CERN-TH.5042/88
SHEP 87/88-1

A LATTICE STUDY OF NUCLEON STRUCTURE

G. Martinelli
CERN - Geneva

and

C.T. Sachrajda
Physics Department, The University
Southampton SO9 5NH, United Kingdom

ABSTRACT

We present the formalism necessary to compute matrix elements of local operators between nucleon states by numerical simulation of QCD on the lattice. We then compute the lowest two moments of the non-singlet structure function, the electric form factor of the proton and the σ -term.

CERN-TH.5042/88
SHEP 87/88-1
May 1988

1. - INTRODUCTION

One of the most important experimental results of the late 1960's was that only about one half of the proton's momentum (in the infinite momentum frame) is carried by its charged constituents [1]. This result played a significant rôle in the motivation for QCD, and today we understand that the balance is carried by the gluons. In this paper we present the first lattice calculation of the average momentum carried by the valence quarks in a proton. In addition, we compute the second moment of the quark distribution function of the proton, its electric form factor and the σ -term. The calculations are performed on a $20 \times 10^2 \times 40$ lattice, using Wilson fermions [2] in the quenched approximation at $\beta = 6.0$. We use the same gluon configurations as were used in our earlier study of the pion's structure function and electromagnetic form factor [3,4]. The details about the gluon configurations, the fermion action (and conventions for the γ -matrices), the study of the pion and rho masses and the calibration of the lattice spacing can be found in Refs. [3] and [5].

The plan of this paper is as follows. Section 2 contains the formalism necessary to compute matrix elements of the form $\langle p | O(0) | p \rangle$ where $O(0)$ is a local operator bilinear in the quark fields. This extends the theoretical framework developed earlier for pion states [3]. In Section 3 we present our results and Section 4 contains our conclusions.

2. - CORRELATION FUNCTIONS AND OPERATOR MATRIX ELEMENTS

The principal calculations reported below are of matrix elements of local operators between states containing a single nucleon. In Ref. [3] the corresponding calculations for pion states were described in detail. In this section we extend the formalism to include nucleons.

The matrix elements are obtained by computing the Euclidean three-point correlation function

$$C(t_x, t_y) = \int d^3x d^3y \langle 0 | T [\bar{J}_{\gamma'}(0) O(y) J_{\gamma}(x)] | 0 \rangle e^{i p \cdot x} e^{i q \cdot y}$$

(2.1)

where J is an interpolating operator for the nucleon (which, for definiteness, for the rest of this section we assume to be a proton), γ and γ' are spinor labels, t_x and t_y are the time components of x and y and we integrate only over the space coordinates. Inserting a complete set of states between each pair of operators in (2.1) and taking $0 < t_y < t_x$, with t_y and $t_x - t_y$ sufficiently large so that only the

lightest state contributes significantly, we have

$$C(t_x, t_y) = - \frac{e^{-E't_y} e^{-E(t_x - t_y)}}{4EE'}$$

$$\sum_{s, s'} \langle 0 | J_\gamma(0) | p, s \rangle \langle p, s | O(0) | p', s' \rangle \langle p', s' | \bar{J}_\gamma(0) | 0 \rangle \quad (2.2)$$

where $|p, s\rangle$ represents a proton states with four-momentum p and a third component of spin s . p and p' are given by

$$p = (E, \underline{p}) \quad E = \sqrt{\underline{p}^2 + m^2} \quad (2.3)$$

and

$$p' = (E', \underline{p} + \underline{q}) \quad E' = \sqrt{(\underline{p} + \underline{q})^2 + m^2} \quad (2.4)$$

where m is the mass of the proton. The required matrix element $\langle p, s | O(0) | p', s' \rangle$ is one of the factors on the right-hand side of (2.2), while the remaining factors can be obtained by computing two-point correlation functions, as will be explained below.

Throughout this paper, we will use the following interpolating operator:

$$J_\gamma = \epsilon^{ijk} (u^i C \gamma^5 d^j) u_\gamma^k \quad (2.5)$$

where i, j, k are colour labels and C is the charge conjugation matrix. This isospin- $\frac{1}{2}$ interpolating operator has been shown in previous lattice studies of hadron spectroscopy to have a significant overlap with the proton [6,7]. Using Lorentz- and parity invariance, we can write

$$\langle 0 | J_\alpha(0) | p, s \rangle = \sqrt{Z_p} N_\alpha^{(s)}(p) \quad (2.6)$$

where Z_p is a scalar quantity and $N_\alpha^{(s)}(p)$ is the proton's spinor, which satisfies the Dirac equation. Z_p and m can be obtained from the two-point Euclidean correlation function

$$K(t_x) = \int d^3x \langle 0 | T [\bar{J}_\alpha(0) J_\beta(x)] | 0 \rangle e^{i \underline{p} \cdot \underline{x}} \quad (2.7)$$

For $\underline{p} = 0$, inserting a complete set of states between the interpolating operators

J and taking t_x to be sufficiently large so that the single proton state is the dominant one, we have:

$$K(t_x) = \frac{Z_P}{2} (1 + \gamma^0)_{\beta\alpha} e^{-mt_x} \quad (2.8)$$

We have neglected in (2.8) the contributions from heavier states, in particular that from the opposite parity baryon, $N^*(\frac{1}{2}^-)$, which is equal to

$$Z_{N^*} \frac{(1 - \gamma^0)_{\beta\alpha}}{2} e^{-m't_x}$$

One can further suppress its contribution by an appropriate choice of spinor indices (e.g., $\alpha = \beta = 1$ or $\alpha = \beta = 2$). Thus from the behaviour of $K(t_x)$ with t_x , both m and Z_P can be determined and from Eq. (2.2) we can extract the matrix elements $\langle p, s | O | p', s' \rangle$.

The evaluation of $K(t_x)$ and $C(t_x, t_y)$ requires the computation of the diagrams in Fig. 1 and Fig. 2 respectively. In these figures, the propagators are quark propagators in a gluon background field (the background gluon field configuration being determined by the Monte Carlo program). The crosses denote the insertion of the interpolating operator (2.5) and the shaded circles that of the operator whose matrix element we wish to compute.

For the diagram in Fig. 1 it is sufficient to have the set of quark propagators from an arbitrary lattice point to the origin ($\{S(x, 0)\}$). Throughout this work we keep the up and down quark masses equal and hence the propagators $S(x, 0)$ are identical for both flavours. In order to compute the diagrams in Fig. 2 we need the sets of generalized propagators $\{\Sigma_d\}$ and $\{\Sigma_u\}$ defined by (see Fig. 3)

$$\begin{aligned} [\Sigma_d(0, y; t_x, p)]_{\beta\sigma}^{xy} &= -\epsilon^{ABC} \epsilon^{A'x c'} (C\gamma^5)_{\alpha\beta} (C\gamma^5)_{\alpha'p} \\ &\sum_x \left\{ e^{ip \cdot x} S_{\beta\sigma}^{BY}(x, y) \left[S_{\alpha\alpha'}^{AA'}(x, 0) S_{\gamma\gamma'}^{cc'}(x, 0) \right. \right. \\ &\quad \left. \left. - S_{\alpha\gamma'}^{Ac'}(x, 0) S_{\gamma\alpha'}^{cA'}(x, 0) \right] \right\} \end{aligned} \quad (2.9)$$

and

$$\begin{aligned}
 [\Sigma_u(0, y; t_x, p)]_{\rho\sigma}^{xy} = & -\epsilon^{ABC} (C\gamma^5)_{\alpha\beta} \sum_x e^{i p \cdot x} S_{\beta\beta'}^{BB'}(x, 0) \\
 & \left\{ \epsilon^{x'B'C'} (C\gamma^5)_{\beta\beta'} [S_{\alpha\sigma}^{AY}(x, y) S_{\gamma\gamma'}^{CC'}(x, 0) - S_{\alpha\gamma}^{AC'}(x, 0) S_{\gamma\sigma}^{CY}(x, y)] \right. \\
 & + \epsilon^{A'B'x} (C\gamma^5)_{\alpha'\beta'} \delta_{\gamma'\gamma} [S_{\alpha\alpha'}^{AA'}(x, 0) S_{\gamma\sigma}^{CY}(x, y) \\
 & \left. - S_{\alpha\sigma}^{AY}(x, y) S_{\gamma\alpha'}^{CA'}(x, 0)] \right\}
 \end{aligned} \tag{2.10}$$

The upper case Latin superscripts and the lower case Greek subscripts (both primed and unprimed) are colour and spinor labels respectively. The interpolating operators for the proton were chosen to be $\bar{J}_Y(0)$ and $J_Y(x)$ [with J defined in Eq. (2.5)], so that the spinor indices γ and γ' are external labels. There is a different Σ_d and Σ_u for each pair (γ, γ') .

To illustrate the significance of the Σ_u 's and Σ_d 's, consider the matrix element $\langle p | \bar{\psi} \Gamma \psi | p \rangle$ where ψ can be an up or down quark and Γ is one of the 16 matrices of Dirac theory. The three-point correlation function (2.1) corresponding to the operator $\bar{\psi} \Gamma \psi$, computed in the quenched approximation, is the weighted average over all configurations of

$$\sum_{\frac{y}{T}} t_z [\Sigma_\psi(0, y; t_x, p) \Gamma S(y, 0)] e^{i q \cdot y} \tag{2.11}$$

where $\psi = u$ or d . Thus the set $\{\Sigma_\psi(0, y; t_x, p), \psi = u \text{ and } d\}$, together with the set of quark propagators $\{S(y, 0)\}$, is sufficient to compute the matrix elements of a large class of local operators (including, as will be shown below, operators containing covariant derivatives).

The Σ_d 's and Σ_u 's satisfy equations which can be solved using the same numerical techniques as those used to generate the usual quark propagators $\{S(x, 0)\}$ [6]. For example, Σ_d satisfies

$$\begin{aligned}
 & \sum_y \sum_d (0, y; t_x, \underline{p})_{\rho\sigma}^{\alpha\gamma} \Delta_{\sigma\tau}^{\gamma\tau}(y, z) \\
 &= -e^{i\underline{p} \cdot \underline{z}} \in^{A\tau C} \in^{A'\tau C'} (C\gamma_5)_{\alpha\tau} (C\gamma_5)_{\alpha'\tau'} \delta_{t_x, t_z} \cdot \\
 & \left[S_{\alpha\alpha'}^{AA'}(z, 0) S_{\gamma\gamma'}^{CC'}(z, 0) - S_{\alpha\gamma'}^{AC'}(z, 0) S_{\gamma\alpha'}^{CA'}(z, 0) \right]
 \end{aligned} \tag{2.12}$$

where $\Delta_{\sigma\tau}^{\gamma\tau}(y, z)$ is the kernel of the fermionic action, i.e.,

$$\sum_z \Delta_{\alpha\beta}^{\alpha\gamma}(x, z) S_{\beta\gamma}^{\gamma A}(z, 0) = \delta^{\alpha A} \delta_{\alpha\gamma} \delta^4(x) \tag{2.13}$$

We then solve Eq. (2.12) iteratively, using the standard Gauss-Siedel algorithm.

Each set $\{\Sigma_u(0, y; t_x, \underline{p})\}$ or $\{\Sigma_d(0, y; t_x, \underline{p})\}$ depends on t_x , on $\underline{p}$ and on the choice of spinor indices, γ and γ' , in the interpolating operators $J_\gamma, \bar{J}_\gamma$, (this dependence on γ and γ' is implicit throughout the above discussion). Therefore, we have to choose each of these parameters in a way which will optimized the amount of information obtainable. We now discuss each of these parameters in turn:

- a) $\underline{p}$: we make the same choice as in the pion case [3], $\underline{p} = (\pi/10, 0, 0)$, for precisely the same reasons. Among the quantities which we wish to calculate are the moments of deep inelastic structure functions, i.e., the forward matrix elements of lowest twist operators between nucleon states. As was discussed in detail in Ref. [3], in order to avoid power divergences in the matrix elements of these operators, one has to give a non-zero spatial momentum to the proton. To minimize the systematic errors associated with the granularity of the lattice, we have chosen $\underline{p}$ to have the smallest non-zero value consistent with our lattice, $\underline{p} = (\pi/10, 0, 0)$.
- b) t_x : t_x must be chosen in such a way that there exist some values of t_y with t_y and $t_x - t_y$ sufficiently large to isolate the single nucleon's contribution. From earlier studies we deduce that we require t_y and $t_x - t_y$ both to be $\gtrsim 8$, which implies that $t_x \gtrsim 16$. Ideally we would like to take t_x to be somewhat larger than 16, in order to be able to vary t_y and verify that the behaviour of the three-point correlation function with t_y is indeed given by Eq. (2.2). For the pion, this was possible and instructive [2,3]. For the proton, however, we find that the statistical errors are considerably larger than for the pion, since the mass is larger and the signal decreases more rapidly with the distance. Thus we have chosen $t_x = 16$.
- c) γ and γ' : the choice of the spinor indices γ and γ' depends on the physics one wishes to study. In this exploratory study we sum over the two cases $\gamma = \gamma' = 1$ and $\gamma = \gamma' = 2$. This, as will be shown below, enables us to

compute the low moments of the unpolarized deep inelastic structure functions, the electric form factor of the proton at several values of momentum transfer and the sigma term. In order to compute other interesting quantities, such as the matrix elements of the axial vector current or pseudoscalar density, the magnetic form factor of the nucleon or the recently-measured lowest moment of the polarized structure function g_1 [7], other choices of γ and γ' must be made. By summing only over the two values $\gamma = \gamma' = 1$ and 2, rather than over all four diagonal terms, we are able to suppress further the contribution of the opposite parity baryons (with masses m') to the correlation functions (as was the case in the two-point correlation function discussed above).

So far the discussion has been general to all matrix elements of local operators between single nucleon states. We now concentrate on the three main topics studied in this paper and discuss each of these in turn.

1) Deep Inelastic Structure Functions

We will compute the first two moments of the non-singlet structure function of the nucleons (for unpolarized scattering). We evaluate the matrix element of the operator

$$O_2 = i \bar{\psi} [\gamma^4 \overleftrightarrow{D}^4 + \gamma^1 \overleftrightarrow{D}^4] \psi \quad (2.14)$$

(with $\psi = u$ or d), from which we deduce $\langle x \rangle$, the average fraction of momentum (in the infinite momentum frame) carried by the valence quarks in the proton, and the matrix elements of $O_3 = O_3^{411} - \frac{1}{2}(O_3^{422} + O_3^{433})$ where

$$O_3^{\mu\nu\rho} = i^3 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \overleftrightarrow{D}^\rho \psi + \text{permutations} \quad (2.15)$$

(again with $\psi = u$ or d and symmetrized over the Lorentz indices), from which we deduce $\langle x^2 \rangle$. The insertion of the operators O_2 and O_3 in Eq. (2.2) gives

$$\begin{aligned} C_{SF}^{(2)}(t_x, t_y) &= \sum_{\gamma=\gamma'=1}^2 \sum_{x,y} \langle 0 | T [\bar{J}_\gamma(0) O_2(y) J_\gamma(x)] | 0 \rangle e^{i p \cdot x} \\ &= - \frac{2 \langle x \rangle_{LATT} (E+m)}{E} p_z Z_p e^{-E t_x} \end{aligned} \quad (2.16)$$

and

$$\begin{aligned}
 C_{SF}^{(3)}(t_x, t_y) &= \sum_{\gamma=\gamma'=1}^2 \sum_{x,y} \langle 0 | T [\bar{J}_\gamma(0) O_3(y) J_\gamma(x)] | 0 \rangle e^{i p \cdot x} \\
 &= - \frac{4 \langle x^2 \rangle_{LATT} (E+m)}{E} p^2 Z_p e^{-E t_x}
 \end{aligned}
 \tag{2.17}$$

where $p = \pi/10$, $E = \sqrt{m^2 + p^2}$ and Z_p is defined in Eq. (2.6). The subscript "LATT" reminds us that the quantities are computed with the corresponding operators renormalized in the lattice renormalization scheme, and at a renormalization scale equal to the lattice spacing [3,4]. Using perturbation theory, we are then able to deduce $\langle x \rangle$ and $\langle x^2 \rangle$ in standard continuum renormalization schemes (such as the $\overline{MS}$ scheme).

ii) The Electric Form Factor of the Proton

On the lattice the allowed momenta are discrete, and hence only a finite number of momentum transfers can be considered. The momentum of the final state proton, p , is fixed to be $(\pi/10, 0, 0)$ and we consider the following cases $q = p, 2p, -p, -2p$ and $(0, \pi/5, 0)$ where q is the momentum carried by the current. We expect the systematic errors to be the smallest for the two cases $q = -p$ and $-2p$, since in these cases, both the initial and final state protons have momentum less than or equal to $\pi/10$.

The correlation functions which we compute are

$$\begin{aligned}
 C_{FF}^\mu(t_x, t_y) &= \sum_{\gamma=\gamma'=1}^2 \sum_{x,y} \langle 0 | T [\bar{J}_\gamma(0) V^\mu(y) J_\gamma(x)] | 0 \rangle \\
 &\quad \cdot e^{i p \cdot x} e^{i q \cdot y}
 \end{aligned}
 \tag{2.18}$$

where $V^\mu(y)$ is the electromagnetic current. In general, the matrix element depends on the two form-factors F_1 and F_2 , defined by (we follow the notation of Ref. [8])

$$\begin{aligned}
 \langle p, s | V^\mu(0) | p', s' \rangle &= \bar{u}^{(s)}(p) \left[\gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} \right. \\
 &\quad \left. \kappa F_2(q^2) \right] u^{(s')}(p')
 \end{aligned}
 \tag{2.19}$$

with the normalization chosen so that $F_2(0) = 1$ and $F_1(0) = 1(0)$ for the proton (neutron) and $q \equiv p-p'$. In the present calculations, given that the non-relativistic approximation $E = m+p^2/2m$ holds to very good precision and that we sum over $\gamma = \gamma' = 1$ and 2, the correlation function is proportional to the electric form factor^{*}), defined by

$$G_E = F_1 + \frac{kq^2}{4m^2} F_2 \quad (2.20)$$

It is easy to show that

$$C_{FF}^4(t_x, t_y) = -\frac{Z_P}{2EE'} e^{-E't_y} e^{-E(t_x-t_y)} \cdot$$

$$\left[(E+m)(E'+m) - P_1 P_1' \right] G_E(q^2) \quad (2.21)$$

and

$$C_{FF}^1(t_x, t_y) = -\frac{i Z_P}{2EE'} e^{-E't_y - E(t_x-t_y)} \cdot$$

$$\left(E P_1' + E' P_1 + m(P_1 + P_1') \right) \left[G_E(q^2) - \frac{P_1(P_1')^2}{(P_1 + P_1')^2} \frac{k F_2}{4m^2} \right] \quad (2.22)$$

The factor of i in (2.22) comes from the continuation to Euclidean space.

iii) The Sigma Term

In order to obtain the forward matrix elements of $\bar{u}u$ and $\bar{d}d$ we compute the correlation functions

$$C_{\bar{q}q}(t_x, t_y) = \sum_{\gamma=\gamma'=1}^2 \sum_{\underline{x}, \underline{y}} \langle 0 | T \left[\bar{J}_{\gamma'}(0) \bar{q}(\underline{y}) q(\underline{y}) J_{\gamma}(\underline{x}) \right] | 0 \rangle \cdot$$

$$e^{i p \cdot \underline{x}} = -\frac{E+m}{2E^2} Z_P e^{-Et_x} \langle p | \bar{q}(0) q(0) | p \rangle \quad (2.23)$$

where $q = u$ or d , and the result is independent of the spin.

^{*}) Except for a small correction for $\mu = 1$ and $\underline{q} = (0, \pi/5, 0)$, see Eq. (2.21) below.

3. - NUMERICAL RESULTS

1) Generalities

Our study of the proton is carried out using the same 15 configurations as were used for the pion [3]. The quark propagators were computed at three values of the Wilson parameter $K = 0.1515$, 0.1530 and 0.1545 , and the critical K , K_c , was found to be $0.1563(2)$. For reference, we start the presentation of the results by tabulating the values of the proton mass in Table 1 at the three values of K . These masses were obtained by calculating the two-point correlation function (2.7) in the time interval $t_x = 8-13$. For completeness we also include the values of the masses of the pion and rho mesons using the same configurations [3]^{*)}.

K	$m_\pi a$	$m_\rho a$	$m_p a$	m_p/m_ρ
0.1515	0.50 ± 0.03	0.55 ± 0.05	0.92 ± 0.08	1.7 ± 0.1
0.1530	0.43 ± 0.03	0.48 ± 0.06	0.82 ± 0.09	1.7 ± 0.1
0.1545	0.36 ± 0.04	0.42 ± 0.09	0.75 ± 0.10	1.8 ± 0.2
0.1563(2)	-	0.34 ± 0.05	0.64 ± 0.11	1.8 ± 0.1

Table 1: π , ρ and proton masses in dimensionless units, together with their extrapolated values at $K_c = 0.1563$.

From the rho mass at K_c , we determined the lattice spacing and found (for a discussion see Refs. [3] and [5])

$$a^{-1} = (2.2 \pm 0.4) \text{ GeV} \quad (3.1)$$

The quoted error is the statistical error only. In common with all previous computations (e.g., Ref. [9] and references therein), we find that the ratio of the proton to rho masses is too high. This in turn implies that there is a significant systematic uncertainty in the value of a^{-1} . If the proton's mass, rather than that of the rho meson, is used to set the scale, a^{-1} is found to be around 1.6 GeV, a value significantly smaller than that in (3.1). This uncertainty in the scale is not so important for the moments of the structure functions where we are calculating dimensionless quantities which vary only logarithmically with the scale. On the contrary, this uncertainty is more significant in quantities such as the electromagnetic form factors which are

*) The meson masses in Table 1 were obtained by fitting in the range $t_x = 9-14$.

expected to vary significantly with the scale, and in dimensionful quantities such as the mesons' decay constants. Below we shall present our results using

(3.1) to set the scale, but the reader should note the systematic uncertainty in a^{-1} .

ii) The Charge of the Proton

A useful numerical check in our study of the pion's structure was provided by the computation of the forward matrix element $\langle \pi(p) | \tilde{V}_\mu(0) | \pi(p) \rangle$, where $|\pi(p)\rangle$ represents a pion state with four-momentum p and $\tilde{V}_\mu$ is the conserved vector current^{*} (with appropriate flavour quantum numbers),

$$\tilde{V}_\mu(x) = \frac{1}{2} \left[\bar{q}(x) (\gamma_\mu - 1) U_\mu(x) q(x+\hat{\mu}) + \bar{q}(x+\hat{\mu}) (\gamma_\mu + 1) U_\mu^\dagger(x) q(x) \right] \quad (3.2)$$

From this matrix element, we can deduce the charge of the π^+ , say. The result should be free from most systematic uncertainties which affect the remaining calculations (such as quenching or the presence of terms proportional to positive powers of the lattice spacing). We obtained the result $Q = 1$ with a statistical error of about 5%. We now repeat the calculation for the proton.

To calculate the charge Q we compute C_{FF}^μ , Eq. (2.18), with $\underline{q} = 0$ and choose V^μ to be the conserved electromagnetic current [i.e., (3.2) with weights $2/3$ and $-1/3$ for the up and down quark terms]. Then

$$C_{FF}^\mu(t_x=16, t_y) = \frac{P^\mu(E+m)}{E^2} Z_P Q e^{-16E} \quad (3.3)$$

In Table 2 we report the results for $t_y = 8$ obtained by using the values of m and Z_P from the two-point function.

^{*}) In some cases it is convenient or necessary to use $\frac{1}{2}[\tilde{V}_\mu(x) + \tilde{V}_\mu(x-\mu)]$ which transforms irreducibly under the hypercubic group.

Using (3.3) with $\mu = 4$

K	Q_u	Q_d
0.1515	$4/3(0.93 \pm 0.23)$	$-1/3(0.94 \pm 0.24)$
0.1530	$4/3(0.70 \pm 0.31)$	$-1/3(0.72 \pm 0.28)$

Using (3.3) with $\mu = 1$

K	Q_u	Q_d
0.1515	$4/3(0.87 \pm 0.15)$	$-1/3(0.92 \pm 0.10)$
0.1530	$4/3(0.89 \pm 0.29)$	$-1/3(1.08 \pm 0.16)$

Table 2: The charges of the up and down quarks in the proton.

The results are consistent with the known values $Q = Q_u + Q_d = 1$. The errors, although not overwhelming, are somewhat larger than the 5% errors found in the analogous calculation for the pion [3]. Below, we shall see that this is a general feature. Since statistical errors rapidly increase at larger values of K, we have not computed $\Sigma_{u,d}$ at $K = 0.1545$ and the results in the following will always refer to $K = 0.1515$ and 0.1530 only.

We performed the exercise above as a numerical check on our techniques, which involved determining the mass of the proton and Z_p from $K(t_x)$ and using (3.3) to obtain Q. However, the values of Q_u and Q_d reported in Table 2 are not the most precise ones which can be obtained. In our earlier calculations [3,4] we have frequently noted strong correlations in the statistical fluctuations in different correlation functions. Hence it is often advantageous to take appropriate ratios of these correlation functions. Here we can calculate the two-point correlation function $K(t_x)$ at $t_x = 16$, but for a proton with momentum $\underline{p} = (\pi/10, 0, 0)$. The ratio of C_{FF}^μ in (3.3) for $\mu = 4$, to this two-point correlation function, is just Q. In this way, we find $Q = 1.00 \pm 0.01$ at $K = 0.1515$ and 1.00 ± 0.05 at $K = 0.1530$. The errors are markedly reduced, as a result of the Ward identity obeyed by the conserved current (for any background field).

iii) $\underline{Z}_V^{LOC}$

We define $\underline{Z}_V^{LOC}$ by the ratio of forward matrix elements

$$Z_V^{LOC} = \frac{\langle p | \tilde{V}_\mu(0) | p \rangle}{\langle p | V_\mu^{LOC}(0) | p \rangle} \quad (3.4)$$

where $\tilde{V}_\mu$ is the conserved vector current (3.2) and V_μ^{LOC} is the local vector current

$$V_\mu^{LOC} = \bar{q}(x) \gamma_\mu q(x) \quad (3.5)$$

For $\mu = 4$, we find

$$Z_V^{LOC} = 0.76 \pm 0.01 \quad \text{at } k = 0.1515$$

and

$$Z_V^{LOC} = 0.73 \pm 0.01 \quad \text{at } k = 0.1530 \quad (3.6)$$

For $\mu = 1$, the fluctuations are a little larger,

$$Z_V^{LOC} = 0.72 \pm 0.03 \quad \text{at } k = 0.1515$$

and

$$Z_V^{LOC} = 0.74 \pm 0.10 \quad \text{at } k = 0.1530 \quad (3.7)$$

For sufficiently small lattice spacings Z_V^{LOC} is independent of the states between which the conserved and local vector currents are sandwiched [10,11], and can be calculated in perturbation theory [12-14]. The results reported in (3.6) and (3.7) are similar to those obtained with the currents sandwiched between pion states, $Z_V^{LOC} \sim 0.71$ [3]. They differ, however, from the one-loop perturbative result (0.83) [12-14] and from the value obtained with the currents sandwiched between the rho meson and the vacuum (0.57) [15]. These discrepancies, due to the presence of terms of order a (a = lattice spacing) which arise in the renormalization of the local current, provide an indication of some of the systematic uncertainties in our calculations. We do not believe that these discrepancies are due to the quenched approximation since (a) the one-loop perturbative result is independent of the number of quark flavours, so that the contribution from the neglected loops should not be large, and (b) the measured values of Z_V^{LOC} do depend on the states between which the currents are placed. Further support for this interpretation comes from the fact that the value of 0.57 obtained by placing the current between the rho meson and the vacuum increases to 0.62 as β is increased from 6.0 to 6.2 [16], i.e., as a^{-1} is increased from about 2 to 3 GeV.

iv) Deep Inelastic Structure Functions

We now use (2.16) and (2.17) to determine the first two moments of the protons' structure function. We perform the calculations separately for the up and down quarks. Using the values of m and Z_p obtained by fitting $K(t_x)$ in the range $t_x = 8-13$, we find:

for the up quark:

$$\begin{aligned} \langle x \rangle_{\text{LATT}} &= 0.45 \pm 0.04 & \text{at } k = 0.1515 \\ \langle x \rangle_{\text{LATT}} &= 0.50 \pm 0.02 & \text{at } k = 0.1530 \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \langle x^2 \rangle_{\text{LATT}} &= 0.13^{+0.02}_{-0.06} & \text{at } k = 0.1515 \\ \langle x^2 \rangle_{\text{LATT}} &= 0.13^{+0.01}_{-0.05} & \text{at } k = 0.1530 \end{aligned} \quad (3.9)$$

and for the down quark:

$$\begin{aligned} \langle x \rangle_{\text{LATT}} &= 0.22 \pm 0.02 & \text{at } k = 0.1515 \\ \langle x \rangle_{\text{LATT}} &= 0.25 \pm 0.03 & \text{at } k = 0.1530 \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \langle x^2 \rangle_{\text{LATT}} &= 0.08^{+0.02}_{-0.06} & \text{at } k = 0.1515 \\ \langle x^2 \rangle_{\text{LATT}} &= 0.12^{+0.01}_{-0.05} & \text{at } k = 0.1530 \end{aligned} \quad (3.11)$$

The perturbative correction relating the operator $O_2^{\mu\nu}$ of Eq. (2.14) renormalized in the lattice renormalization scale to that renormalized in the $\overline{\text{MS}}$ scheme is negligibly small [17]. Thus the values given in (3.8) and (3.10) can be taken as being $\langle x \rangle_u$ and $\langle x \rangle_d$, the average fraction of momentum carried by the up and down valence quarks, as defined from the operator $O_2^{\mu\nu}$ in the $\overline{\text{MS}}$ scheme at a renormalization scale of about 2 GeV. Within our errors, $\langle x \rangle_u$ is twice $\langle x \rangle_d$.

$$\frac{\langle x \rangle_u}{\langle x \rangle_d} = 2.0 \pm 0.25 \quad \text{at } k = 0.1515$$

$$\frac{\langle x \rangle_u}{\langle x \rangle_d} = 2.0 \pm 0.50 \quad \text{at } k = 0.1530 \quad (3.12)$$

Of course we should extrapolate the results in (3.8) and (3.10) to the chiral limit $K_c = 0.1563$ (2) [see Ref. (3)]. Within the errors our results are consistent with being independent of K , where $\langle x \rangle_v$ is the fraction of momentum carried by the valence quarks. These values are larger than those measured experimentally,

$$\langle x \rangle_v = \langle x \rangle_u + \langle x \rangle_d \simeq 0.4 \quad \text{at } \mu \simeq 2 \text{ GeV} \quad (3.13)$$

From Eqs. (3.8) and (3.10), we obtain values for $\langle x \rangle_v$ of 0.67 ± 0.05 at $K = 0.1515$ and $0.75^{+0.02}_{-0.10}$ at $K = 0.1530$. Part of the discrepancy (but less than about 0.1 in $\langle x \rangle_v$) can be attributed to the fact that there are no sea quarks in our quenched approximation.

Using the tadpole dominance of lattice perturbation theory [18], we estimate that the values of $\langle x^2 \rangle$ in the $\overline{\text{MS}}$ scheme are about 13% smaller than those in the lattice renormalization scheme which are given in Eqs. (3.9) and (3.11). Given the large statistical errors in our results for $\langle x^2 \rangle$, we do not comment on them further, other than to note that they are reasonable.

For the pion we found that we were able to reduce the errors a little by dividing the correlation functions $C_{SF}^{(2)}$ and $C_{SF}^{(3)}$ [defined in (2.16) and (2.17)] by the three-point correlation function C_{FF}^μ corresponding (3.3) to the conserved vector current (3.2):

$$\frac{C_{SF}^{(2)}}{C_{FF}^4} = 2 \langle x \rangle p$$

$$\frac{C_{SF}^{(3)}}{C_{FF}^4} = 4 \langle x^2 \rangle p \quad (3.14)$$

where $p = \pi/10$. The results obtained using the latter method are given in Table 3.

UP QUARK			DOWN QUARK		
K	$\langle x \rangle_{\text{LATT}}$	$\langle x^2 \rangle_{\text{LATT}}$	K	$\langle x \rangle_{\text{LATT}}$	$\langle x^2 \rangle_{\text{LATT}}$
0.1515	0.48 ± 0.05	0.14 ± 0.05	0.1515	0.24 ± 0.04	0.08 ± 0.04
0.1530	$0.7^{+0.1}_{-0.3}$	$0.19^{+0.04}_{-0.08}$	0.1530	$0.35^{+0.05}_{-0.15}$	$0.17^{+0.03}_{-0.09}$

Table 3: Values of $\langle x \rangle_{\text{LATT}}$ and $\langle x^2 \rangle_{\text{LATT}}$ obtained using (3.14) and (3.15)

For the proton, this technique provides us with a consistency check, but no improvement to the precision of the results in (3.8) to (3.11).

v) G_E

We determine the electric form factor G_E by computing the three-point correlation function C_{FF}^μ [defined in Eq. (2.18)] and using Eqs. (2.21) and (2.22). For the vector current, we take the conserved one (3.2). The three-momentum of the final state protons is fixed to be $\underline{p} = (\pi/10, 0, 0)$ and we evaluate G_E for the five values of momentum transfer corresponding to the initial state proton having three-momentum $\underline{p}'$, equal to $2\underline{p}$, $3\underline{p}$, $\underline{0}$, $-\underline{p}$ and $(\pi/10, \pi/5, 0)$. The results are shown in Table 4; where possible we present results obtained using both $\mu = 4$ and $\mu = 1$.

$$K = 0.1515$$

$\underline{p}'$	$-q^2 a^2$	$G_E(q^2)$ using C_{FF}^4	$G_E(q^2)$ using C_{FF}^1
$2\underline{p}$	0.078	0.53 ± 0.19	0.69 ± 0.17
$3\underline{p}$	0.28	0.26 ± 0.15	0.43 ± 0.04
$\underline{0}$	0.096	0.77 ± 0.04	0.84 ± 0.18
$-\underline{p}$	0.39	0.36 ± 0.01	-
$(\pi/10, \pi/5, 0)$	0.36	0.49 ± 0.08	-

$$K = 0.1530$$

$\underline{p}'$	$-q^2 a^2$	$G_E(q^2)$ using C_{FF}^4	$G_E(q^2)$ using C_{FF}^1
$2\underline{p}$	0.075	0.33 ± 0.26	0.77 ± 0.25
$3\underline{p}$	0.26	0.38 ± 0.09	0.54 ± 0.17
$\underline{0}$	0.095	0.90 ± 0.16	1.12 ± 0.33
$-\underline{p}$	0.39	0.39 ± 0.06	-
$(\pi/10, \pi/5, 0)$	0.35	0.60 ± 0.18	-

Table 4: Values of $G_E(q^2)$ obtained by computing C_{FF}^μ with $\mu = 4$ or 1.

The results in the Table were derived using the formula

$$G_E(q^2) = \left(\frac{C_{FF}^4(16,8)}{K(8)} \right)_{q \neq 0} \left(\frac{K(8)}{C_{FF}^4(16,8)} \right)_{q=0}.$$

$$\frac{2E(E'+m)}{[(E+m)(E'+m)+pp']} \quad (3.15)$$

and the analogous formula for $\mu = 1$, [see Eqs. (2.21) and (2.22)]. Within the errors, the results are independent of t_y , in the range 7-9. We use Eq. (3.15) because it is less sensitive to the details of the fitting procedures. The large fluctuations in C_{FF}^4 at $q = 0$ at $K = 0.1530$ (cf. Table 2), induce large errors for the corresponding G_E .

Given the errors, we do not attempt to extrapolate to the chiral limit. Instead we plot G_E in Fig. 4 against $|q|/m_\rho$, where m_ρ is the mass of the rho

obtained on the lattice at the same value of the quark mass (see Table 1). The solid curve is the prediction obtained using the vector meson dominance model

$$g_E(q^2) = \frac{1}{1 + \frac{q^2}{m_\rho^2}} \quad (3.16)$$

Our results have the correct trend of the experimental results [19]. The points lie a little below the vector dominance prediction, and if we replace m_ρ by λm_ρ in (3.16), then the best fit gives

$$\lambda = 0.85 \pm 0.05 \quad (3.17)$$

to be compared to $\lambda \sim 0.65$ which corresponds to the best fit to the results of Ref. [19].

vi) The σ -term

We determine $\langle p, s | \bar{u}(0)u(0) | p, s \rangle$ and $\langle p, s | \bar{d}(0)d(0) | p, s \rangle$ by using Eq. (2.23) and find that at $K = 0.1515$

$$\langle p | \bar{u}u | p \rangle_a = 2.7 \pm 0.7 \quad (3.18)$$

$$\langle p | \bar{d}d | p \rangle_a = 1.4 \pm 0.3 \quad (3.19)$$

and at $K = 0.1530$

$$\langle p | \bar{u}u | p \rangle_a = 1.1^{+1.5}_{-0.5} \quad (3.20)$$

$$\langle p | \bar{d}d | p \rangle_a = 0.5^{+0.4}_{-0.2} \quad (3.21)$$

where we have suppressed the spin label s , since the results are independent of s . For D/F , defined by

$$\frac{D}{F} = \frac{\langle p | \bar{u}u | p \rangle - 2\langle p | \bar{d}d | p \rangle}{\langle p | \bar{u}u | p \rangle} \quad (3.22)$$

we obtain

$$D/F = -0.03 \pm 0.23 \quad \text{at } K = 0.1515 \quad (3.23)$$

and

$$\frac{D}{F} = +0.04 \pm 0.29 \quad \text{at } K = 0.1530 \quad (3.24)$$

The results for $\langle p | \bar{u}u | p \rangle$ and $\langle p | \bar{d}d | p \rangle$ in Eqs. (3.18)-(3.20) are compatible with those of Refs. [20] and [21] but with larger statistical errors. In Refs. [20] and [21] a different technique was used to evaluate these matrix elements. Given the large fluctuations the D/F term is, within the error, compatible with zero, and it is not possible to attempt any extrapolation of the matrix elements of the scalar density to the chiral limit.

With respect to Ref. [20], we are using here a much larger lattice where it is easier to isolate the proton state. To evaluate the σ -term ($\sigma = M_q \langle p | \bar{u}u + \bar{d}d | p \rangle$, where M_q is the bare quark mass), we may then make use of the Ward identity [20]:

$$\frac{dm}{d\left(\frac{1}{2Ka}\right)} = \langle p | \bar{u}u + \bar{d}d | p \rangle \quad (3.25)$$

where m is the proton mass, K is the Wilson hopping parameter and a is the lattice spacing.

From our determination of the proton mass we find:

$$\frac{dm}{d\left(\frac{1}{2Ka}\right)} = 2.7 \pm 0.6 \quad (3.26)$$

Taking for M_q the determination of Ref. [15], $M_q = 1/2a(1/k - 1/k_c) \approx 4.6$ MeV, we obtain

$$\sigma = (12 \pm 3) \text{ MeV}$$

in agreement with the determinations of Refs. [20].

4. - CONCLUSIONS

We have presented the formalism necessary to compute matrix elements of local operators between nucleon states. We then used this formalism to compute the lowest two moments of the (unpolarized) structure functions and the electric form factor of the proton, as well as the σ -term. With the exception of the σ -term, which was computed in Refs. [20,21], these are the first lattice computations of these fundamental quantities.

In general, the results are consistent with expectations. For the σ -term, the statistical errors in the determination of the matrix elements of the scalar density forbid for the moment any precise evaluation. The value of the σ -term, extracted from the derivative of the proton mass with respect to the quark mass, is in agreement with the results of Refs. [20] and [21] and confirm that the σ -term on the lattice is too small in the quenched approximation. The values which we obtain for $\langle x \rangle_v$, the average momentum carried by valence quarks inside a proton, are larger than the experimental result (0.67 ± 0.05 at $K = 0.1515$ and $0.75^{+0.02}_{-0.01}$ at $K = 0.1530$, to be compared with the experimental result of about 0.4). Our central values for $\langle x \rangle_v$ are rather large, even taking into account the systematic effects entailed in the quenched approximation. The results for the electric form factor presented in Table 4 are in reasonable agreement with experiments and support vector meson dominance.

The results of this exploratory study of the nucleon are encouraging. The statistical errors are fairly small and the major uncertainties seem to come from systematic effects. The general trend of our results seems to indicate that we can compute matrix elements of operators between nucleon states with an uncertainty of 30%-50%. To check whether we have not underestimated the statistical errors and to determine more precisely to what extent our results are affected by systematic effects (in particular those coming from the quenched approximation which seems to be worse for nucleons than for mesons), we are planning to double the number of configurations in collaboration with M.L. Paciello and B. Taglienti.

ACKNOWLEDGEMENTS

The computations described in this paper were performed using the CRAY-XMP at the Rutherford-Appleton Laboratory, and used about 180 hours of CPU time. We are grateful to the SERC for a research grant which made these computations possible, and to the INFN for some financial support.

REFERENCES

- [1] E.D. Bloom et al., Phys. Rev. Lett. 23 (1969) 930;
M. Briedenbach et al., Phys. Rev. Lett. 23 (1969) 935.
- [2] K. Wilson, "New Phenomena in Subnuclear Physics", Erice 1975, A. Zichichi ed., (Plenum Press, New York, 1977).
- [3] G. Martinelli and C.T. Sachrajda, CERN preprint TH.4909/87 (1987), to be published in Nucl. Phys. B.
- [4] G. Martinelli and C.T. Sachrajda, Phys. Lett. B196 (1987) 181.
- [5] M.B. Gavela et al., CERN preprint TH.4905/87 (1987), to be published in Nucl. Phys. B.
- [6] R. Gupta et al., Phys. Lett. 164B (1985) 347.
- [7] J. Ashman et al., CERN preprint EP/87-230 (1987), to be published in Phys. Lett. B.
- [8] J.D. Bjorken and S.D. Drell, "Relativistic Quantum Mechanics" (McGraw Hill, 1965).
- [9] M. Fukujita, in Proceedings of the 1987 International Symposium on Lattice Gauge Theory: Lat '87 (Seillac), Nucl. Phys. B (Proc. Suppl.) 4 (1988).
- [10] L.H. Karsten and J. Smit, Nucl. Phys. B183 (1981) 103.
- [11] M. Bochicchio et al., Nucl. Phys. B262 (1985) 445.
- [12] B. Meyer and C. Smith, Phys. Lett. 123B (1983) 62.
- [13] G. Martinelli and Y.C. Zhang, Phys. Lett. 123B (1983) 77.
- [14] R. Groot, J. Hoek and J. Smit, Nucl. Phys. B237 (1984) 111.
- [15] L. Maiani and G. Martinelli, Phys. Lett. 168B (1986) 265.
- [16] M.B. Gavela et al., CERN preprint TH.4930/87 (1987), to be published in Phys. Lett. B.
- [17] G. Martinelli and Y.C. Zhang, Phys. Lett. 125B (1983) 9.
- [18] A.S. Kronfeld and D.M. Photiadis, Phys. Rev. D31 (1985) 2939.
- [19] G. Häfner et al., Nucl. Phys. B114 (1976) 505 and references therein.
- [20] L. Maiani et al., Nucl. Phys. B293 (1987) 420.
- [21] S. Güsken et al., CERN preprint TH.5027/88 (1988).

FIGURE CAPTIONS

Fig. 1 : Two-point correlation function $K(t_x)$.

Fig. 2 : Three-point correlation function $C(t_x, t_y)$.

Fig. 3 : The "generalized quark propagators" Σ_u and Σ_d .

Fig. 4 : A comparison of our results with the experimental results of Ref. [19]. The solid line corresponds to the vector meson dominance approximation [Eq. (3.16)]. The dashed line is the best fit to our results [Eq. (3.17)].

Fig. 1

Fig. 2

$$\Sigma_u(y, 0; t_x, p) = \sum_{\underline{x}} e^{ip \cdot \underline{x}}$$

$$\Sigma_d(y, 0; t_x, p) = \sum_{\underline{x}} e^{ip \cdot \underline{x}}$$

Fig. 3

Fig. 4