EIGENVALUE EQUATIONS FOR KRICHEVER-NOVIKOV ALGEBRAS

D.B. Fairlie and P. Fletcher^{*)}

Department of Mathematical Sciences
University of Durham, Durham DH1 3LE, England

and

J. Nuyts⁺⁾

CERN - Geneva

A B S T R A C T

We construct non-trivial examples of Krichever-Novikov algebras. Our construction involves parameters which satisfy eigenvalue conditions. For the linear case and an r -term algebra the eigenvalues are $0, 1, \dots, r-3$.

^{*)} Supported by a Durham University Research Studentship.

⁺⁾ Permanent address: Service de Physique Théorique, Université de l'Etat à Mons, 19 Avenue Maistriau, 7000 Mons, Belgium.

1. Introduction

Recently, Krichever and Novikov^[1,2] have proposed as an algebraic extension of conformal invariance on an arbitrary genus Riemann surface the study of algebras of the type

$$[N_m, N_n] = \sum_{k=0}^{r-1} C^k(m, n) N_{m+n-r+1+2k} \quad (1.1)$$

where the genus of the surface is $(r-1)/3$, for $r = 1 \bmod 3$. These will be referred to as Krichever-Novikov (KN) algebras. The work of Krichever and Novikov has been taken up by Bonora *et al.*^[3] and Nepomechie *et al.*^[4] The simplest and best-known case of a KN algebra is the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + (m^3 - m)c\delta_{m+n,0} \quad (1.2)$$

which describes conformal invariance on a sphere (genus 0). This algebra is said to be $\mathbb{Z}$ -graded, and the KN algebras (1.1) generalised-graded. In this article we take a purely algebraic point of view and raise the question of the classification of the possible structure constants, and their functional dependence on m, n permitted by the Jacobi identities. This is a problem similar to the classification of finite Lie algebras, and we must take care that any solutions are not simply isomorphic to the Virasoro algebra.

The case $r = 1$ of a KN algebra (without central extension) is in general

$$[L_m, L_n] = C(m, n)L_{m+n} \quad (1.3)$$

and one solution for the structure constant is

$$C(m, n) = (m-n) \frac{\alpha(m)\alpha(n)}{\alpha(m+n)} \quad (1.4)$$

where the arbitrary function $\alpha(m)$ can be renormalised away by dividing L_m by $\alpha(m)$. We believe that this is the only solution, within the assumption of an ansatz for $C(m, n)$ as the ratio of multinomials in m, n – but we know no general proof of (1.4). However it turns out that there are many different possible sets of structure constants $C^k(m, n)$ for a KN algebra with $r > 1$.

Let us call such an algebra a three, four, etc term algebra according to the number of terms on the right hand side of (1.1). The determination of allowable structure constants which satisfy the Jacobi identities is an extremely difficult non-linear problem: the number of such identities which arise in the r -term case being $2r - 1$.

Fortunately there is a class of allowable algebras for which the problem may be transmuted to a simpler, though still difficult form, *i.e.* those for which the operators N_m may be re-expressed as a finite sum of contiguous even or odd Virasoro generators L_m :

$$N_m = \sum_{k=0}^{r-1} a^k(m) L_{m-r+1+2k} \quad (1.5)$$

The conditions for N_m to generate a KN algebra are still $2r - 1$ equations, but the r unknown functions depend on one variable $a^k(m)$ rather than two $C^k(m, n)$. One may conjecture that all KN algebras may be expressed as (1.5): evidence for this viewpoint will be given at the end of the article. If the relationship (1.5) can be inverted to express the L 's in terms of a sum (possibly semi-infinite) of the N 's then the algebra is equivalent to the full Virasoro algebra, otherwise, and we shall exhibit examples, it is an infinite sub-algebra.

These algebras may be categorised according to the degree of m in the $a^k(m)$. The constraints for a closed algebra become more complicated as this increases. We shall use a convenient representation of the Virasoro generators:

$$L_m = z^{1-m} \frac{d}{dz} \quad (1.6)$$

Within this representation (1.5) becomes a power series in z :

$$N_m = \sum_{k=0}^{r-1} a^k(m) z^{r-1-2k} z^{1-m} \frac{d}{dz} \quad (1.7)$$

2. Constant Coefficients

The KN algebras for which the $a^k(m)$ are independent of m are straightforward, and correspond to the well-known Virasoro case – the structure constants $C^k(m, n)$ are just $(m - n)$ up to multiplicative constants. Indeed it is clear from the following theorem that all KN algebras of this form may be expressed as a sum of L 's.

Theorem 1. *If the coefficients $a^k(m)$ in (1.5) do not depend on m then the N_m form a KN algebra satisfying (1.1) with $C^k(m, n) = (m - n)a^k$.*

Proof: If the $a^k(m)$ are constants, (1.7) becomes

$$\begin{aligned} N_m &= \sum_{k=0}^{r-1} a^k z^{r-1-2k} z^{1-m} \frac{d}{dz} \\ &= g(z) z^{1-m} \frac{d}{dz} \end{aligned} \quad (2.1)$$

defining $g(z)$ as a power series. When forming the commutator $[N_m, N_n]$ terms symmetric in m, n cancel, leaving only the part where z^{1-n} is differentiated.

$$\begin{aligned} [N_m, N_n] &= (m - n) (g(z))^2 z^{1-m-n} \frac{d}{dz} \\ &= (m - n) \sum_{k=0}^{r-1} a^k g(z) z^{1-(m+n-r+1+2k)} \frac{d}{dz} \\ &= (m - n) \sum_{k=0}^{r-1} a^k N_{m+n-r+1+2k} \end{aligned} \quad (2.2)$$

Comparing with (1.1) gives the result.

Alternatively, if $p(z)$ is an elliptic or hyperelliptic function of z satisfying $(p'(z))^2 = g(p(z))$ where $g(p)$ is a polynomial of finite degree in p , then

$$N_m = p'(z) (p(z))^{1-m-r+1} \frac{d}{dz} \quad (2.3)$$

$$= g(p) p^{1-m-r+1} \frac{d}{dp} \quad (2.4)$$

Commuting (2.3) gives a sum of N 's. If $g(p)$ is a polynomial in p^2 , the N_m form a KN algebra, with constant coefficients. Putting $L_m = p^{1-m} \frac{d}{dp}$ makes (2.4) equivalent to (1.5).

3. Linear Coefficients

If the $a^k(m)$ depend on m the problem is much harder. When forming the commutator, $g(z)$ is a function of m so there are more antisymmetric terms and the above proof for the constant case no longer works. We need to find the constraints on the $a^k(m)$ to get a KN algebra. The Jacobi identities are satisfied automatically as we are making the assumption that N_m is a sum of L 's. The condition for a KN algebra is that the N 's satisfy (1.1) for some structure constants $C^k(m, n)$. The way to check this is to express N_m and N_n as sums of L 's, commute them, and ensure that the resulting sum of L 's may be expressed as a sum of N 's.

$$\begin{array}{ccc} [N, N] & \stackrel{?}{=} & \Sigma N \\ \downarrow & & \uparrow \\ [\Sigma L, \Sigma L] & \rightarrow & \Sigma L \end{array}$$

We shall start by considering the following ansatz for KN algebras linear in m :

$$N_m = ((m + f)z + (m - f)\frac{1}{z}) \left(z + \frac{1}{z}\right)^{r-2} z^{1-m} \frac{d}{dz} \quad (3.1)$$

and we shall find the possible values of the parameter f for a closed algebra. Note that this is invariant up to sign under $z \mapsto -\frac{1}{z}$, $m \mapsto -m$. We shall refer to this as the parity automorphism. It is the remnant of the automorphisms of the Virasoro algebra generated by $L_m \mapsto \pm \frac{1}{\lambda} L_{\lambda m}$. The original KN algebra has a different normalisation for the N 's. In their papers^[1] the normalisation is chosen so that the end structure constants are given by:

$$\begin{aligned} C^{r-1}(m, n) &= (m - n) \\ C^0(m, n) &= (m - n) \frac{\alpha(m)\alpha(n)}{\alpha(m + n - r + 1)} \end{aligned} \quad (3.2)$$

The conditions of parity invariance are then

$$N_{-m} = \mp \alpha(-m) N_m \quad \text{and} \quad \alpha(m)\alpha(-m) = -1 \quad (3.3)$$

These equations imply that the general structure of $\alpha(m)$ is given by

$$\alpha(m) = A^m \prod_i \frac{m - \beta_i}{m + \beta_i} \quad (3.4)$$

where the β_i and A are constants. Our ansatz corresponds to the case where there is only one factor in the product. The merit of this ansatz is that the $2r - 1$ conditions

for closure reduce to $r + 1$ linear equations for r unknowns, giving a single consistency requirement. The commutator may be calculated

$$[N_m, N_n] = (m - n) \left\{ ((m + f)(n + f)z + (m - f)(n - f)\frac{1}{z}) \left(z + \frac{1}{z}\right) - 4f(f - 1) \right\} \left(z + \frac{1}{z}\right)^{2r-4} z^{1-m-n} \frac{d}{dz} \quad (3.5)$$

or, as a power series in z

$$[N_m, N_n] = (m - n) \sum_{k=0}^r \begin{pmatrix} \binom{r-1}{k}(m + f)(n + f) \\ + \binom{r-1}{k-1}(m - f)(n - f) \\ - \binom{r-2}{k-1}4f(f - 1) \end{pmatrix} z^{r-2k} \left(z + \frac{1}{z}\right)^{r-2} z^{1-m-n} \frac{d}{dz} \quad (3.6)$$

(1.1) may be also be expressed as a power series in z , using the same ansatz for N_m (3.1), and rearranging the summation

$$[N_m, N_n] = \sum_{k=0}^r \begin{pmatrix} C^k(m, n)(m + n - r + 2k + 1 + f) \\ + C^{k-1}(m, n)(m + n - r + 2k - 1 - f) \end{pmatrix} z^{r-2k} \left(z + \frac{1}{z}\right)^{r-2} z^{1-m-n} \frac{d}{dz} \quad (3.7)$$

where $C^{-1}(m, n) = C^r(m, n) = 0$.

The set of $r + 1$ equations obtained by equating coefficients in (3.6) and (3.7) may be written as a determinant which must be zero for all m, n for a closed algebra. We shall illustrate this for the case $r = 5$, the general case being evident from this example. Removing the factor of $(m - n)$, and defining:

$$\begin{aligned} \lambda &= (m + f)(n + f) \\ \mu &= (m - f)(n - f) \\ \nu &= -4f(f - 1) \end{aligned} \quad (3.8)$$

We will treat these as arbitrary parameters in the first stage of the analysis. Then for $r = 5$ the determinant is

$$\begin{vmatrix} m+n-4+f & 0 & 0 & 0 & 0 & \binom{4}{0}\lambda \\ m+n-4-f & m+n-2+f & 0 & 0 & 0 & \binom{4}{1}\lambda + \binom{4}{0}\mu + \binom{3}{0}\nu \\ 0 & m+n-2-f & m+n+f & 0 & 0 & \binom{4}{2}\lambda + \binom{4}{1}\mu + \binom{3}{1}\nu \\ 0 & 0 & m+n-f & m+n+2+f & 0 & \binom{4}{3}\lambda + \binom{4}{2}\mu + \binom{3}{2}\nu \\ 0 & 0 & 0 & m+n+2-f & m+n+4+f & \binom{4}{4}\lambda + \binom{4}{3}\mu + \binom{3}{3}\nu \\ 0 & 0 & 0 & 0 & m+n+4-f & \binom{4}{4}\mu \end{vmatrix}$$

This may be reduced by row and column operations. Put $R1' = R1 - R2 + R3 - R4 + \dots$

Then it is evident that the combinatorial factors in the last column cancel and that $f = 0$ is an eigenvalue. Then perform $C5' = C5 + C4$, $C4' = C4 + C3$, etc., giving

$$\begin{vmatrix} 2f & 0 & 0 & 0 & 0 & 0 \\ m+n-4-f & 2m+2n-6 & m+n-2+f & 0 & 0 & \binom{4}{1}\lambda + \binom{4}{0}\mu + \binom{3}{0}\nu \\ 0 & m+n-2-f & 2m+2n-2 & m+n+f & 0 & \binom{4}{2}\lambda + \binom{4}{1}\mu + \binom{3}{1}\nu \\ 0 & 0 & m+n-f & 2m+2n+2 & m+n+2+f & \binom{4}{3}\lambda + \binom{4}{2}\mu + \binom{3}{2}\nu \\ 0 & 0 & 0 & m+n+2-f & 2m+2n+6 & \binom{4}{4}\lambda + \binom{4}{3}\mu + \binom{3}{3}\nu \\ 0 & 0 & 0 & 0 & m+n+4-f & \binom{4}{4}\mu \end{vmatrix}$$

And finally $R5' = R5 - R6$, $R4' = R4 - R5'$, ..., yielding

$$\begin{vmatrix} 2f & 0 & 0 & 0 & 0 & 0 \\ m+n-4-f & m+n-4+f & 0 & 0 & 0 & \binom{3}{0}\lambda \\ 0 & m+n-2-f & m+n-2+f & 0 & 0 & \binom{3}{1}\lambda + \binom{3}{0}\mu + \binom{2}{0}\nu \\ 0 & 0 & m+n-f & m+n+f & 0 & \binom{3}{2}\lambda + \binom{3}{1}\mu + \binom{2}{1}\nu \\ 0 & 0 & 0 & m+n+2-f & m+n+2+f & \binom{3}{3}\lambda + \binom{3}{2}\mu + \binom{2}{2}\nu \\ 0 & 0 & 0 & 0 & m+n+4-f & \binom{3}{3}\mu \end{vmatrix}$$

This is then $2f$ times a 5×5 sub-determinant which is equivalent to the determinant in the 4-term case with f shifted to $f+1$. Thus the 5-term matrix has a zero at $f = 0$, and also at zeros of the 4-term case shifted by one. It is clear that the same row and column operations will work for the r -term case, which has zeros at $f = 0, f'+1$ where the f' are the zeros of the $(r-1)$ -term case. The 3-term determinant has a single zero at $f = 0$, completing the inductive proof that the r -term case has zeros at $f = 0, 1, \dots, r-3$, regardless of λ , μ , and ν . It should be noted that the $f = 0$ case is not interesting, as the factor m may be normalised away. Explicit calculation of the 3-term determinant with λ , μ , and ν as in (3.8) show that the case with $r = 3$ has extra zeros at $f = 1, 2$. These zeros do not carry over in the induction above as the shift in f removes them.

This simple family may be extended by noting that the only properties of the binomial coefficients $\binom{n}{i}$ used in the above proof are that $\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$ and $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$. There are other sets of coefficients that satisfy these conditions - those in the expansion of $(z + \frac{1}{z})^{n-q} (z - \frac{1}{z})^q$, which we shall refer to as $\binom{n}{i}^{(q)}$. When $q = 0$ these reduce to the ordinary binomial coefficients. Note that $\binom{n}{n-i}^{(q)} = (-1)^q \binom{n}{i}^{(q)}$, i.e. the parity automorphism still holds, and also that $\binom{n-1}{i-1}^{(q-1)} - \binom{n-1}{i}^{(q-1)} = \binom{n}{i}^{(q)}$.

A very similar inductive proof then gives for the family:

$$N_m = ((m+f)z + (-1)^p(m-f)\frac{1}{z}) (z + \frac{1}{z})^{r-2-q} (z - \frac{1}{z})^q z^{1-m} \frac{d}{dz} \quad (3.9)$$

which has three parameters, r , q ($q \leq r-2$) and p ($p = 0, 1$) the algebra is closed for the following values of f :

$$f = \begin{cases} 0 & \text{if } p = 0, q = r-2 \\ 0, 1, 2 & \text{if } p = 0, q = r-3 \\ 0, 1, \dots, r-3-q & \text{if } p = 0, q > r-3 \\ 0, 1, \dots, q-1 & \text{if } p = 1 \end{cases}$$

The solution (3.9) has positive parity if $r-1-q+p$ is even. If we abandon the parity requirement then we can find more general solutions with two free parameters, *e.g.* in the four term case

$$N_m = (m+f)L_{m+3} + c(m+f+\frac{2}{3})L_{m+1} + \frac{c^2}{3}(m+f+\frac{4}{3})L_{m-1} + \frac{c^3}{27}(m+f+2)L_{m-3} \quad (3.10)$$

This is simply a transformation of the previous solution, of the form

$$L_m \mapsto 3c^{m/2} L_{m+f+2} \quad (3.11)$$

The parity requirement is tantamount to the imposition of unitarity or anti-unitarity. Thus the eigenvalue condition plays a similar rôle to the restrictions on the c -number for unitary representations of the Virasoro algebra found by Belavin, Polyakov and Zamolodchikov^[5] and Friedan, Qiu and Shenker.^[6]

4. Quadratic Coefficients

This form of construction generalises to second order in m . For example, one ansatz which gives a KN algebra with quadratic coefficients of m in the Virasoro generator expansion may be written as:

$$N_m = [(m^2 + am + b)z + (m^2 - am + b)\frac{1}{z}] (z + \frac{1}{z})^{r-2} z^{1-m} \frac{d}{dz} \quad (4.1)$$

where a, b satisfy eigenvalue equations. The eigenvalues for $r \leq 6$ are as follows:

$$r = 3 : \quad a = 2, b \text{ arbitrary}$$

$$r = 5 : \quad a = 2, b \text{ arbitrary}$$

$$r = 6 : \quad \begin{cases} a = 2, b \text{ arbitrary} \\ a = 3, b = 2 \end{cases}$$

All the other quadratic KN algebras we have found are of the form

$$N_m = [(m^2 + am + b)z^2 + 2(m^2 + c) + (m^2 - am + b)\frac{1}{z^2}] (z + \frac{1}{z})^{r-3} z^{1-m} \frac{d}{dz} \quad (4.2)$$

$$\text{where } c = (a - 1)b - 3a^2 + 11a - 14$$

The eigenvalues for this case are

$$r = 4 : \quad \begin{cases} a = 2, b \text{ arbitrary} \\ a = 3, b = 2 \\ a = 4, b = 3 \\ a = 4, b = 5 \end{cases}$$

$$r = 5 : \quad a = 2, b \text{ arbitrary}$$

$$r = 6 : \quad \begin{cases} a = 2, b \text{ arbitrary} \\ a = 3, b = 2 \\ a = 5, b = 6 \end{cases}$$

These results were obtained using the algebraic manipulation program REDUCE. The case (4.1) may be subsumed under the same form as (4.2) by setting $c = b$ instead. For

both cases, the linear equations which must be solved for the structure constants take the form:

$$\begin{pmatrix} A(m+n-r+1+2j)C^j(m,n) \\ + B(m+n-r-1+2j)C^{j-1}(m,n) \\ + A(-(m+n-r-3+2j))C^{j-2}(m,n) \end{pmatrix} = (m-n) \sum_{i=0}^4 \binom{r-3}{j-i} R_i \quad (4.3)$$

where

$$A(m) = m^2 + am + b$$

$$B(m) = 2(m^2 + c)$$

$$R_0 = A(m)A(n)$$

$$R_1 = A(m)B(n) + A(n)B(m) + \frac{2}{(m-n)}(B(m)A(n) - B(n)A(m))$$

$$R_2 = B(m)B(n) + A(m)A(-n) + A(n)A(-m) + \frac{4}{(m-n)}(A(n)A(-m) - A(m)A(-n))$$

$$R_3 = B(m)A(-n) + B(n)A(-m) + \frac{2}{(m-n)}(B(n)A(-m) - B(m)A(-n))$$

$$R_4 = A(-m)A(-n)$$

and where $j = 0, \dots, r+1$ and $C^{-2}(m,n) = C^{-1}(m,n) = C^r(m,n) = C^{r+1}(m,n) = 0$.

We are still searching for an analytic understanding of these equations.

5. Discussion

A pertinent question arises as to whether these algebras are in fact merely transformations of the original Virasoro algebra. In those cases where (1.5) can be inverted to solve for L_m in terms of a sum, albeit infinite, of N 's then this just creates a representation for L_m in terms of pseudo-differential operators.^[7] For the linear solutions the $\alpha^k(m)$ have zeroes, which prevent this inversion for all L 's and thus they are genuinely different. (The matrix of the set of linear equations (1.5) from $m = -\infty$ to a given m is lower triangular: if the diagonal contains any zeros, it is singular).

Returning to the general problem of whether the KN generators can be expressed in terms of a finite sum of Virasoro generators, we adduce some evidence for this hypothesis from an examination of the structure constants for the second highest term. In the KN normalisation the leading constant, as we have remarked, is simply $(m-n)$. At the

next stage the Jacobi identity is linear in the structure constant $C^{r-2}(m, n)$, and is, explicitly:

$$[((m-n)C^{r-2}(m+n+r-1, p) + (m+n+r-3-p)C^{r-2}(m, n)] + \text{cyclic} = 0 \quad (5.1)$$

Putting $p = 1 - r$ we can solve for $C^{r-2}(m, n)$ in terms of $C^{r-2}(m, 1 - r)$ through

$$C^{r-2}(m, n) = \frac{1}{2} \left\{ (m-n)C^{r-2}(m+n+r-1, 1-r) - (m-n-2)C^{r-2}(m, 1-r) - (m-n+2)C^{r-2}(n, 1-r) \right\} \quad (5.2)$$

The remarkable property of this solution is that it is also a solution of (5.1) for all m, n, p . Furthermore if one looks for a representation for N_m of the form of (1.5) the coefficients are given by

$$\begin{aligned} a^{r-1}(m) &= 1 \\ a^{r-2}(m) &= C^{r-1}(m, 1-r) + (m+r)a^{r-2}(1-r) \end{aligned} \quad (5.3)$$

Of course this is only a first order identification, but it holds promise that similar connections may be deduced for further terms in the expansion. This conjecture is supported by computer experiments using REDUCE.

We have not discussed the possible central terms in the algebra. Such terms will be automatically induced by whatever representation is employed for the Virasoro generators.

ACKNOWLEDGEMENTS

We should like to thank Cosmas Zachos for many useful conversations. PF thanks the University of Durham for a Research Studentship and JN thanks CERN for their hospitality.

REFERENCES

1. I. M. Krichever and S. P. Novikov, *Funk Anal i Ego Pril* **21:2** (1987) 46
I. M. Krichever and S. P. Novikov, *Funk Anal i Ego Pril* **21:4** (1987) 47
2. L. Alvarez-Gaumé, C. Gomez, G. Moore and C. Vafa, 'Strings in the operator formalism', preprint CERN-TH-4883/87
3. L. Bonora, M. Bregola, P. Cotta-Ramusino and M. Martellini, *Phys Lett B* **205** (1988) 53
L. Bonora, M. Martellini, M. Rinaldi and J. Russo, *Phys Lett B* **206** (1988) 444
L. Bonora, M. Rinaldi, J. Russo and K. Wu, 'The Sugawara construction on genus g Riemann surfaces', preprint SISSA 38/EP
4. R. Nepomechie, L. Mezincescu and C. Zachos, '(Super)conformal Algebra on the (Super)torus', preprint ANL-HEP-PR-88-23
5. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nuclear Physics B* **241** (1984) 333
6. D. Friedan, Z. Qiu and S. H. Shenker, *Phys Rev Lett* **52** (1984) 1575
7. A. Beilinson, Yu. Manin and V. Schechtman, 'Sheaves of the Virasoro and Neveu-Schwarz algebras', Steklov Institute preprint (1986)