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An Improved Lattice Fermion Action from Block-Diagonalisation *

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Abstract

We derive an improved lattice fermion action using approximate block-diagonalisation. Following Real Space Renormalisation Group ideas, an effective action for the infra-red modes is obtained by integrating out the heavy modes. We expand this effective action perturbatively in the contribution of the heavy modes. With a systematic improvement of this expansion, we hope to extract continuum properties of QCD from coarse lattices.

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1. Introduction

Some years ago, two of us proposed a method (called approximate block-diagonalisation [1]) aimed at constructing an improved lattice fermion action, which would allow an efficient computation of the infra-red behaviour of QCD. The method consists of approximately integrating out the high frequency modes within a block (e.g., of size 2^4 or 4^4) on a “fine” lattice, leaving behind an effective action on a blocked lattice. The process preserves the long distance behaviour, but generates more complicated lattice couplings (e.g., next to nearest neighbour ones). The approximation can be systematically improved, the expansion parameter being the inverse of the lattice mass of the heavy modes.

The method was first developed for blocks of size 2^4 (“scale 2 blocking”) and applied to the computation of hadron masses [2]. Quenched configurations on a $24^3 \times 48$ lattice were blocked twice with the scale 2 geometry, and quark propagators were computed on the resulting $6^3 \times 12$ blocked configurations with the effective fermion action. Even though such a blocked lattice is pretty small, the results showed promising features :

- The signal to noise ratio in the hadron propagators is significantly improved compared to computations with the standard Wilson action at similar couplings on similar size lattices.
- Smaller quark masses ($m_\pi/m_\rho \sim 0.4$) can be reached at comparatively smaller correlation lengths. This reduces finite size effects and allows us to work closer to the chiral limit.

In the present paper, we derive the effective fermion action for another block geometry — scale $\sqrt{3}$ blocking. This blocking scheme was first introduced by Cordery, Gupta and Novotny [3] for pure gauge theories. It has been applied to a renormalisation group analysis of $SU(2)$ and $SU(3)$ lattice gauge theories [4]. Its extension to gauge-Higgs system was carried out in ref.[5]. Here we extend it further to gauge-fermion systems.

The construction and the geometry of the $\sqrt{3}$ blocked lattice is reviewed in section 2, where we introduce our notation. The reformulation of the Wilson action on the fine lattice in terms of an effective fermion action on the $\sqrt{3}$ blocked lattice is presented in sections 3 and 4. The effective fermion action so generated contains couplings of all ranges. However, as will be demonstrated in section 4, long range couplings are suppressed by powers of “heavy mode masses”. The simplest truncation scheme is elaborated in section 5. It leads to two types of couplings, nearest neighbour and three-space diagonal. We show how the repeated application of this truncated blocking drives the ratios of couplings to a fixed point. In section 6, we present our conclusions and perspectives.

2. The geometry of the $\sqrt{3}$ blocked lattice

We start from a “fine” hypercubic lattice (FHL) with spacing a [†] and lattice sites $x^{(0)} = (x_1, x_2, x_3, x_4)$. Every blocking step consists of decomposing the lattice into non-overlapping blocks of 9 sites : the block site and its 8 neighbouring sites (in 4 dimensions) on the fine lattice. We denote the sites of the n^{th} level blocked lattice by $x^{(n)}$. For the direction vectors on these lattices, we adopt the notation $e_j^{(n)}$, where the index n represents the blocking level and the direction index j takes on values in $\{-8, \dots, 0, \dots, 8\}$. We choose

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[†] For abbreviation, we set $a = 1$ in what follows.

the basis :

$$\begin{aligned}
e_1^{(0)} &= (1, 0, 0, 0) , \quad e_2^{(0)} = (0, 1, 0, 0) , \quad e_3^{(0)} = (0, 0, 1, 0) , \quad e_4^{(0)} = (0, 0, 0, 1) ; \\
e_1^{(1)} &= (1, 1, 1, 0) , \quad e_2^{(1)} = (-1, 1, 0, 1) , \quad e_3^{(1)} = (1, 0, -1, 1) , \quad e_4^{(1)} = (0, -1, 1, 1) ; \\
e_0^{(n)} &= 0 , \quad e_{j+4}^{(n)} = e_j^{(n+1)} , \quad \text{for } j \in \{1, \dots, 4\} ; \\
e_{-j}^{(n)} &= -e_j^{(n)} , \quad e_j^{(n+2)} = 3e_j^{(n)} , \quad \text{for } j \in \{-8, \dots, 0, \dots, 8\} ; \\
(e_j^{(n)}, e_k^{(n)}) &= 3^n \delta_{jk} , \quad \text{for } j, k \in \{1, \dots, 4\} .
\end{aligned} \tag{2.1}$$

All direction vectors are thus expressed in the FHL coordinates. With each site $x^{(n+1)}$ on the $(n+1)^{th}$ level lattice, we associate a block $B(x^{(n+1)})$ of nine sites on the n^{th} level lattice, consisting of the site itself and its eight nearest neighbours :

$$x^{(n)} = x^{(n+1)} + e_j^{(n)} , \quad \text{with } j \in \{-4, \dots, 0, \dots, 4\} . \tag{2.2}$$

The key step in block diagonalisation is to express the fermion matrix on the n^{th} level in terms of the coordinates on the $(n+1)^{th}$ level, using a fermion field that has an extra internal degree of freedom corresponding to the 9 sites within a block. To this effect, we first express nearest neighbour Kronecker δ 's on FHL in terms of those on the once blocked lattice (OBL) :

$$\delta_{x^{(0)}, x^{(0)} + e_i^{(0)}} = \sum_{j=-8}^8 T_{i;j}^l \delta_{x^{(1)}, x^{(1)} + e_j^{(1)}} , \tag{2.3}$$

where

$$x^{(0)} = x^{(1)} + e_{j'}^{(0)} , \quad x^{(0)} = x^{(1)} + e_j^{(0)} .$$

The T_i^l are 9×9 matrices in the space of the "internal coordinates" $j', j \in \{-4, \dots, 0, \dots, 4\}$ of the block $B(x^{(1)})$. Their matrix elements are :

$$T_{i;j'}^l = \begin{cases} 1 , & \text{if } e_i^{(1)} = e_{j'}^{(0)} + e_l^{(0)} - e_j^{(0)} ; \\ 0 , & \text{otherwise .} \end{cases} \tag{2.4}$$

The OBL is hypercubic but has its axes rotated with respect to the FHL. The next $\sqrt{3}$ block transformation constructs the sites $x^{(2)}$ on the twice blocked lattice (TBL) from the sites $x^{(1)}$ on OBL. We choose it to be different from the one defined in eq.(2.1), such that the axes of TBL are realigned with those of FHL but with a lattice spacing 3 times larger:

$$\begin{aligned}
e_1^{(2)} &= e_1^{(1)} - e_2^{(1)} + e_3^{(1)} = 3e_1^{(0)} , \quad e_2^{(2)} = e_1^{(1)} + e_2^{(1)} - e_4^{(1)} = 3e_2^{(0)} , \\
e_3^{(2)} &= e_1^{(1)} - e_3^{(1)} + e_4^{(1)} = 3e_3^{(0)} , \quad e_4^{(2)} = e_2^{(1)} + e_3^{(1)} + e_4^{(1)} = 3e_4^{(0)} .
\end{aligned} \tag{2.5}$$

Again the Kronecker δ 's on OBL can be expressed in terms of those on TBL :

$$\delta_{x^{(1)}, x^{(1)} + e_i^{(1)}} = \sum_{j=-8}^8 \tilde{T}_{i;j}^l \delta_{x^{(2)}, x^{(2)} + e_j^{(2)}} , \tag{2.6}$$

where

$$\tilde{T}_{i,j'j}^l = \begin{cases} 1, & \text{if } e_i^{(2)} = e_j^{(1)} + e_l^{(1)} - e_{j'}^{(1)}; \\ 0, & \text{otherwise.} \end{cases} \quad (2.7)$$

Since the TBL is hypercubic and aligned with the FHL, this 2-step blocking procedure can now be repeated. The two distinct matrices, T_i^l and $\tilde{T}_i^l$ completely encode the geometry of the transformation.

3. The Wilson action as seen from the $\sqrt{3}$ blocked lattice

In this section, we rewrite the Wilson action on FHL in terms of the new fermion and gluon fields on the $\sqrt{3}$ blocked lattice :

$$\sum_{x^{(0)}, x^{(1)}} \bar{\psi}(x^{(0)}) \Delta(x^{(0)}, x^{(0)}) \psi(x^{(0)}) = \sum_{x^{(1)}, x^{(1)}} \sum_{j', j} \bar{\chi}_{j'}(x^{(1)}) \Delta(x^{(1)j'}, x^{(1)j}) \chi_j(x^{(1)}) . \quad (3.1)$$

Given the fermion matrix $\Delta(x^{(0)}, x^{(0)})$ on FHL, our task is to express it as $\Delta(x^{(1)j'}, x^{(1)j})$ in terms of the block coordinates and internal indices $j', j \in \{-4, \dots, 0, \dots, 4\}$. The nine species of fermion fields $\chi_j(x^{(1)})$ are the original fermion fields $\psi(x^{(0)})$, $x^{(0)} \in B(x^{(1)})$, parallel transported to the block site $x^{(1)}$ using the gauge connection[†] $U(x^{(1)}, x^{(0)})$:

$$\chi_j(x^{(1)}) = U(x^{(1)}, x^{(0)}) \psi(x^{(0)}), \quad \text{with } x^{(0)} = x^{(1)} + e_j^{(0)}, \quad j \in \{-4, \dots, 0, \dots, 4\} . \quad (3.2)$$

The new gauge fields on the $\sqrt{3}$ blocked lattice describing the interactions of χ_j are :

$$U(x^{(1)j'}, x^{(1)j}) = U(x^{(1)}, x^{(0)}) U(x^{(0)}, x^{(0)}) U(x^{(0)}, x^{(1)}) , \quad (3.3)$$

where $U(x^{(0)}, x^{(0)})$ denotes the link variable between two neighbouring sites on FHL, with

$$x^{(0)} = x^{(0)} + e_l^{(0)} = x^{(1)} + e_{j'}^{(0)} ,$$

$$x^{(1)} = x^{(1)} + e_i^{(1)}, \quad e_i^{(1)} = e_j^{(0)} + e_l^{(0)} - e_{j'}^{(0)}, \quad i \in \{-8, \dots, 0, \dots, 8\} . \quad (3.4)$$

We can now reformulate the Wilson fermion matrix,

$$\begin{aligned} \Delta(x^{(0)}, x^{(0)}) &= 2m \delta_{x^{(0)}, x^{(0)}} - \sum_{l=1}^4 c_l^{(0)}(x^{(0)}, x^{(0)}) + \sum_{l=1}^4 \gamma_l s_l^{(0)}(x^{(0)}, x^{(0)}) , \\ c_i^{(n)}(x^{(n)}, x^{(n)}) &= U(x^{(n)}, x^{(n)}) (\delta_{x^{(n)}, x^{(n)} + e_i^{(n)}} + \delta_{x^{(n)}, x^{(n)} - e_i^{(n)}} - 2\delta_{x^{(n)}, x^{(n)}}) , \\ s_i^{(n)}(x^{(n)}, x^{(n)}) &= U(x^{(n)}, x^{(n)}) (\delta_{x^{(n)}, x^{(n)} + e_i^{(n)}} - \delta_{x^{(n)}, x^{(n)} - e_i^{(n)}}) , \end{aligned} \quad (3.5)$$

in terms of the new gauge fields, eq.(3.3), on the $\sqrt{3}$ blocked lattice :

$$\Delta(x^{(1)j'}, x^{(1)j}) = (M + \Gamma)_{j'j} \delta_{x^{(1)j'}, x^{(1)j}} + \tilde{\Delta}(x^{(1)j'}, x^{(1)j}) , \quad (3.6)$$

[†] The reader may note that there exists a special version of axial gauge — the “ $\sqrt{3}$ block gauge” — in which all these parallel transporters are fixed to unity.

$$\begin{aligned}\tilde{\Delta}(x'^{(1)}j', x^{(1)}j) &= \sum_{i=1}^8 (-T_i^+ + \Gamma_i^+)_{j'j} C_{ij'j}(x'^{(1)}, x^{(1)}) \\ &+ \sum_{i=1}^8 (-T_i^- + \Gamma_i^-)_{j'j} S_{ij'j}(x'^{(1)}, x^{(1)}) ,\end{aligned}\quad (3.7)$$

where

$$M = (2m+8)\mathbf{1} - 2\sum_{i=1}^8 T_i^+ - T_0^+ , \quad \Gamma = 2\sum_{i=1}^8 \Gamma_i^+ + \Gamma_0^+ , \quad (3.8)$$

$$T_i^\pm = \frac{1}{2} \sum_{l=1}^4 (T_i^l + T_i^{-l} \pm T_{-i}^l \pm T_{-i}^{-l}) , \quad (3.9)$$

$$\Gamma_i^\pm = \frac{1}{2} \sum_{l=1}^4 \gamma_l (T_i^l - T_i^{-l} \pm T_{-i}^l \mp T_{-i}^{-l}) , \quad (3.10)$$

$$\begin{aligned}C_{ij'j}(x'^{(1)}, x^{(1)}) &= U(x'^{(1)}j', x^{(1)}j)(\delta_{x'^{(1)}, x^{(1)}+e_i^{(1)}} + \delta_{x'^{(1)}, x^{(1)}-e_i^{(1)}} - 2\delta_{x'^{(1)}, x^{(1)}}) , \\ S_{ij'j}(x'^{(1)}, x^{(1)}) &= U(x'^{(1)}j', x^{(1)}j)(\delta_{x'^{(1)}, x^{(1)}+e_i^{(1)}} - \delta_{x'^{(1)}, x^{(1)}-e_i^{(1)}}) .\end{aligned}\quad (3.11)$$

The matrices $T_i^\pm$, $\Gamma_i^\pm$ can be read off tables 1,2. They have the following symmetry properties :

$$T_{ij'j}^\pm = \pm T_{ijj'}^\pm = \pm T_{i-j'-j}^\pm , \quad (3.12)$$

$$\Gamma_{ij'j}^\pm = \mp \Gamma_{ijj'}^\pm = \mp \Gamma_{i-j'-j}^\pm , \quad (3.13)$$

$$\Gamma_{ij'j}^\pm = \gamma_l T_{ijj'}^\pm , \quad \text{if } e_i^{(1)} = e_i^{(0)} + e_l^{(0)} - e_{j'}^{(0)} . \quad (3.14)$$

Eqs.(3.6-7) show that the matrix $(M + \Gamma)/2$ is the mass matrix for the 9 fermion modes. M is a 9×9 matrix in the space of the internal coordinates and a unit matrix in the spin space. Γ is a 36×36 matrix affecting both the internal coordinates and the Dirac indices. However, as one can see from table 2, Γ^2 is a unit matrix in the spin space. Thus in the space of internal coordinates, we have

$$M = (2m+9)\mathbf{1} - E , \quad \Gamma^2 = -9\mathbf{1} + E , \quad E_{j'j} = 1 \quad \forall j', j \in \{-4, \dots, 0, \dots, 4\} . \quad (3.15)$$

The eigenvalues and eigenvectors of M and Γ^2 are easily found. There is a “zero” eigenmode:

$$M\phi_0 = 2m\phi_0 , \quad \Gamma^2\phi_0 = 0 , \quad \phi_0 = \frac{1}{3}(1, 1, 1, 1, 1, 1, 1, 1, 1) . \quad (3.16)$$

and 8 “heavy” eigenmodes with degenerate eigenvalues :

$$M\phi_\mu = (2m+9)\phi_\mu , \quad \Gamma^2\phi_\mu = -9\phi_\mu . \quad (3.17)$$

Therefore, the matrix $(M + \Gamma)$ appearing in eq.(3.6) has three different eigenvalues :

$$2m, \quad 2m + 9 + 3i, \quad 2m + 9 - 3i. \quad (3.18)$$

The projection operators P_0, P_+, P_- onto the corresponding eigenstates are given in appendix A, eqs.(A.1,15-18).

Having identified the light and heavy modes within the block, we now choose the light mode as the blocked fermion field :

$$\psi(x^{(1)}) = \frac{1}{3} \sum_{j=-4}^4 \chi_j(x^{(1)}) \quad (3.19)$$

Next we construct the effective action for this blocked fermion field by integrating out the effects of the heavy modes.

4. The effective fermion action on the $\sqrt{3}$ blocked lattice

The fermion matrix for the 9 fermion modes, eq.(3.6-7), is of the form :

$$\Delta = \begin{pmatrix} \Delta_{00} & \Delta_{0h} \\ \Delta_{h0} & \Delta_{hh} \end{pmatrix} \quad (4.1)$$

Here, Δ_{00} is the projection of Δ on the “zero mode” :

$$\Delta_{00} = P_0 \Delta P_0, \quad (4.2)$$

and

$$\Delta_{hh} = (P_+ + P_-) \Delta (P_+ + P_-) \quad (4.2)$$

is its projection on the “heavy mode” block with “mass eigenvalues” $2m + 9 \pm 3i$. The off-diagonal blocks describe the interaction between the zero mode and the heavy ones :

$$\Delta_{0h} = P_0 \tilde{\Delta} (P_+ + P_-), \quad \Delta_{h0} = (P_+ + P_-) \tilde{\Delta} P_0. \quad (4.5)$$

It can be inferred from eq.(3.7) that the projection of $\tilde{\Delta}$ on P_0 , which controls the infra-red contribution of $\tilde{\Delta}$, vanishes in the formal continuum limit $a \rightarrow 0$. So to extract the infra-red behaviour of the theory, we look for an expansion of the fermion matrix in which these off-diagonal blocks are treated perturbatively.

The heavy mode contribution to the fermion action can be isolated by means of the identity :

$$\det \begin{pmatrix} \Delta_{00} & \Delta_{0h} \\ \Delta_{h0} & \Delta_{hh} \end{pmatrix} = \det(\Delta_{eff}) \det(\Delta_{hh}), \quad (4.6)$$

where the effective fermion matrix for the light mode is

$$\Delta_{eff} = \Delta_{00} - \Delta_{0h} \Delta_{hh}^{-1} \Delta_{h0}. \quad (4.7)$$

With this separation, $\det(\Delta_{hh})$ can be dropped, since it does not affect the expectation values of blocked variables. The presence of Δ_{hh}^{-1} in the remaining matrix Δ_{eff} generates all sorts of couplings on the blocked lattice as can be seen from the expansion :

$$\Delta_{hh}^{-1} = ((M + \Gamma)_{hh} + \tilde{\Delta}_{hh})^{-1} = (M + \Gamma)_{hh}^{-1} \sum_{n=0}^{\infty} (-1)^n (\tilde{\Delta}_{hh} (M + \Gamma)_{hh}^{-1})^n . \quad (4.8)$$

Higher order terms in this expansion contain longer range interactions. However, the heavy mass denominators suppress their coefficients exponentially with the distance. A truncated expansion therefore can be systematically improved to desired accuracy by including higher order terms.

In the limit $m \rightarrow 0$, we have the result :

$$\Delta_{eff} = P_0 \Delta P_0 - \frac{1}{10} P_0 \tilde{\Delta} (P_+ + P_-) \tilde{\Delta} P_0 + \frac{i}{30} P_0 \tilde{\Delta} (P_+ - P_-) \tilde{\Delta} P_0 + \dots . \quad (4.9)$$

We analyse the leading term in the next section, while the first order corrections are investigated in appendix B.

5. The zeroth order truncation of the effective fermion action and its fixed point limit

The first term (“zeroth order truncation”) in the expansion (4.9) of the effective fermion action, $\bar{\psi}(x^{(1)}) Tr[\Delta_{eff}(x^{(1)}, x^{(1)})] \psi(x^{(1)})$ is :

$$Tr[P_0 \Delta(x^{(0)}, x^{(0)}) P_0] = 2m \delta_{x^{(1)}, x^{(1)}} - \sum_{i=1}^8 r_i c_i^{(1)}(x^{(1)}, x^{(1)}) + \sum_{i=1}^8 \Gamma_i s_i^{(1)}(x^{(1)}, x^{(1)}) , \quad (5.1)$$

where using tables 1 and 2,

$$r_i = \frac{2}{3} , \Gamma_i = \frac{2}{9} \gamma_i^{(1)} , \text{ if } i \in \{1, \dots, 4\} , \quad (5.2)$$

$$r_i = \frac{1}{9} , \Gamma_i = \frac{1}{27} \gamma_i^{(1)} , \text{ if } i \in \{5, \dots, 8\} . \quad (5.3)$$

The $\gamma_i^{(n)}$ here are the Dirac matrices with respect to the axes of the blocked lattices :

$$\begin{aligned} \gamma_i^{(n)} &= (\gamma, e_i^{(n)}) , \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) , \quad \text{for } i \in \{1, \dots, 8\} ; \\ \{\gamma_i^{(n)}, \gamma_j^{(n)}\} &= 3^n \cdot 2\delta_{ij} , \quad \text{for } i, j \in \{1, \dots, 4\} . \end{aligned} \quad (5.4)$$

The zeroth order effective action, eq.(5.1), contains two types of couplings on the OBL : nearest neighbours ($i=1, \dots, 4$) and three-space diagonals[†] ($i=5, \dots, 8$). The effective gauge

[†] These diagonals connect the nearest neighbours on the TBL.

fields appearing in the combinations $c_i^{(1)}, s_i^{(1)}$ (cf. eq.(3.5) with $x'^{(1)} = x^{(1)} \pm e_i^{(1)}$) are defined as follows :

- For $i \in \{1, \dots, 4\}$, $U(x'^{(1)}, x^{(1)})$ is the path ordered product of the link matrices, averaged over the six topologically equivalent 3-link paths connecting the sites $x'^{(1)}$ and $x^{(1)}$. This is precisely the definition of the block link on the OBL used in pure gauge MCRG studies.
- For $i \in \{5, \dots, 8\}$, $U(x'^{(1)}, x^{(1)})$ is the path ordered product of the link matrices along the straight line path connecting the sites $x'^{(1)}$ and $x^{(1)}$. This is the seventh independent path which was ignored in the construction of the block link (cf. ref.[5]).

The zeroth order truncation described above has the simple property that both T_i^l and $\tilde{T}_i^l$ give rise to the same projections. We can therefore easily iterate the blocking procedure. Thus, in this lowest order of approximation, the effective fermion matrix after n blocking steps becomes :

$$\Delta^{(n)} = 2m \cdot 1 + A_1^{(n)} \sum_{i=1}^4 c_i^{(n)} + A_3^{(n)} \sum_{i=5}^8 c_i^{(n)} + A_2^{(n)} \sum_{i=1}^4 \gamma_i^{(n)} s_i^{(n)} + A_4^{(n)} \sum_{i=5}^8 \gamma_i^{(n)} s_i^{(n)} , \quad (5.5)$$

where the couplings satisfy the recurrence relations :

$$A_1^{(n+1)} = \frac{2}{3} A_1^{(n)} + A_3^{(n)} , \quad A_3^{(n+1)} = \frac{1}{9} A_1^{(n)} , \quad (5.6)$$

$$A_2^{(n+1)} = \frac{2}{9} A_2^{(n)} + A_4^{(n)} , \quad A_4^{(n+1)} = \frac{1}{27} A_2^{(n)} . \quad (5.7)$$

It is seen that the Wilson type and the Dirac type couplings evolve independently. The system of equations is closed, and the recurrence relations for the ratios of couplings :

$$\alpha^{(n)} = A_1^{(n)} / A_3^{(n)} , \quad \beta^{(n)} = A_2^{(n)} / A_4^{(n)} , \quad (5.8)$$

have only one attractive fixed point solution :

$$\alpha^{(\infty)} = 3(1 + \sqrt{2}) , \quad \beta^{(\infty)} = 9 . \quad (5.9)$$

As a final restriction, we fix the relative strength of the Wilson type to Dirac type couplings, by imposing the constraint that the fixed point fermion action be the same as the standard Wilson action in the naive continuum limit[§] :

$$A_1^{(\infty)} + 3A_3^{(\infty)} = A_2^{(\infty)} + 3A_4^{(\infty)} \implies A_1^{(\infty)} / A_2^{(\infty)} = 2\sqrt{2}/3 . \quad (5.10)$$

The overall multiplicative constant corresponding to wave-function renormalisation remains undetermined at this stage. This constant is irrelevant to the calculation of physical

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[§] This choice is motivated by the fact that the spinor space projection operator structure of the Wilson action eliminates from the transfer matrix unphysical modes appearing near the ultra-violet cutoff [6].

observables, and the choice $A_2^{(\infty)} = 3/4$ matches the standard convention. This completes the specification of the zeroth order truncated effective Wilson fermion action.

6. Discussion

We have shown in this paper how to construct an effective fermion action on a $\sqrt{3}$ blocked lattice. Without any truncation, the effective fermion action contains many types of couplings. In zeroth order truncation, only nearest neighbour and three-space diagonal couplings arise, and repeated blocking of the lattice drives the ratios of the couplings to a fixed point (cf. eq.(5.9)). Our fixed point action does not have the full symmetry of the hypercubic lattice, since we use only 8 three-space diagonal neighbours of a site out of the possible 32. This loss of symmetry is due to the fact that within the $\sqrt{3}$ geometry, one can construct four equivalent block transformations, which differ from each other by rotations. Such situations are known to occur in Real Space Renormalisation Group applications to condensed matter systems. If so desired, the full lattice symmetry can easily be recovered by averaging the block action over the different block transformations.

The dimensionless expansion parameter that governs the corrections to the zeroth order truncation scheme is the ratio of the inverse lattice spacing to heavy mode masses. The magnitude of this expansion parameter reflects the efficiency of the renormalisation group transformation – the smaller its value the faster the convergence. We have an optimal situation here that all the heavy mode masses are degenerate in absolute value. This is a consequence of the symmetry of the $\sqrt{3}$ blocking scheme, which results in the appearance of the matrix E in the formulae.

The first order corrections to the effective fermion action are evaluated in appendix B, with the Wilson action as the starting point. The new correction terms are quite small due to the small expansion parameter. However, the set of recurrence relations is not closed once these new couplings are included and the procedure cannot be iterated.

There are two ways of looking at our results. One is to try to relate the calculation on the blocked lattice to a direct calculation on the fine lattice. This was the line of thought followed in ref.[1]. The improvement in the results is then seen as the hadron operators on the blocked lattice having a much better overlap with the physical hadron wave-functions, than the standard local operators on the fine lattice — by about two orders of magnitude. But also the truncated renormalisation prescription means that the infra-red properties of the fine and blocked lattices are not identical, and an estimate of the error is desirable.

We emphasize that there is an alternative view-point. The fixed point action we have arrived at can be interpreted as a renormalisation group improved lattice discretisation of the continuum action. In this language, we seek the relation of our results to the continuum ones and not to those on the fine lattice. Any improvement in the results is then a consequence of having a better action. Of course, one should question whether this action involving more complicated couplings is really better (i.e., approaches the continuum limit faster) than the standard Wilson action. We have used this “fixed point” fermion action in refs.[7,8] (within the quenched approximation) to compute the hadron spectrum, the F and D parameters governing the mass splittings within the baryon octet and the nucleon σ -term. The results of these computations nicely demonstrate that the fixed point fermion action is superior to the standard Wilson action : The hadron correlators on

the blocked lattice have a much better signal to noise ratio and smaller values of m_π/m_ρ can be reached.

Appendix A : The eigenvectors of the matrix Γ

According to its definition, eq.(3.8), the matrix Γ is anti-Hermitian and operates in a 36 dimensional spin $\otimes$ internal space. As was already shown in section 3, Γ has three distinct eigenvalues :

a) A zero mode

$$\Gamma\phi_0 = 0 \quad , \quad \phi_0 = \frac{1}{3}(\chi, \chi, \chi, \chi, \chi, \chi, \chi, \chi, \chi) \quad , \quad P_0 = \frac{1}{9}E \quad , \quad (A.1)$$

with χ being an arbitrary Dirac spinor normalised to one. The zero mode thus has a 4-fold degeneracy in the 36 dimensional space.

b) Two heavy modes

$$\Gamma\phi_\pm = \pm 3i\phi_\pm \quad . \quad (A.2)$$

Solving this eigenvalue equation using table 2, and the ansatz

$$\phi_\pm = (\chi_\pm(-4), \chi_\pm(-3), \chi_\pm(-2), \chi_\pm(-1), \chi_\pm(0), \chi_\pm(1), \chi_\pm(2), \chi_\pm(3), \chi_\pm(4)) \quad , \quad (A.3)$$

we get

$$\chi_\pm(j) = \frac{1}{2} \sum_{k=1}^4 (\pm \frac{i}{3} \theta_{jk} + \delta_{jk}) \eta_k \quad , \quad j = 1, \dots, 4 \quad (A.4)$$

$$\chi_\pm(-j) = \frac{1}{2} \sum_{k=1}^4 (\pm \frac{i}{3} \theta_{jk} - \delta_{jk}) \eta_k \quad , \quad j = 1, \dots, 4 \quad (A.5)$$

$$\chi_\pm(0) = \pm \frac{i}{3} \sum_{k=1}^4 \gamma_k \eta_k \quad . \quad (A.6)$$

The matrix θ has the form :

$$\theta = \begin{pmatrix} -\gamma_1 & -\gamma_3 - \gamma_4 & -\gamma_2 + \gamma_4 & -\gamma_2 + \gamma_3 \\ -\gamma_3 + \gamma_4 & -\gamma_2 & -\gamma_1 - \gamma_4 & \gamma_1 - \gamma_3 \\ -\gamma_2 - \gamma_4 & -\gamma_1 + \gamma_4 & -\gamma_3 & -\gamma_1 + \gamma_2 \\ \gamma_2 + \gamma_3 & \gamma_1 + \gamma_3 & \gamma_1 + \gamma_2 & -\gamma_4 \end{pmatrix} \quad . \quad (A.7)$$

The four Dirac spinors η_j , $j = 1, \dots, 4$ are constrained only by a normalisation condition :

$$\sum_{j=1}^4 (\eta_j, \eta_j) = 1 \quad . \quad (A.8)$$

Therefore, each of the two heavy modes with eigenvalues $\pm 3i$ has a 16-fold degeneracy. The eigenvectors for the same eigenvalue are orthogonal, if the corresponding sets of spinors are orthogonal :

$$\sum_{j=1}^4 (\eta_j, \eta'_j) = 0 . \quad (A.9)$$

The projection operators $P_{\pm}$ on the eigenspace of the heavy modes can now be deduced :

$$(P_{\pm})_{jk} = \frac{1}{4} \left(\frac{1}{9} \theta \theta^{\dagger} + 1 \pm \frac{i}{3} (\theta - \theta^{\dagger}) \right)_{jk} , \quad j, k = 1, \dots, 4 , \quad (A.10)$$

$$(P_{\pm})_{j-k} = \frac{1}{4} \left(\frac{1}{9} \theta \theta^{\dagger} - 1 \mp \frac{i}{3} (\theta + \theta^{\dagger}) \right)_{jk} , \quad j, k = 1, \dots, 4 , \quad (A.11)$$

$$(P_{\pm})_{-jk} = \frac{1}{4} \left(\frac{1}{9} \theta \theta^{\dagger} - 1 \pm \frac{i}{3} (\theta + \theta^{\dagger}) \right)_{jk} , \quad j, k = 1, \dots, 4 , \quad (A.12)$$

$$(P_{\pm})_{-j-k} = \frac{1}{4} \left(\frac{1}{9} \theta \theta^{\dagger} + 1 \mp \frac{i}{3} (\theta - \theta^{\dagger}) \right)_{jk} , \quad j, k = 1, \dots, 4 , \quad (A.13)$$

Eq.(A.7) for θ yields

$$\theta \theta^{\dagger} = 9 \cdot 1 - 2E , \quad (\theta^{\dagger} \theta)_{jk} = 9 \cdot 1 - 2\gamma_j \gamma_k . \quad (A.14)$$

The computation of higher order terms in the effective fermion action, eq.(4.9), involves the matrix elements of $(P_{+} \pm P_{-})$. We find, in particular

$$P_{+} + P_{-} = 1 - \frac{1}{9} E = 1 - P_0 , \quad (A.15)$$

$$(P_{+} - P_{-})_{jk} = -(P_{+} - P_{-})_{-j-k} = \frac{i}{6} (\theta - \theta^{\dagger})_{jk} , \quad j, k = 1, \dots, 4 , \quad (A.16)$$

$$(P_{+} - P_{-})_{j-k} = -(P_{+} - P_{-})_{-jk} = -\frac{i}{6} (\theta + \theta^{\dagger})_{jk} , \quad j, k = 1, \dots, 4 , \quad (A.17)$$

$$(P_{+} - P_{-})_{0j} = -(P_{+} - P_{-})_{j0} = -(P_{+} - P_{-})_{0-j} = (P_{+} - P_{-})_{-j0} = \frac{i}{3} \gamma_j , \quad j = 1, \dots, 4 . \quad (A.18)$$

Appendix B : The effective fermion action at first order

We compute here the second and third terms in the effective action, eq.(4.9), to $\mathcal{O}(a^2)$. For this purpose, it is sufficient to take into account the contributions of the S -term in eq.(3.7), and substitute the j -dependent gauge fields by the effective gauge fields described in section 5. In this approximation, the first order corrections to the effective action (4.9) reduce to :

$$\begin{aligned} & Tr \left[-\frac{1}{10} P_0 \tilde{\Delta} (P_{+} + P_{-}) \tilde{\Delta} P_0 + \frac{i}{30} P_0 \tilde{\Delta} (P_{+} - P_{-}) \tilde{\Delta} P_0 \right] \\ &= \sum_{i,k=1}^8 a_{ik} s_i^{(1)}(x'^{(1)}, z^{(1)}) s_k^{(1)}(z^{(1)}, x^{(1)}) . \end{aligned} \quad (B.1)$$

The field independent coefficients a_{ik} are calculated from the matrices T_i^- , Γ_i^- listed in tables 1,2 and from the projectors (A.15-18) by means of the formula :

$$a_{ik} = \text{Tr} \left[-\frac{1}{10} P_0 (-T_i^- + \Gamma_i^-) (P_+ + P_-) (-T_k^- + \Gamma_k^-) P_0 \right. \\ \left. + \frac{i}{30} P_0 (-T_i^- + \Gamma_i^-) (P_+ - P_-) (-T_k^- + \Gamma_k^-) P_0 \right] \quad (B.2)$$

For the actual computation, it is convenient to use the symmetry properties (3.12-13) and to introduce the combinations ($j = 1, \dots, 4$) :

$$T_{i,j} = \sum_{l=1}^4 (T_{i,lj}^- - T_{i,l-j}^-) = (e_i^{(1)}, e_j^{(0)}) \times \begin{cases} 1, & \text{for } i = 1, \dots, 4; \\ \frac{1}{6}, & \text{for } i = 5, \dots, 8; \end{cases} \quad (B.3)$$

$$\Gamma_{i,j} = \sum_{l=1}^4 (\Gamma_{i,lj}^- + \Gamma_{i,l-j}^-) \\ = \begin{cases} \frac{1}{2} \gamma_i^{(1)} (1 - \delta_{5-i,j}) - \frac{1}{2} \gamma_j^{(0)} (e_i^{(1)}, e_j^{(0)}), & \text{for } i = 1, \dots, 4; \\ \frac{1}{2} \gamma_j^{(0)} \delta_{i-4,j}, & \text{for } i = 5, \dots, 8. \end{cases} \quad (B.4)$$

These matrices are listed in tables 3,4. Eq.(B.2) can be rewritten in terms of them :

$$a_{ik} = \frac{1}{45} \sum_{j=1}^4 (T_{i,j} T_{k,j} - \Gamma_{i,j} \Gamma_{k,j}) + \frac{1}{405} \sum_{l,j=1}^4 (2\Gamma_{i,l} \Gamma_{k,j} - T_{i,l} \theta_{lj}^\dagger \Gamma_{k,j} - \Gamma_{i,l} \theta_{lj} T_{k,j}) \quad (B.5)$$

With the abbreviations

$$d_{ik} = (e_i^{(1)}, e_k^{(1)}), \quad \sigma_{ik} = \frac{1}{2} [\gamma_i^{(n)}, \gamma_k^{(n)}], \quad \text{for } i, k = 1, \dots, 8, \quad (B.6)$$

the evaluation of the r.h.s. leads to the following results :

i	k	$405 \cdot a_{ik}$
≤ 4	≤ 4	$\frac{17}{2} d_{ik} + \sigma_{ik} - \frac{1}{12} [\sigma_{i,9-i} d_{k,9-i} - \sigma_{k,9-k} d_{i,9-k}]$
≥ 4	≥ 4	$\frac{1}{9} d_{ik} + \frac{1}{9} \sigma_{ik} - \frac{1}{36} [\sigma_{i,9-i} d_{k,9-i} - \sigma_{k,9-k} d_{i,9-k}]$
≤ 4	≥ 4	$\frac{13}{6} d_{ik} - \sigma_{ik} (\frac{1}{3} + \frac{1}{4} d_{i,9-k}) + \frac{1}{108} \sum_{j=5}^8 \sigma_{j,9-j} d_{ij} d_{k,9-j}$
≥ 4	≤ 4	$\frac{13}{6} d_{ik} - \sigma_{ik} (\frac{1}{3} + \frac{1}{4} d_{9-i,k}) + \frac{1}{108} \sum_{j=1}^4 \sigma_{j,9-j} d_{ij} d_{k,9-j}$

(B.7)

Comparing this with the zeroth order truncation of the fermion action (cf. eq.(5.1)), we see that the first order corrections have introduced interactions between sites $x^{(n)}$ and

$$x^{(n)} \pm e_i^{(n)} \pm e_k^{(n)}, \quad i, k = 1, \dots, 8. \quad (B.8)$$

At $\mathcal{O}(a^2)$, these new couplings consist of contact terms giving rise to fermion mass renormalisation, as well as the operators $\bar{\psi}\partial^2\psi$ and $\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$.

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	-4	-3	-2	-1	0	1	2	3	4
-4	0	$q_3^\pm$	$q_4^\pm$	$q_2^\pm$	q_0	$q_3^\pm$	$q_2^\pm$	$q_4^\pm$	$q_8^\pm$
-3	$\pm q_3^\pm$	0	$q_4^\pm$	$\pm q_3^\pm$	q_0	$q_1^\pm$	$q_1^\pm$	$q_7^\pm$	$q_4^\pm$
-2	$\pm q_4^\pm$	$\pm q_4^\pm$	0	$q_2^\pm$	q_0	$q_1^\pm$	$q_6^\pm$	$q_1^\pm$	$q_2^\pm$
-1	$\pm q_2^\pm$	$q_3^\pm$	$\pm q_2^\pm$	0	q_0	$q_5^\pm$	$q_1^\pm$	$q_1^\pm$	$q_3^\pm$
0	q_0	q_0	q_0	q_0	0	q_0	q_0	q_0	q_0
1	$\pm q_3^\pm$	$\pm q_1^\pm$	$\pm q_1^\pm$	$\pm q_5^\pm$	q_0	0	$q_2^\pm$	$\pm q_3^\pm$	$q_2^\pm$
2	$\pm q_2^\pm$	$\pm q_1^\pm$	$\pm q_6^\pm$	$\pm q_1^\pm$	q_0	$\pm q_2^\pm$	0	$q_4^\pm$	$q_4^\pm$
3	$\pm q_4^\pm$	$\pm q_7^\pm$	$\pm q_1^\pm$	$\pm q_1^\pm$	q_0	$q_3^\pm$	$\pm q_4^\pm$	0	$q_3^\pm$
4	$\pm q_8^\pm$	$\pm q_4^\pm$	$\pm q_2^\pm$	$\pm q_3^\pm$	q_0	$\pm q_2^\pm$	$\pm q_4^\pm$	$\pm q_3^\pm$	0

Table 1 : The matrix $2 \sum_{i=1}^8 q_i^\pm T_i^\pm + q_0 T_0^+$.

	-4	-3	-2	-1	0	1	2	3	4
-4	0	$q_3^\pm(+\gamma_1)$	$q_4^\pm(+\gamma_3)$	$q_2^\pm(+\gamma_2)$	$q_0(-\gamma_4)$	$q_3^\pm(-\gamma_3)$	$q_2^\pm(-\gamma_1)$	$q_4^\pm(-\gamma_2)$	$q_8^\pm(+\gamma_4)$
-3	$\pm q_3^\pm(-\gamma_1)$	0	$q_4^\pm(+\gamma_4)$	$\pm q_3^\pm(-\gamma_4)$	$q_0(-\gamma_3)$	$q_1^\pm(+\gamma_2)$	$q_1^\pm(+\gamma_1)$	$q_7^\pm(+\gamma_3)$	$q_4^\pm(-\gamma_2)$
-2	$\pm q_4^\pm(-\gamma_3)$	$\pm q_4^\pm(-\gamma_4)$	0	$q_2^\pm(+\gamma_4)$	$q_0(-\gamma_2)$	$q_1^\pm(+\gamma_3)$	$q_6^\pm(+\gamma_2)$	$q_1^\pm(+\gamma_1)$	$q_2^\pm(-\gamma_1)$
-1	$\pm q_2^\pm(-\gamma_2)$	$q_3^\pm(+\gamma_4)$	$\pm q_2^\pm(-\gamma_4)$	0	$q_0(-\gamma_1)$	$q_5^\pm(+\gamma_1)$	$q_1^\pm(+\gamma_3)$	$q_1^\pm(+\gamma_2)$	$q_3^\pm(-\gamma_3)$
0	$q_0(+\gamma_4)$	$q_0(+\gamma_3)$	$q_0(+\gamma_2)$	$q_0(+\gamma_1)$	0	$q_0(-\gamma_1)$	$q_0(-\gamma_2)$	$q_0(-\gamma_3)$	$q_0(-\gamma_4)$
1	$\pm q_3^\pm(+\gamma_3)$	$\pm q_1^\pm(-\gamma_2)$	$\pm q_1^\pm(-\gamma_3)$	$\pm q_5^\pm(-\gamma_1)$	$q_0(+\gamma_1)$	0	$q_2^\pm(+\gamma_4)$	$\pm q_3^\pm(-\gamma_4)$	$q_2^\pm(+\gamma_2)$
2	$\pm q_2^\pm(+\gamma_1)$	$\pm q_1^\pm(-\gamma_1)$	$\pm q_6^\pm(-\gamma_2)$	$\pm q_1^\pm(-\gamma_3)$	$q_0(+\gamma_2)$	$\pm q_2^\pm(-\gamma_4)$	0	$q_4^\pm(+\gamma_4)$	$q_4^\pm(+\gamma_3)$
3	$\pm q_4^\pm(+\gamma_2)$	$\pm q_7^\pm(-\gamma_3)$	$\pm q_1^\pm(-\gamma_1)$	$\pm q_1^\pm(-\gamma_2)$	$q_0(+\gamma_3)$	$q_3^\pm(+\gamma_4)$	$\pm q_4^\pm(-\gamma_4)$	0	$q_3^\pm(+\gamma_1)$
4	$\pm q_8^\pm(-\gamma_4)$	$\pm q_4^\pm(+\gamma_2)$	$\pm q_2^\pm(+\gamma_1)$	$\pm q_3^\pm(+\gamma_3)$	$q_0(+\gamma_4)$	$\pm q_2^\pm(-\gamma_2)$	$\pm q_4^\pm(-\gamma_3)$	$\pm q_3^\pm(-\gamma_1)$	0

Table 2 : The matrix $2 \sum_{i=1}^8 q_i^\pm \Gamma_i^\pm + q_0 \Gamma_0^+$.

$\begin{smallmatrix} i \\ j \end{smallmatrix}$	1	2	3	4	5	6	7	8
1	2	-2	2	0	1	0	0	0
2	2	2	0	-2	0	1	0	0
3	2	0	-2	2	0	0	1	0
4	0	2	2	2	0	0	0	1

Table 3 : The matrix $2T_{i;j}$ (cf. eq.(B.3)).

$\begin{smallmatrix} i \\ j \end{smallmatrix}$	1	2	3	4	5	6	7	8
1	$+\gamma_2 + \gamma_3$	$+\gamma_2 + \gamma_4$	$-\gamma_3 + \gamma_4$	0	γ_1	0	0	0
2	$+\gamma_1 + \gamma_3$	$-\gamma_1 + \gamma_4$	0	$+\gamma_3 + \gamma_4$	0	γ_2	0	0
3	$+\gamma_1 + \gamma_2$	0	$+\gamma_1 + \gamma_4$	$-\gamma_2 + \gamma_4$	0	0	γ_3	0
4	0	$-\gamma_1 + \gamma_2$	$+\gamma_1 - \gamma_3$	$-\gamma_2 + \gamma_3$	0	0	0	γ_4

Table 4 : The matrix $2\Gamma_{i;j}$ (cf. eq.(B.4)).