A^a) - e \hat{D}^\mu (A^a A_a) - ie \bar{\psi} \cdot A \gamma_5 \psi] + \frac{9}{16} (\bar{\psi}_a \Gamma^{ab} \psi_b \bar{\phi}_c \Gamma^{cd} \phi_d \\
 & - 4 \bar{\psi}_a \Gamma^{ac} \psi_b \bar{\phi}_c \Gamma^{bd} \phi_d + \bar{\psi}_a \Gamma^{cd} \psi_b \bar{\phi}_c \Gamma^{ab} \phi_d) + e^{-1} \partial_\mu [e \bar{z} \cdot \gamma z^\mu \\
 & - \frac{3}{2} e \bar{\psi}_\nu \Gamma^{\mu\nu\rho} (S + i \gamma_5 P) \phi_\rho - \frac{1}{2} e \bar{z}_a \Gamma^{a\mu c} A \gamma_5 \psi_c] .
 \end{aligned} \tag{4.6}$$

The super Gauss-Bonnet theorem is obtained when we take the combination:

$$\mathcal{L}(\bar{W}_{ab} W^{ab}) + \frac{4}{9} [\mathcal{L}(W_a W^a) + \frac{1}{2} \mathcal{L}(\bar{R}R)] = \partial_\mu K_{(r)}^\mu + i \partial_\mu K_{(i)}^\mu \tag{4.7}$$

which is a total divergence. The real part is the four-divergence of

$$K_{(r)}^\mu = \epsilon^{\mu\nu\rho\sigma} \left(\frac{1}{2} \Omega_{\nu\rho\sigma}^{(GB)} + \frac{4i}{3} \bar{\psi}_\nu \phi_\rho A_\sigma - 4 \bar{\phi}_\nu \gamma_5 D_\rho \psi_\sigma \right) + \frac{4}{9} e \left[e_a^\mu D_b (A^b A^a) + \right.$$

$$\begin{aligned}
 & - \hat{D}^\mu (A_a A^a) - i \bar{\psi} \cdot A \gamma_5 z^\mu + \bar{z} \cdot \gamma z^\mu - \frac{3}{2} \bar{\psi}_\nu \Gamma^{\mu\nu\rho} (S + i \gamma_5 P) \phi_\rho \\
 & - \frac{1}{2} \bar{z}_\nu \Gamma^{\nu\mu\rho} A \gamma_5 \psi_\rho] \quad (4.8)
 \end{aligned}$$

and the imaginary part is the divergence of:

$$K_{(i)}^\mu = -i \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{2} \Omega_{\nu\rho\sigma}^{(LOR)} + \frac{1}{3} \Omega_{\nu\rho\sigma}(A) - 4 \bar{\phi}_\nu \mathcal{D}_\rho \psi_\sigma + 2 \bar{\phi}_\nu \gamma_\rho \phi_\sigma \right). \quad (4.9)$$

The Gauss-Bonnet three-form $\Omega_{\nu\rho\sigma}^{GB}$ is given by

$$\Omega_{\nu\rho\sigma}^{GB} = \frac{1}{2} \epsilon_{abcd} (\omega_{[\mu}^{ab} R_{\nu\rho]}^{cd} - 2 \omega_{[\mu}^{ab} \omega_{\nu}^{ce} \omega_{\rho]e}^d), \quad (4.10)$$

and satisfies:

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu \Omega_{\nu\rho\sigma}^{GB} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} = e (R_{abcd} R^{cdab} - 4 R_{ab} R^{ba} + R^2). \quad (4.11)$$

In obtaining the super Gauss-Bonnet theorem (4.7), the following identity must be used:

$$\begin{aligned}
 & \hat{R}_{abcd} \hat{R}^{cdab} - 4 \hat{R}_{ab} \hat{R}^{ba} + \hat{R}^2 - e^{-1} \epsilon^{\mu\nu\rho\sigma} R_{\nu\rho ab} \bar{\phi}_\mu \gamma_5 \Gamma^{ab} \psi_\sigma \\
 & = R_{abcd} R^{cdab} - 4 R_{ab} R^{ba} + R^2 + 2 \bar{\psi}_a \Gamma^{db} \psi_b \bar{\phi}_d \Gamma^{ac} \phi_c - \frac{1}{2} \bar{\psi}_a \Gamma^{cd} \psi_b \bar{\phi}_c \Gamma^{ab} \phi_d + \\
 & - \frac{1}{2} \bar{\psi}_a \Gamma^{ab} \psi_b \bar{\phi}_c \Gamma^{cd} \phi_d. \quad (4.12)
 \end{aligned}$$

5. Gauss-Bonnet and Chern-Simons Multiplets.

A vector multiplet $V = [C, Z, H, K, B_b, \Lambda, D]$ can be projected into a chiral one, which corresponds to the superspace operation $(\overline{D}\overline{D}-8R)V$, as shown in Ref. 19, whose formulae can be symthesized as:

$$\begin{aligned} \Sigma(V) = & \left[-H + iK + \frac{2}{3} Cu^* , \right. \\ & - i\Lambda_L - i \left(\hat{D} + \frac{1}{2} K \right) Z_R + \frac{1}{3} u^* Z_L - \frac{1}{3} K Z_R - \frac{2}{3} C \gamma \cdot z_R , \\ & D + \hat{D}C + i\hat{D}_a B^a + \frac{2}{3} C(B - i\hat{D}_a A^a) + \frac{2}{3} A^a (B_a - i\hat{D}_a C) + \frac{2}{3} u(H-iK) \\ & \left. - \frac{1}{3} u^*(H+iK) + \frac{2i}{3} \bar{Z}_L \gamma \cdot z + \frac{1}{3} \bar{Z}_R \gamma \cdot z \right] . \end{aligned} \quad (5.1)$$

If we apply the density formula (2.11) to the chiral projection $\Sigma(V)$ (5.1), one obtains a complex Lagrangian whose real part is the D-density formula $\mathcal{L}_D$ in (2.16) plus a divergence¹⁹ and whose imaginary part is a total divergence. Explicitly,

$$\begin{aligned} \mathcal{L}_h(\Sigma(V)) = & \mathcal{L}_D(V) + \partial_\mu \left[e \left(\hat{D}^\mu C + \frac{1}{2} \bar{Z}_\gamma \Gamma^{\mu\nu} \psi_\nu \right) \right] \\ & + i\partial_\mu \left[e \left(B^\mu - \frac{2}{3} CA^\mu - \frac{1}{2} \bar{Z} \Gamma^{\mu\nu} \psi_\nu \right) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} C \bar{\psi}_\nu \gamma_\rho \psi_\sigma \right] . \end{aligned} \quad (5.2)$$

The third basic type of multiplet in supergravity is the linear multiplet,²⁰ which is a vector multiplet whose chiral projection vanishes.²¹ This means that only the C, Z and B_b components are independent and the vector obeys the constraint

$$\hat{D}_a \left(B^a - \frac{2}{3} C A^a \right) + \frac{1}{2} \bar{Z} \gamma \cdot z = 0 \quad (5.3)$$

which can be solved in terms of an antisymmetric tensor $b_{\mu\nu}$ ²¹

$$B^\mu = \frac{2}{3} C A^\mu - \frac{i}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma} (\partial_\nu b_{\rho\sigma} + C \bar{\psi}_\nu \gamma_\rho \psi_\sigma - i \bar{Z} \gamma_S \Gamma_{\nu\rho} \psi_\sigma) . \quad (5.4)$$

The dependent components H, K, Λ and D are easily read off from (5.1).

Eq. (5.2) provides an alternative way to present the super Gauss-Bonnet theorem, if we just replace C, Z and B_μ by the corresponding components of the multiplet $\frac{4}{9} \left(W_a W^a + \frac{1}{2} R \bar{R} \right)$

$$\mathcal{L}_h \left[\bar{W}_{ab} W^{ab} + \frac{4}{9} \Sigma \left(W_a W^a + \frac{1}{2} R \bar{R} \right) \right] = \partial_\mu (e V_{(r)}^\mu) + i \partial_\mu (e V_{(i)}^\mu) \quad (5.5)$$

with

$$\begin{aligned} V_{(r)}^\mu &= e^{-1} K_{(r)}^\mu + \frac{4}{9} \left[\hat{D}^\mu \left(A_a A^a + \frac{1}{2} |u|^2 \right) + i \bar{\psi}_\nu \gamma_S \Gamma^{\nu\mu} A \cdot z \right. \\ &\quad \left. - \frac{1}{2} \bar{\psi}_\nu \Gamma^{\nu\mu} (S - i \gamma_S P) \gamma \cdot z \right] , \\ V_{(i)}^\mu &= e^{-1} K_{(i)}^\mu + \frac{4}{9} \left[B^{\text{new } \mu} + \frac{1}{2} u \hat{D}^\mu u^* - \frac{i}{2} \bar{Z} \cdot \gamma \gamma_S \gamma^\mu \gamma \cdot z \right. \\ &\quad \left. - \frac{2}{3} \left(A_a A^a + \frac{1}{2} |u|^2 \right) A^\mu + \frac{i}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma} \left(A_a A^a + \frac{1}{2} |u|^2 \right) \bar{\psi}_\nu \gamma_\rho \psi_\sigma \right. \\ &\quad \left. - \bar{\psi}_\nu \Gamma^{\nu\mu} A \cdot z - \frac{i}{2} \bar{\psi}_\nu \gamma_S \Gamma^{\nu\mu} (S - i \gamma_S P) \gamma \cdot z \right] , \end{aligned} \quad (5.6)$$

and $K_{(r)}^\mu$, $K_{(i)}^\mu$ have been defined in (4.8), (4.9) while B_b^{new} is given in (3.26).

Now we turn to the Chern-Simons and Gauss-Bonnet multiplets. Following Ref. 6 we define the Lorentz Chern-Simons multiplet Ω by its chiral projection:

$$\Sigma(\Omega^{CS}) = \bar{W}_{ab} W^{ab} . \quad (5.7)$$

The Gauss-Bonnet Chern-Simons multiplet $\Omega_{(i)}$ will be defined by the modification of (5.7) implied by (5.5):

$$\Sigma(\Omega_{(i)}) = \bar{W}_{ab} W^{ab} + \frac{4}{9} \Sigma \left(W_a W^a + \frac{1}{2} R \bar{R} \right) , \quad (5.8)$$

while the Gauss-Bonnet multiplet $\Omega_{(r)}$ will satisfy instead:

$$\Sigma(\Omega_{(r)}) = i \bar{W}_{ab} W^{ab} + \frac{4}{9} i \Sigma \left(W_a W^a + \frac{1}{2} R \bar{R} \right) . \quad (5.9)$$

All these multiplets are defined up to a linear multiplet of course. In order to find Ω , $\Omega_{(i)}$, $\Omega_{(r)}$, one must proceed to solve Eqs. (5.7), (5.8) and (5.9) following the method described in Ref. 6. Suppose we are looking for the solution $\Omega = [C^\Omega, Z^\Omega, H^\Omega, K^\Omega, B_b^\Omega, \Lambda^\Omega, D^\Omega]$ to the problem

$$\Sigma(\Omega) = [z, \chi_L, h] = Y \quad (5.10)$$

where the multiplet $Y = [z, \chi_L, h]$, $\chi_L = \frac{1}{2} (1 + \gamma_5) \chi$, is given and furthermore we want Ω real. From (5.1) and (5.10) one gets

$$H^\Omega = \frac{2}{3} C^\Omega_S - \frac{1}{2} (z + z^*)$$

$$K^\Omega = -\frac{2}{3} C^\Omega_P - \frac{i}{2} (z-z^*)$$

$$\Lambda^\Omega = i\gamma_5 \chi - \left(\hat{D} - \frac{i}{2} \not{K} \gamma_5 \right) Z^\Omega + \frac{1}{3} (S+i\gamma_5 P - i\not{K} \gamma_5) Z^\Omega + \frac{2}{3} iC^\Omega \gamma_5 \gamma \cdot z$$

$$D^\Omega = \frac{1}{2} (h+h^*) - \hat{D}C^\Omega - \frac{2}{3} C^\Omega_B - \frac{2}{3} A^a B_a^\Omega - \frac{2}{9} C^\Omega |u|^2 + \frac{1}{6} (zu+z^* u^*)$$

$$- \frac{i}{6} \bar{Z}^\Omega \gamma_5 \gamma \cdot z . \quad (5.11)$$

Also there is a constraint on B_a^Ω :

$$- \frac{1}{2} (h-h^*) = \hat{D}^a B_a^\Omega - \frac{2}{3} C^\Omega \hat{D}_a A^a - \frac{2}{3} A^a \hat{D}_a C^\Omega + \frac{i}{2} (zu-z^* u^*) + \frac{1}{2} \bar{Z}^\Omega \gamma \cdot z . \quad (5.12)$$

In order to solve this constraint we exploit the fact that if Ω is a solution to (5.10), so is $\Omega-L$ where L is a linear multiplet

$$L = [C^L, Z^L, H^L, K^L, B_b^L, \Lambda^L, D^L] . \quad (5.13)$$

Thus one can choose:

$$C^L = C^\Omega , \quad Z^L = Z^\Omega \quad (5.14)$$

but, since Ω and L are different multiplets, the supercovariant derivatives differ:[†]

[†]This gives explicit ψ_u terms in the last two components of the multiplet which are missing in Eq. (5.25) of Ref. 6.

$$\hat{D}_a C^L - \hat{D}_a C^\Omega = 0$$

$$\hat{D}_a Z^\Omega - \hat{D}_a Z^L = -\frac{i}{2} z \psi_{aR} + \frac{i}{2} z^* \psi_{aL} + \frac{1}{2} (B^\Omega - B^L) \psi_a$$

$$\hat{D}C^\Omega - \hat{D}C^L = \frac{1}{4} z \bar{\psi}_R^a \psi_a + \frac{1}{4} z^* \bar{\psi}_L^a \psi_a . \quad (5.15)$$

Then, if we call $V_a = B_a^\Omega - B_a^L$, the constraint (5.12) becomes:

$$\partial_\mu (e V^\mu) = \text{Im } \mathcal{L}_h(\Sigma(\Omega)) . \quad (5.16)$$

Eq. (5.16) implies Eq. (5.10) has a local solution if and only if the imaginary part of the density for the chiral multiplet in (5.10) is a total divergence which in turn determines V^μ . The multiplets in (5.7) and (5.8) satisfy this condition, so we can readily write particular solutions for Ω^{CS} and $\Omega_{(i)}$ in (5.7), (5.8) from the particular solution to the general problem (5.10):

$$\Omega = \left[0, 0, -\frac{1}{2} (z+z^*), -\frac{1}{2} (z-z^*), V_b, \right.$$

$$i\gamma_5 \chi + \frac{i}{2} z \gamma \cdot \psi_R - \frac{i}{2} z^* \gamma \cdot \psi_L - \frac{1}{2} \gamma^a \not{V} \psi_a ,$$

$$\left. \frac{1}{2} (h+h^*) + \frac{1}{6} (zu+z^*u^*) - \frac{2}{3} A^a V_a - \frac{1}{4} z \bar{\psi}_R^a \psi_a - \frac{1}{4} z^* \bar{\psi}_L^a \psi_a \right] , \quad (5.17)$$

and one only needs to replace z , χ and h by the components of the corresponding multiplets on the right-hand side of Eqs. (5.7) and (5.8), which can be extracted from (5.1) and the results of Section 3, as well as:

$$V^\mu = e^{-1} K_{(i)}^\mu \quad \text{for } \Omega^{\text{CS}}, \quad (5.18)$$

$$V^\mu = V_{(i)}^\mu \quad \text{for } \Omega_{(i)}.$$

In order to obtain the Gauss-Bonnet multiplet $\Omega_{(r)}$ in (5.9), we need the solution to the modified problem:

$$\Sigma(\Omega') = iY. \quad (5.19)$$

Following exactly the same procedure used to solve (5.10), one finds the particular solution to (5.19)

$$\begin{aligned} \Omega' = & \left[0, 0, -\frac{1}{2} (z-z^*), \frac{1}{2} (z+z^*), V_b, \right. \\ & -\chi - \frac{1}{2} \gamma \cdot \psi_R z - \frac{1}{2} \gamma \cdot \psi_L z^* - \frac{1}{2} \gamma^a \psi_a, \\ & \left. \frac{i}{2} (h-h^*) + \frac{i}{6} (zu-z^*u^*) - \frac{2}{3} A^a V_a - \frac{1}{4} z \bar{\psi}_R^a \psi_a + \frac{1}{4} z^* \bar{\psi}_L^a \psi_a \right] \end{aligned} \quad (5.20)$$

if and only if

$$\partial_\mu (eV^\mu) = \text{Re } \mathcal{L}_h(Y). \quad (5.21)$$

From Eqs. (5.21), (5.5) and (5.9), we get:

$$V^\mu = V_{(r)}^\mu \quad \text{for } \Omega_{(r)} \quad (5.22)$$

and the components z, χ, h in (5.20) are the ones of the multiplet

$\bar{W}_{ab} W^{ab} + \frac{4}{9} \Sigma \left(W_a W^a + \frac{1}{2} R\bar{R} \right)$, which are easily derived from (S.1) and the formulae of Section 3.

Finally, let us mention that

$$\Omega^{CS} + \frac{4}{9} \left(W_a W^a + \frac{1}{2} R\bar{R} \right) \quad (5.23)$$

is also a solution to the equation (S.8) for $\Omega_{(i)}$ as it is obvious from (S.7) and (S.8). However, the solution from (S.17) looks simpler than (5.23) while the difference between the two is a linear multiplet.

When we add the multiplets Ω^{CS} , $\Omega_{(i)}$ or $\Omega_{(r)}$ to a linear multiplet L , we obtain generalizations of the field strength for the antisymmetric tensor $b_{\mu\nu}$ in (5.4) giving three different versions of the Green-Schwarz mechanism. The one corresponding to Ω^{CS} has been described in detail in Ref. 6.

6. Conclusion.

We have completed the derivation of the Ricci multiplet in old-minimal supergravity, as well as presented a complete and explicit proof of the super-Gauss-Bonnet theorem. Our main results rely on the full calculation of the fermionic terms of these multiplets which have never been reported before. In particular, we bring the attention of the reader to Eqs. (4.7), (4.8), and (4.9) which give the full expression of the supertopological densities. Also the full component expression of the Chern-Simons multiplet $\Omega_{(i)}$ (Eqs. (5.17), (5.18)) and the Gauss-Bonnet multiplet $\Omega_{(r)}$ (Eqs. (5.20), (5.22)) are new results. These multiplets as well as Ω^{CS} couple to linear multiplets through different supersymmetric generalizations of the Green-Schwarz mechanism.

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Appendix A: Relations between conformal quantities and Poincaré quantities.

The Q-curvature of conformal supergravity is given by^{4,22,23}

$$R_{\mu\nu}^{(Q)} = -2 \left[D_{[\mu} (\omega(e, \psi, b)) + \frac{i}{2} \gamma_5 A_{[\mu} \right] \psi_{\nu]} - 2\gamma_{[\mu} \varphi_{\nu]} - b_{[\mu} \psi_{\nu]} \quad (A.1)$$

where b_μ is the dilatation gauge field which is independent, while the Lorentz connection $\omega_\mu^{ab}(e, \psi, b)$ and the S-supersymmetry gauge field φ_μ are dependent as a result of solving the constraints of conformal supergravity which give:

$$\omega_\mu^{ab}(e, \psi, b) = \omega_\mu^{ab}(e, \psi) + 2e^{[a}{}_\mu b^{b]} \quad (A.2)$$

$$\varphi_\mu = \frac{1}{2} \left(\mathcal{R}_\mu - \frac{1}{3} \gamma_\mu \gamma \cdot \mathcal{R} \right)$$

$$\mathcal{R}^\mu = e^{-1} \epsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \left[\partial_\rho + \frac{1}{4} \omega_\rho^{ab}(e, \psi, b) \Gamma_{ab} + \frac{1}{2} \gamma_5 A_\rho + \frac{1}{2} b_\rho \right] \psi_\sigma .$$

Actually φ_μ can be expressed in terms of z_μ :⁶

$$\varphi_\mu = \frac{1}{3} z_\mu + \frac{1}{2} (\eta + \not{x}) \psi_\mu \quad (A.3)$$

and it is a simple matter to verify that $R_{\mu\nu}^{(Q)}$ is in fact independent of b_μ .

The relation between $R_{\mu\nu}^{(Q)}$ and $\psi_{\mu\nu}^{(cov)}$ has been given in the text:

$$R_{ab}^{(Q)} = -\psi_{ab}^{(cov)} + \frac{2}{3} \gamma_{[a} z_{b]} . \quad (A.4)$$

The covariantized M-curvature of conformal supergravity is:

$$R_{\mu\nu}^{(cov)}(M_{ab}) = R_{\mu\nu}(M_{ab}) - \bar{\psi}_{[\mu} \gamma_{\nu]} R_{ab}^{(Q)} \quad (A.5)$$

where $R_{\mu\nu}(M_{ab})$ is given by gauging the superconformal algebra

$$R_{\mu\nu}(M^{ab}) = R_{\mu\nu}^{ab}[\omega(e, \psi, b)] - 8e_{[\mu}^{[a} f_{\nu]}^{b]} - \bar{\psi}_{[\mu} \Gamma^{ab} \phi_{\nu]} , \quad (A.6)$$

$R_{\mu\nu}^{ab}[\omega(e, \psi, b)]$ being the Riemann tensor constructed with the connection in (A.2). The conformal boosts gauge field f_{μ}^b is also a dependent field and can be expressed in terms of Poincaré quantities⁴

$$\begin{aligned} f_{a\mu} = & -\frac{1}{6} e_{\mu}^c (B_{ca} - i\tilde{F}_{ca}) - \frac{1}{36} (|u|^2 + A_c A^c) e_{a\mu} + \frac{1}{18} A_{\mu} A_a \\ & + \frac{1}{12} \bar{\psi}_{\mu} z_a - \frac{1}{2} D_{\mu} b_a + \frac{1}{2} b_{\mu} b_a - \frac{1}{4} b_c b^c e_{a\mu} \end{aligned} \quad (A.7)$$

where the connection in $D_{\mu} b_a$ is $\omega(e, \psi)$. When one considers the traceless combination to construct the Weyl tensor, the f_{μ}^a terms vanish identically. Furthermore, by replacing (A.7) into (A.6) one sees that the curvature $R_{\mu\nu}(M^{ab})$ is actually independent of $b_{\mu}^{\dagger}$ and since, as already mentioned, $R_{\mu\nu}(Q)$ is also independent of b_{μ} , then $R_{\mu\nu}^{(cov)}(M_{ab})$ does not depend on b_{μ} either. Hence

$$\begin{aligned} R_{ab}(M^{cd}) = & \hat{R}_{ab}^{cd} + \frac{4}{3} \delta_{[a}^{[c} \left[B_{b]}^{d]} - i\tilde{F}_{b]}^{d]} + \frac{1}{6} \delta_{b]}^{d]} (|u|^2 + A_e A^e) \right. \\ & \left. - \frac{1}{3} A_{b]} A^{d]} \right] \end{aligned} \quad (A.8)$$

where

[†]To our knowledge this has not been pointed out in the literature.

$$\hat{R}_{ab}{}^{cd} = R_{ab}{}^{(M^{cd})} \Big|_{f^a_{\mu}=b_{\mu}=0} = R_{ab}{}^{cd} - \bar{\psi}_{[a} \Gamma^{cd} \phi_{b]} \quad (A.9)$$

and

$$\phi_{\mu} = \varphi_{\mu} \quad (b_{\nu} = 0) \quad . \quad (A.10)$$

Combining (A.9), (A.10) and (A.3) we get for $\hat{R}_{ab}{}^{cd}$ the expression given in the text. For the covariantized curvature, we get simply:

$$\begin{aligned} R_{ab}^{(cov)}(M^{cd}) &= \hat{R}_{ab}^{(cov)cd} + \frac{4}{3} \delta^{[c}_{[a} \left[B_{b]}^{d]} - i \tilde{F}_{b]}^{d]} \right. \\ &\quad \left. + \frac{1}{6} \delta^{d]}_{b]} (|u|^2 + A_e A^e) - \frac{1}{3} A_{b]} A^{d]} \right] . \end{aligned} \quad (A.11)$$

From the cancellation of f^a_{μ} and b_{μ} terms we see that the Weyl tensor is in fact the traceless part of $\hat{R}_{abcd}^{(cov)}$.¹⁶

An interesting property of $\hat{R}_{\mu\nu}{}^{ab}$ is that the antisymmetric part of its contraction $\hat{R}_{\mu\rho} = e_a{}^{\nu} e_{b\rho} \hat{R}_{\mu\nu}{}^{ab}$ is

$$\hat{R}_{[\mu\nu]} = -\bar{\psi}_{[\mu} \phi_{\nu]} - \frac{1}{4} \bar{\psi} \cdot \gamma R_{\mu\nu}(Q) \quad (A.12)$$

from which it is easily derived the identity given in the text:

$$B_{[ab]} = \frac{1}{2} \tilde{F}_{ab} \quad . \quad (A.13)$$

The complete expression for $\hat{R}_{\mu\nu}$ is:

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} \bar{\psi}_{\mu} \Gamma_{\nu c} \phi^c + \frac{1}{2} \bar{\psi}^c \Gamma_{c\nu} \phi_{\mu} \quad (A.14)$$

which has the trace:

$$\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu} = R + \bar{\psi}_a \Gamma^{ab} \phi_b . \quad (A.15)$$

Two more conformal curvatures which do not depend on the b_μ field are the A-curvature

$$R_{\mu\nu}(A) = \frac{2}{3} (2\partial_{[\mu} A_{\nu]} - 3i\bar{\psi}_{[\mu} \gamma_5 \psi_{\nu]}) \quad (A.16)$$

and the D-curvature which is related to it:

$$R_{\mu\nu}(D) = -\frac{1}{2} \tilde{R}_{\mu\nu}(A) . \quad (A.17)$$

As mentioned in the text, there is the relation:⁴

$$R_{ab}(A) = \frac{2}{3} \hat{F}_{ab} . \quad (A.18)$$

The covariantized S-curvature is:

$$R_{\mu\nu}^{(cov)}(S) = R_{\mu\nu}(S) + \frac{1}{2} (\gamma_5 R_{\lambda[\mu}(A) + \tilde{R}_{\lambda[\mu}(A)) \gamma^{\lambda} \psi_{\nu]} \quad (A.19)$$

while $R_{\mu\nu}(S)$ is simply:

$$R_{\mu\nu}(S) = 2D_{[\nu}(\omega(e, \psi, b)) \phi_{\mu]} - 2\gamma_a f^a_{[\mu} \psi_{\nu]} + b_{[\mu} \phi_{\nu]} + i\gamma_5 A_{[\mu} \phi_{\nu]} . \quad (A.20)$$

The dependence of these curvatures on the dilatation field can be displayed by writing $R_{\mu\nu}^{(cov)}(S)$ in terms of Poincaré quantities:

$$R_{ab}^{(cov)}(S) = -\frac{2}{3} \left(\hat{D}_{[a} - \frac{i}{2} \gamma_5 A_{[a} + \frac{1}{2} \eta \gamma_{[a} \right) z_{b]} + \frac{1}{2} (\eta + \mathcal{B}) R_{ab}(Q) . \quad (A.21)$$

Then we can see that even though $R_{ab}^{(cov)}(S)$ is not independent of b_μ , the Weyl multiplet is, because the constraint on $R_{\mu\nu}(Q)$ (see Appendix B) implies:

$$T_{abcd} \gamma^\mu R^{cd}(Q) = 0 \quad (A.22)$$

so that $T_{ab}^{cd} R_{cd}^{(cov)}(S)$ is independent of b_μ . With all these elements it is a trivial matter to see that the action of conformal supergravity is independent of the dilatation gauge field b_μ .²²

Appendix B: Bianchi identities.

The basic property of the Q-curvature is the constraint²³

$$\gamma^\nu R_{\mu\nu}(Q) = 0 \quad (B.1)$$

from which we can derive several identities more:

$$\Gamma^{ab} R_{ab}(Q) = 0, \quad R_{ab}(Q) + \gamma_5 \tilde{R}_{ab}(Q) = 0,$$

$$T_{abcd} R^{cd}(Q) = R_{ab}(Q), \quad T_{abcd} \gamma^\mu R^{cd}(Q) = 0,$$

$$\gamma_{[a} R_{bc]}(Q) = 0, \quad \Gamma^{ab} \gamma_c R_{ab}(Q) = 0,$$

$$\Gamma^{ab} \gamma_\mu \gamma_\nu R_{ab}(Q) = -8 R_{\mu\nu}(Q). \quad (B.2)$$

These identities provide additional ones when we consider quadratic objects in $R_{ab}(Q)$:

$$\bar{\psi}^c \gamma^a \psi_d \bar{R}^{bd}(Q) \gamma_a R_{bc}(Q) = 0,$$

$$\bar{\psi}^c \psi_d \bar{R}^{bd}(Q) R_{bc}(Q) = \frac{1}{4} \bar{\psi}^c \psi_c \bar{R}^{bd}(Q) R_{bd}(Q),$$

$$\bar{\psi}^c \gamma_5 \psi_d \bar{R}^{bd}(Q) \gamma_5 R_{bc}(Q) = \frac{1}{4} \bar{\psi}^c \gamma_5 \psi_c \bar{R}^{bd}(Q) \gamma_5 R_{bd}(Q),$$

$$\bar{\psi}_a \Gamma^{ac} \psi^d \bar{R}_{bd}(Q) R_c{}^b(Q) = -\frac{1}{4} \bar{\psi}_a \Gamma^{ac} \psi_c \bar{R}_{bd}(Q) R^{bd}(Q).$$

For the covariantized M-curvature, the basic constraint is:

$$R_{ac}^{(cov)}(M_b^c) = \frac{i}{2} \tilde{R}_{ab}(A) \quad (B.4)$$

which trivially implies

$$R_{ab}^{(cov)}(M^{ab}) = 0 . \quad (B.5)$$

These are complemented by two more:

$$R_{ab}^{(cov)}(M^{cd}) - R^{(cov)cd}(M_{ab}) = -2i\delta_{[a}^{[c} \tilde{R}_{b]}^{d]}(A) \quad (B.6)$$

$$\epsilon^{ebcd} R_{ab}^{(cov)}(M_{cd}) = iR_a^e(A)$$

and another one implied by the last one

$$\epsilon^{abcd} R_{ab}^{(cov)}(M_{cd}) = 0 . \quad (B.7)$$

The basic Bianchi identity for $R_{ab}^{(cov)}(S)$ is:

$$\Gamma^{ab} \Gamma_{cd} R_{ab}^{(cov)}(S) = 0 \quad (B.8)$$

from which we can obtain a set of additional ones:

$$\Gamma^{ab} R_{ab}^{(cov)}(S) = 0 , \quad R_{ab}^{(cov)}(S) - \gamma_5 \tilde{R}_{ab}^{(cov)}(S) = 0 ,$$

$$\Gamma_{[a}^c R_{b]c}^{(cov)}(S) = 0 , \quad \Gamma_{[ab} R_{cd]}^{(cov)}(S) = 0 ,$$

$$\Gamma_a^{bc} R_{bc}^{(cov)}(S) + 2\gamma^{bc} R_{ab}^{(cov)}(S) = 0 . \quad (B.9)$$

The identities for $R_{ab}^{(Q)}$ together with the relation (A.4) imply corresponding identities for $\psi_{ab}^{(cov)}$

$$\gamma^b \psi_{ab}^{(cov)} = -z_a - \frac{1}{3} \Gamma_{ab} z^b$$

$$\Gamma^{ab} \psi_{ab}^{(cov)} = -2\gamma \cdot z$$

$$\psi_{ab}^{(cov)} + \gamma_S \tilde{\psi}_{ab}^{(cov)} = \frac{2}{3} \left(\gamma_{[a} z_{b]} + \frac{1}{2} \epsilon_{abcd} \gamma_S \gamma^c z^d \right)$$

$$\gamma_{[a} \psi_{bc]}^{(cov)} = \frac{2}{3} \Gamma_{[ab} z_{c]}$$

$$T_{abcd} \psi^{(cov)cd} = \psi_{ab}^{(cov)} - \frac{2}{3} \gamma_{[a} z_{b]}$$

$$T_{abcd} \gamma_e \psi^{(cov)cd} = \frac{4}{3} T_{abed} z^d$$

$$\Gamma^{ab} \gamma_c \psi_{ab}^{(cov)} = 2 \left(z_c - \frac{1}{3} \Gamma_{cb} z^b \right)$$

$$\Gamma^{ab} \Gamma_{cd} \psi_{ab}^{(cov)} + 8 \psi_{cd}^{(cov)} = \frac{2}{3} (8 \gamma_{[c} z_{d]} + \Gamma_{cd} \gamma \cdot z) . \quad (B.10)$$

In addition, $\psi_{ab}^{(cov)}$ obeys the useful differential Bianchi identity:

$$\left(\hat{D}_{[c} + \frac{1}{2} \gamma_S A_{[c} - \frac{1}{2} \gamma_{[c} \eta \right) \psi_{ab]}^{(cov)} = 0 . \quad (B.11)$$

By using the relation (A.11) between $R_{ab}^{(cov)}(M_{cd})$ and $\hat{R}_{abcd}^{(cov)}$ as well as (B.4) through (B.7), we can derive a set of Bianchi identities for $\hat{R}_{abcd}^{(cov)}$:

$$\hat{R}_{abcd}^{(cov)} - \hat{R}_{cdab}^{(cov)} = 0 , \quad \epsilon^{ebcd} \hat{R}_{abcd}^{(cov)} = 0$$

$$\begin{aligned}\hat{R}_{ab}^{(cov)} &= \hat{R}_{ac}^{(cov)c}{}_b = \frac{1}{3} (2B_{ab} - i\tilde{F}_{ab}) + \frac{1}{3} (B + |u|^2) \eta_{ab} \\ &\quad - \frac{2}{9} (A_a A_b - A_c A^c \eta_{ab})\end{aligned}\quad (B.12)$$

and from the latter we deduce as well

$$\begin{aligned}\hat{R}_{[ab]}^{(cov)} &= 0 \\ \hat{R}^{(cov)} &= \hat{R}_a^{(cov)a} = 2B + \frac{4}{3} |u|^2 + \frac{2}{3} A_c A^c.\end{aligned}\quad (B.13)$$

Aside from these we have the differential identity mentioned in the text:

$$\begin{aligned}\hat{D}_a B - \hat{D}^b B_{ab} &= -\frac{1}{2} \hat{D}_a \left(|u|^2 + \frac{1}{3} A_c A^c \right) - \frac{1}{3} \hat{D}_b (A_a A^b) \\ &\quad + \bar{z}^c \psi_{ac}^{(cov)} + \frac{1}{6} \bar{z}_a \gamma \cdot z.\end{aligned}\quad (B.14)$$

Lastly, from the identities for $R_{ab}^{(cov)}(S)$ and the relation (A.21) we can derive identities for $\hat{D}_a z_b$ like:

$$\begin{aligned}\Gamma^{ab} \Gamma_{cd} \left(\hat{D}_a - \frac{1}{2} \gamma_S A_a \right) z_b &= -\frac{3}{4} \Gamma^{ab} \Gamma_{cd} \eta \psi_{ab}^{(cov)} \\ \Gamma^{ab} \left(\hat{D}_a - \frac{1}{2} \gamma_S A_a + \frac{1}{2} \eta \gamma_a \right) z_b &= 0 \\ \epsilon_{abcd} \left(\hat{D}^c - \frac{1}{2} \gamma_S A^c \right) z^d &= 2\gamma_S \left(\hat{D}_{[a} - \frac{1}{2} \gamma_S A_{[a} \right) z_{b]} - (S - i\gamma_S P) \gamma_S \psi_{ab}^{(cov)} \\ &\quad - \eta \left(\gamma_S \gamma_{[a} z_{b]} + \frac{1}{2} \epsilon_{abcd} \gamma^c z^d \right).\end{aligned}\quad (B.15)$$

Appendix C: Additional transformation rules.

For reference, we provide the transformation rules of $\hat{D}_\mu z_a$, $\hat{D}_b A_a$ and $\hat{D}_a \hat{D}_b A_c$ from which one can derive some of the transformations used in Section 3.

$$\begin{aligned}
 \delta \hat{D}_\mu z_a &= \frac{1}{2} \left[\hat{D}_\mu H_a i\gamma_5 - \hat{D}_\mu K_a - \gamma^b \hat{D}_\mu B_{ab} + i\gamma^b \gamma_5 \hat{D}_\mu \hat{D}_b A_a \right. \\
 &\quad - \frac{1}{3} \Gamma_{ab} \gamma_5 \hat{D}_\mu \{ (S - i\gamma_5 P) A^b \} \left. \right] \epsilon - \frac{1}{4} \left[H_a i\gamma_5 - K_a - B_{ab} \gamma^b + i\gamma^b \gamma_5 \hat{D}_b A_a \right. \\
 &\quad - \frac{1}{3} \Gamma_{ab} \gamma_5 (S - i\gamma_5 P) A^b \left. \right] (iA_\mu \gamma_5 - \gamma_\mu \eta) \epsilon + \frac{1}{2} \left\{ -\frac{5}{24} \bar{z}_a z_\mu - \frac{1}{8} \bar{z}_a \gamma_\mu \gamma \cdot z \right. \\
 &\quad + \frac{1}{12} \bar{z}_\mu \gamma_a \gamma \cdot z + \left[-\frac{5}{24} \bar{z}_a \gamma_5 z_\mu - \frac{1}{8} \bar{z}_a \gamma_5 \gamma_\mu \gamma \cdot z + \frac{1}{12} \bar{z}_\mu \gamma_5 \gamma_a \gamma \cdot z \right] \gamma_5 \\
 &\quad + \frac{1}{12} \left[\bar{z}^b \Gamma_{d\mu a} z_b + \frac{1}{2} \bar{z}_a \gamma_\mu \Gamma_{dc} z^c - \bar{z}_\mu \gamma_a \Gamma_{dc} z^c - \frac{5}{2} \bar{z}_a \gamma_\mu z_d \right. \\
 &\quad + \frac{7}{2} \bar{z}_a \gamma_d z_\mu - \bar{z}_d \gamma_a z_\mu \left. \right] \gamma^d + \left[\frac{1}{8} \bar{z}_a \gamma_5 \gamma_\mu \Gamma_{dc} z^c - \frac{1}{12} \bar{z}_\mu \gamma_5 \gamma_a \Gamma_{dc} z^c \right. \\
 &\quad + \frac{1}{24} \bar{z}_a \gamma_5 \gamma_\mu z_d - \frac{1}{8} \bar{z}_a \gamma_5 \gamma_d z_\mu + \frac{1}{12} \bar{z}_d \gamma_5 \gamma_a z_\mu - \frac{1}{6} \bar{z}^b \gamma_5 \eta_{d[c} \gamma_a] z_b e^c{}_\mu \left. \right] \gamma^d \gamma_5 \\
 &\quad + \frac{1}{24} \left[\bar{z}_a \Gamma^{cd} z_\mu + \frac{1}{2} \bar{z}_a \Gamma_{\mu b} \Gamma^{cd} z^b - \bar{z}_\mu \Gamma_{ab} \Gamma^{cd} z^b \right. \\
 &\quad - \left. \bar{z}^b \Gamma^{cd} \Gamma_{\mu a} z_b \right] \Gamma_{cd} \left. \right\} \epsilon + \frac{1}{8} \left\{ \bar{z}^b \gamma_\mu \psi_{ab}^{(\text{cov})} + \bar{z}^b \gamma_5 \gamma_\mu \psi_{ab}^{(\text{cov})} \gamma_5 \right. \\
 &\quad + (2\bar{z}_a \psi_{\mu d}^{(\text{cov})} - \bar{z}^b \gamma_d \gamma_\mu \psi_{ab}^{(\text{cov})}) \gamma^d - (2\bar{z}_a \gamma_5 \psi_{\mu d}^{(\text{cov})} + \bar{z}^b \gamma_d \gamma_5 \gamma_\mu \psi_{ab}^{(\text{cov})}) \gamma^d \gamma_5 \\
 &\quad + \left. \frac{1}{2} \bar{z}^b \Gamma^{cd} \gamma_\mu \psi_{ab}^{(\text{cov})} \Gamma_{cd} \right\} \epsilon + \frac{1}{2} \bar{\epsilon} \gamma^d \psi_\mu \hat{D}_d z_a . \tag{C.1}
 \end{aligned}$$

One can replace μ by a Latin index everywhere provided one drops the last term. The remaining two are:

$$\begin{aligned} \delta \hat{D}_b^A A_a = & \frac{1}{2} \bar{\epsilon} \gamma_5 \left(\hat{D}_b - \frac{1}{2} A_b \gamma_5 \right) z_a - \frac{1}{4} \bar{\epsilon} \eta \gamma_b \gamma_5 z_a \\ & + \frac{1}{2} \bar{\epsilon} (\Gamma_{[ac} z_{b]} - \gamma_b \psi_{ac}^{(cov)}) A^c, \end{aligned} \quad (C.2)$$

and

$$\begin{aligned} \delta \hat{D}_a^A \hat{D}_b^A A_c = & \frac{1}{2} \bar{\epsilon} \left(\hat{D}_a - \frac{1}{2} \gamma_5 A_a - \frac{1}{2} \eta \gamma_a \right) \left[\gamma_5 \left(\hat{D}_b - \frac{1}{2} A_b \gamma_5 \right) - \frac{1}{2} \eta \gamma_b \gamma_5 \right] z_c \\ & + \frac{1}{2} \bar{\epsilon} \left(\hat{D}_a - \frac{1}{2} \gamma_5 A_a - \frac{1}{2} \eta \gamma_a \right) \left[(\Gamma_{[bc} z_{d]} - \gamma_b \psi_{cd}^{(cov)}) A^d \right] \\ & + \frac{1}{2} \bar{\epsilon} (\Gamma_{[ad} z_{e]} - \gamma_a \psi_{de}^{(cov)}) (\delta_b^d \hat{D}^e A_c + \delta_c^d \hat{D}_b^e A^e). \end{aligned} \quad (C.3)$$