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ON THE RADIATIVE CORRECTIONS TO THE ELECTROWEAK PARAMETERS
AND PRECISION TESTS OF THE ELECTROWEAK THEORY

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ABSTRACT

The radiative corrections to the electroweak parameters are reconsidered with an emphasis on analyzing prospects for future tests of the as yet untested parts of the electroweak theory, in particular the "new physics" of vector-boson self-interactions and the Higgs scalar. The vacuum polarization due to the light fermions is treated in the leading-log approximation, while the top-quark is taken into account exactly. A detailed analysis of the errors involved in our approximations shows that vacuum polarization due to bosons is negligible, if $m_H = 100$ GeV, while it may become visible in precision tests in e^+e^- annihilation, if $m_H \approx 1$ TeV. We also give detailed results (as a function of the top-quark mass) on the radiatively corrected parameters used in model-independent fits to neutrino-scattering and in the interpretation of atomic-parity violation experiments. Technically, we diagonalize the γ -Z propagator for any q^2 , and we show, when treating the top-quark vacuum polarization exactly, that the intuitively appealing notion of running coupling constants can be used beyond the leading-log approximation.

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1. Introduction

In a recent paper¹⁾ (hereafter referred to as (I)) on the radiative corrections to the electroweak parameters, we suggested discriminating between the radiative corrections which arise from the empirically well-known vector-boson fermion interactions on the one hand, and the additional radiative corrections which are induced by the "new physics" of the empirically unknown vector-boson self-interactions (VBSI) and the couplings of the vector-bosons to the Higgs scalar on the other hand.

Such a discrimination is necessary, since VBSI may become strong at high energies and may thus become perturbatively incalculable. This happens within the standard $SU(2)_L \times U(1)_Y$ electroweak theory²⁾ for the self-interactions of the longitudinal vector-bosons, if the mass of the Higgs scalar lies above a lower bound of approximately^{3,4)} 1 TeV. Quite generally, a breakdown of perturbation theory for the boson self-interactions is expected to occur in any theory containing a new strong interaction at a high-energy scale, e.g., in technicolor⁵⁾ and composite models.

On the other hand, the contribution of the vector-boson fermion interactions to the radiative corrections of the electroweak parameters is perturbatively calculable by itself. It is independent of the VBSI and the mass of the Higgs scalar, and it is even independent of the notion of spontaneous symmetry breaking, as the vector-boson fermion interactions can credibly be derived within the framework of a globally $SU(2)_{WI}$ invariant vector-boson theory broken by electromagnetism via W -dominance^{6,7)}. Or else, the radiative corrections due to vector-boson ($\gamma, W^\pm, Z^0$) fermion interactions will remain the same in any theory which is consistent with the empirically established neutral current structure and the vector-boson masses.

Within the $SU(2)_L \times U(1)_Y$ standard electroweak theory, for Higgs masses of $m_H \approx 100$ GeV, i.e., excluding extremely small or large values of m_H , the vector-boson fermion interactions, in particular the fermion-loop corrections to the propagators, dominate⁸⁾ the radiative corrections to the electroweak parameters to such an extent¹⁾ that additional corrections, in particular the boson-loop contributions to the propagators, become negligible. Larger deviations from the values of the electroweak parameters corrected by radiative effects due to vector-boson fermion interactions will occur for extreme values of m_H (for $m_H \approx 10$ GeV or $m_H \approx 1$ TeV), or as a consequence of "new physics" beyond the standard model. The magnitude of these deviations set the scale¹⁾ for the accuracy of

future precision experiments at the SLC and at LEP, if these are to quantitatively test the theory beyond the empirically known vector-boson fermion interactions. It seems almost needless to stress that such a quantitative test would be particularly desirable in view of the fact that the renormalizability of the $SU(2)_L \times U(1)_Y$ theory is intimately connected with the as yet untested interactions within the boson sector of the electroweak theory.

The purpose of the present paper is threefold:

- i) To improve on the results of (I) as regards the treatment of the top-quark vacuum polarization. In (I) we used approximate formulae for $m_t \ll M_W$ and $m_t \gg M_W$ ⁹⁾ to obtain a simple interpolation which, however, is somewhat inaccurate for masses of the t-quark, m_t , in the range of $50 \text{ GeV} \lesssim m_t \lesssim 100 \text{ GeV}$. Within the present paper the t-quark vacuum polarization will be treated exactly.
- ii) To give a detailed and more refined analysis of the errors involved in our treatment of the radiative corrections. We will thus be able to reaffirm our previous conclusion¹⁾ that the results of our approximate and fairly simple treatment of the radiative corrections coincide within negligible errors with the results of the much more involved full one-loop calculations obtained within the $SU(2)_L \times U(1)_Y$ theory for a mass of the Higgs scalar of the order of $m_H \approx 100 \text{ GeV}$.
- iii) To present simple formulae for the radiatively corrected parameters which are relevant in neutrino-scattering and atomic-physics parity-violation experiments. Numerical values for these parameters will be given as a function of the mass of the t-quark.

In Section 2 we will present our detailed analysis of the radiative corrections to the vector-boson mass-formulae, deferring all technicalities to the Appendix. In Section 3, we will give the radiatively corrected parameters relevant for neutrino scattering and the atomic-physics parity-violation experiments. Final conclusions will be drawn in Section 4.

2. The Radiatively Corrected Vector-Boson Mass Formulae

Using the conventional parameter¹⁰⁾

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2}, \quad (2.1)$$

where M_W and M_Z denote the masses of the charged and neutral weak vector-bosons, the radiatively corrected formula¹⁰⁻¹⁴⁾ for the charged boson mass takes the well-known form

$$M_W = \left(\frac{\alpha(0)}{\sqrt{2}} \frac{\pi}{G_\mu} \right) \frac{1}{s_W^2 (1 - \Delta r)}. \quad (2.2)$$

The electromagnetic fine-structure constant measured in the low momentum transfer limit ($q^2 \rightarrow 0$) has been denoted by $\alpha(0)$, and G_μ denotes the Fermi-coupling measured in μ -decay. The radiative corrections to the Born-term formula are contained in Δr and will be specified below in our approximation of keeping the dominant vacuum polarization due to lepton and quark loops only. Combining (2.1) and (2.2) we obtain

$$M_Z = \left(\frac{\alpha(0)}{\sqrt{2}} \frac{\pi}{G_\mu} \right) \frac{1}{s_W^2 c_W^2 (1 - \Delta r)}. \quad (2.3)$$

Besides $\alpha(0)$ and G_μ , the Z-boson mass, M_Z , will be the most precisely known parameter⁸⁾ occurring in (2.2) and (2.3) as soon as Z production has been measured at LEP. It has thus become customary to use $\alpha(0)$, G_μ and M_Z as input in order to predict s_W^2 from (2.3) and subsequently determine M_W from (2.1) or (2.2). Direct measurements of M_W as well as measurements of s_W^2 in neutrino scattering and in e^+e^- annihilation at the Z peak will then yield a test of the radiatively corrected theory.

As mentioned in (I) and in the Introduction of the present paper, we will concentrate on the (dominating) contribution to Δr which arises from the vacuum polarization of the leptons and quarks on the vector-boson propagation (see Fig. 1). Evaluating the vacuum polarization due to the leptons and the (light) u, d, c, s, and b quarks in the leading-log approximation¹⁵⁻¹⁷⁾, but treating the vacuum polarization due to the (very massive) top quark exactly, one obtains (compare (A 53) in the Appendix, to which we refer for the technical details of the calculation)

$$\begin{aligned}
 1 - \Delta r = & \frac{\alpha(0)}{\alpha(M^2)} + \frac{c_W^2}{s_W^2} \frac{\sqrt{2} G_\mu}{8\pi^2} [m_t^2 (F_R(\gamma_Z) - \frac{1}{2}) \\
 & + \frac{M_Z^2}{3} (3 - 4s_W^2) \ln \left(\frac{m_t^2}{M^2} \right) \\
 & + m_t^2 (1 - \frac{s_W^2}{c_W^2}) (\frac{1}{2} + \frac{1}{\gamma_W} + (\frac{1}{\gamma_W^2} - 1) \ln |\gamma_W - 1|) \\
 & + \frac{M_Z^2}{3} (6(1 - 2s_W^2)(1 - \frac{1}{\gamma_W}) \ln |\gamma_W - 1| - \frac{32s_W^2}{\gamma_Z} + 8s_W^2(1 + \frac{2}{\gamma_Z}) F_R(\gamma_Z) \\
 & - 3F_R(\gamma_Z))] \quad . \quad (2.4)
 \end{aligned}$$

The electromagnetic coupling constant measured at the scale M^2 , where $M \approx M_W \approx M_Z$, has been denoted by $\alpha(M^2)$ in (2.4). Its numerical value depends on the magnitude of the fermion masses, in particular on the mass of the top quark, m_t . The ratio of $\alpha(0)/\alpha(M^2)$ in (2.4) is given by (compare (A 46) and (A 48))

$$\frac{\alpha(0)}{\alpha(M^2)} = 1 - \frac{\alpha(0)}{3\pi} \left(\sum_{i \neq t} Q_i^2 \ln \left(\frac{M^2}{m_i^2} \right) - \frac{4}{3} T(0, \gamma_Z) \right) \quad , \quad (2.5)$$

where the summation runs over all leptons and quarks of charges Q_i and masses m_i with the exception of the top quark which contributes the term $-(4/3) T(0, \gamma_Z)$, where

$$T(0, \gamma_Z) = \frac{4}{\gamma_Z} - (1 + \frac{2}{\gamma_Z}) F_R(\gamma_Z) + \frac{5}{3} \quad . \quad (2.6)$$

Here, as well as in (2.4), we use the definitions

$$\gamma_Z = \frac{M_Z^2}{m_t^2} \quad , \quad (2.7a)$$

and

$$\gamma_W = \frac{M_W^2}{m_t^2} \quad . \quad (2.7b)$$

As regards the function $F_R(\gamma) \equiv \text{Re } F(\gamma)$ appearing in (2.4) and (2.6), we only note the limiting cases of $F_R(\gamma_Z \rightarrow \infty)$ and $F_R(\gamma_Z \rightarrow 0)$ which are given by

$$F_R(\gamma_Z) \approx \ln \gamma_Z \quad \text{for} \quad \gamma_Z \gg 4 \quad (\text{i.e., } m_t \ll M_Z/2) \quad , \quad (2.8a)$$

and

$$F_R(\gamma_Z) \approx 2 - \frac{\gamma_Z}{6} \quad \text{for} \quad \gamma_Z \ll 4 \quad (\text{i.e., } m_t \gg \frac{M_Z}{2}) \quad (2.8b)$$

and refer to (A25) for the complete definition of $F_R(\gamma_Z)$. Using (2.8) we imme-

diately obtain from (2.6)

$$-\frac{4}{3} T(0, \gamma_Z) = \begin{cases} \frac{4}{3} \ln \frac{M_Z^2}{m_t^2} - \frac{20}{9} \approx \frac{4}{3} \ln \frac{M^2}{m_t^2} , & \text{for } m_t \ll M_Z/2 , \\ 0 , & \text{for } m_t \gg M_Z/2 , \end{cases} \quad (2.9)$$

i.e., we recover the leading logarithm also for the top-quark contribution to $\alpha(0)/\alpha(M^2)$ in the limit of $m_t \rightarrow 0$ as well as a vanishing top-quark contribution to this ratio for $m_t \rightarrow \infty$,

$$\frac{\alpha(0)}{\alpha(M^2)} = \begin{cases} 1 - \frac{\alpha(0)}{3\pi} \sum_i Q_i^2 \ln \frac{M^2}{m_i^2} , & \text{for } m_t \ll M_Z/2 , \\ 1 - \frac{\alpha(0)}{3\pi} \sum_{i \neq t} Q_i^2 \ln \frac{M^2}{m_i^2} , & \text{for } m_t \gg M_Z/2 . \end{cases} \quad (2.10)$$

This result coincides with (3.5) in (I).

Returning to $(1 - \Delta r)$ in (2.4), we note that for $m_t \rightarrow 0$ ($\gamma_{Z,W} \rightarrow \infty$) the first and the third term within the big bracket go to zero individually, while the second term and the fourth term also combine to zero, yielding

$$1 - \Delta r \Big|_{m_t \rightarrow 0} = \frac{\alpha(0)}{\alpha(M^2)} \Big|_{m_t \rightarrow 0} , \quad (2.11)$$

where the right-hand side is given in (2.10). For $m_t \rightarrow \infty$, the third and fourth terms in (2.4) yield a contribution which is independent of m_t and negligible compared with the remaining first and second terms which increase quadratically and logarithmically with m_t , leaving us with

$$1 - \Delta r \Big|_{m_t \rightarrow \infty} = \frac{\alpha(0)}{\alpha(M^2)} \Big|_{m_t \rightarrow \infty} + \frac{c_W^2}{s_W^2} \frac{\sqrt{2} G_\mu}{24\pi} \left(\frac{9}{2} m_t^2 + M_Z^2 (3 - 4s_W^2) \ln \frac{m_t^2}{M^2} \right). \quad (2.12)$$

These results, (2.11) and (2.12), coincide with the expressions used for the numerical evaluation in (I) (compare (3.4) in (I)) where the limiting cases of $m_t \rightarrow 0$ and $m_t \rightarrow \infty$ were used to construct a simple interpolation for the region of $50 \text{ GeV} \lesssim m_t \lesssim 100 \text{ GeV}$. Having established the connection of the expression for $1 - \Delta r$ with our previous result¹⁾, we now turn to the numerical evaluation of (2.4), which contains the top-quark contribution exactly.

For the numerical evaluation of (2.4) we put

$$G_\mu = 1.16634 \times 10^{-5} \text{ GeV}^{-2} , \quad (2.13a)$$

and

$$\alpha(0)^{-1} = 137.035963 . \quad (2.13b)$$

For the quark masses, as in (I), we use the standard values of¹⁸⁾

$$\begin{aligned} m_u &= m_d = 0.1 \text{ GeV} , \\ m_s &= 0.3 \text{ GeV} , \\ m_c &= 1.5 \text{ GeV} , \\ m_b &= 4.5 \text{ GeV} , \end{aligned} \quad (2.14)$$

which were adjusted to describe the observed thresholds in e^+e^- annihilation into hadrons reasonably well. With the scale

$$M = \frac{1}{2} (M_W + M_Z) \quad (2.15)$$

we evaluate the contribution of the light quarks in (2.5) to obtain

$$\alpha(M)^{-1} = 128 + \frac{4\alpha(0)}{9\pi} T(0, \frac{M^2}{m_t^2}) . \quad (2.16)$$

Substituting this result in (2.4) and evaluating subsequently (2.3) and (2.1) for various choices of M_Z and m_t we arrive at the results for s_W^2 and M_W shown in the third and fourth columns of Tables 1 and 2. For subsequent reference we also quote our result for the contribution of the light quarks to Δr , which is given by

$$\Delta r|_{\text{had}} = 0.0304 . \quad (2.17)$$

A test and a refinement of the above procedure of using the quark masses (2.14) when calculating the hadronic vacuum-polarization may be obtained by employing dispersion relations and inserting directly the data for e^+e^- annihilation into hadrons. This calculation has been carried out by Jegerlehner¹⁹⁾ (and others²⁰⁾) who obtains*

$$\Delta r|_{\text{had}} = 0.0326 \pm 0.0007 . \quad (2.18)$$

The difference between (2.17) and (2.18) amounts to a small correction to the quark-loop result for $\Delta r|_{\text{had}}$ which is given by

$$\delta(\Delta r|_{\text{had}}) = 0.0022 \pm 0.0007 . \quad (2.19)$$

It implies a shift in M_W and s_W^2 of

* We note that this result corresponds to using, e.g., $m_u = m_d = m_s \approx 60 \text{ MeV}$ instead of the masses given in (2.14).

$$\begin{aligned}\delta M_W &\approx -40 \pm 10 \text{ MeV} , \\ \delta s_W^2 &\approx +0.0008 \pm 0.0002 .\end{aligned}\quad (2.20)$$

The exact value of this shift slightly depends on the input used for M_Z and m_t . The results for M_W and s_W^2 obtained upon applying the hadronic correction (2.19) are shown in the fifth and sixth columns of Table 1 and Table 2.

For a meaningful comparison of our results for M_W and s_W^2 with the ones obtained in complete one-loop calculations, it will be essential to analyse the accuracy of our above calculations in detail:

i) Comparing the vacuum-polarization contribution of the leptons evaluated in the leading-log approximation with the results of an exact calculation, one finds that the leading-log results for M_W and s_W^2 should be corrected by approximately $\delta M_W \approx -20 \text{ MeV}$ and $\delta s_W^2 \approx +0.0004$, respectively. Each of the lepton doublets contributes an equal amount to this correction.

ii) As to the vacuum polarization of the quarks, the dispersion theory analysis entering our final results eliminates all errors previously introduced by effectively describing the hadronic vacuum polarization in terms of simple quark loops in the leading-log approximation with the quark-mass parameters (2.14). The remaining error in the hadronic contribution to Δr in (2.18) is due to the experimental data used as an input. It implies uncertainties in M_W and s_W^2 which are given by $\delta M_W \approx \pm 10 \text{ MeV}$ and $\delta s_W^2 \approx \mp 0.0002$, respectively.

iii) A different source of error, also related to the fermion-loops, is connected with the choice (2.15) of the scale, M , in (2.4) which defines the renormalization point or, in other words, the mass scale at which the theoretical quantities are identified with the measured ones. Within an isolated treatment of the loop-corrections to the $W^\pm$ propagator, the scale, M , may be conveniently and most naturally identified with $M = M_W$. In the γ -Z case, however, the choice $M = M_Z$ seems more appropriate. According to (2.15) we have chosen $M = (1/2)(M_W + M_Z)$ for our numerical evaluation. Changing this value of M to become $M = M_W$ or $M = M_Z$ induces a difference of $\delta M_W \approx +30 \text{ MeV}$ and $\delta M_W \approx -20 \text{ MeV}$, respectively, with corresponding changes in s_W^2 of $\delta s_W^2 \approx -0.0006$ and $\delta s_W^2 \approx +0.0004$. The variation of $\alpha(M^2)$ with M^2 which leads to these differences of $\delta M_W \approx +30 \text{ MeV}$ or $\delta M_W \approx -20 \text{ MeV}$ is of the order of magnitude of a typical higher-order effect, as

$$\alpha(M_W^2) - \alpha(M_Z^2) = \frac{1}{3\pi} \left(\sum_i Q_i^2 \right) \alpha(M_W^2) - \alpha(M_Z^2) = \frac{20}{9\pi} \alpha(M_W^2) - \alpha(M_Z^2) . \quad (2.21)$$

Uncertainties of the order of $\delta M_W \approx \pm 30 \text{ MeV}$ and $\delta s_W^2 \approx \mp 0.0006$ (which within our

approach are induced by different choices of the scale, M) are consequently not specific to our treatment, but should be present in any one-loop calculation of the radiative corrections to the electroweak parameters.

iv) An additional uncertainty in our predictions for M_W and s_W^2 derives from the neglect of various box- and vertex-diagrams when extracting the Fermi coupling*

$$G_\mu = \frac{\pi}{\sqrt{2}} \frac{\alpha_W(0)}{M_W^2(0)} \quad , \quad (2.22)$$

from the measured μ -decay life-time. While the empirical value of G_μ in (2.13a) was corrected for the photon-exchange contribution shown in Fig. 2a, in our calculation the additional genuine weak corrections corresponding to the diagrams of Figs. 2b-d were discarded.** This may be justified by noting that in processes with momentum transfer $|q^2| \ll M_W^2$ the electromagnetic corrections (which in this case have the magnitude of about α/π) must be much larger than the weak corrections***, the neglected purely weak corrections thus satisfying

$$|\delta M_W| \ll \frac{\alpha}{2\pi} M_W \approx 100 \text{ MeV} \quad , \quad (2.23)$$

and correspondingly

* This basic identification (2.22) implies the radiatively corrected mass formula (2.2) upon relating $M_W(0)$ to the physical W mass, M_W , by taking into account vacuum-polarization effects (compare (A18)).

** We remark in this connection that a gauge-dependent but universal contribution is understood to be subtracted from the vertex diagrams in Figs. 2c-d and included in the bosonic self-energy diagram in Fig. 2e. Thus the sum of the contributions from Figs. 2b-d, as well as the contribution of Fig. 2e, become separately gauge-invariant within the $SU(2)_L \times U(1)_Y$ theory (see, e.g., Kennedy and Lynn²¹).

*** We believe this hypothesis on the weak corrections to the low-energy μ -decay process to be a safe one, even though the vertex corrections of Fig. 2d depend on the empirically unknown VBSI and thus involve "new physics" in the terminology of Section 1 of the present paper. The hypothesis has to be reanalysed, if experimental results indicate strong deviations from the present predictions which may in principle be induced by alternative VBSI.

$$|\delta s_W^2| \ll \left| \frac{2M_W \delta M_W}{M_Z^2} \right| \approx 0.002 \quad . \quad (2.24)$$

On the grounds of (2.23) and (2.24) we thus allow for a conservative uncertainty of $\delta M_W \approx \pm 30$ MeV ($\delta s_W^2 \approx \pm 0.0006$) for the neglected box- and vertex-diagrams shown in Figs. 2b-d.

Adding in quadrature the uncertainties ii) to iv) and applying the correction i) which originates from the use of the leading-log approximation for the leptonic-vacuum polarization, we end up with an uncertainty of our results in Tables 1 and 2 of

$$\delta M_W \approx \begin{matrix} +20 \\ -60 \end{matrix} \text{ MeV} \quad (2.25a)$$

and

$$\delta s_W^2 \approx \begin{matrix} -0.0004 \\ +0.0012 \end{matrix} \quad . \quad (2.25b)$$

Apart from the vacuum polarization due to leptons and hadrons, which was taken into account in our calculation of Δr , there exists the bosonic-vacuum polarization of Fig. 2e which contains the empirically unknown VBSI and the interactions with the Higgs scalar. It is dependent on the mass of the Higgs scalar, m_H , and in a sense constitutes that part of the radiative corrections to the electroweak parameters which is closest to the very heart of the renormalizable electroweak theory. Any deviation between the results of our calculations and the experimental data which is larger than the above uncertainty (2.25) within the $SU(2)_L \times U(1)_Y$ theory is to be attributed to this bosonic-vacuum polarization.

Keeping in mind this significance of a deviation between our results and future data, we now compare our values for M_W with the ones obtained in complete one-loop calculations, which take into account the bosonic-vacuum polarization (as well as the negligibly small box- and vertex-diagrams discussed above). The results obtained for M_W by Jegerlehner¹⁹⁾ and by Lynn and Stuart²²⁾ under the assumption of $m_H = 100$ GeV are shown in the 7th column of Tables 1 and 2. As the estimated¹⁹⁾ error of these results is due to the experimental uncertainty in the hadronic input and the neglect of higher order terms, it is of the same order of magnitude, ≈ 40 MeV, as estimated for our results in (2.25). As explicitly shown in the last column of Tables 1 and 2, the results of our calculation, taking into account the vacuum polarization due to the leptons and the hadrons only, are in good agreement within errors with the complete one-loop results for $m_H = 100$ GeV. In other words, the bosonic-vacuum polarization is negligible for

$m_H = 100$ GeV. For extreme values of m_H , however, the bosonic-vacuum polarization increases substantially to imply contributions to M_W which are given by^{19,22)}

$$\begin{aligned} \Delta M_W &\approx -200 \text{ MeV} \quad (\Delta s_W^2 \approx +0.004) \quad \text{for } m_H = 1 \text{ TeV} \quad , \\ \Delta M_W &\approx +120 \text{ MeV} \quad (\Delta s_W^2 \approx -0.002) \quad \text{for } m_H = 10 \text{ GeV} \quad . \end{aligned} \quad (2.26)$$

The magnitude of these radiative effects due to the bosonic-vacuum polarization sets the scale for the accuracy needed in measurements aiming at a quantitative precision test of the electroweak theory beyond the known vector-boson fermion interactions. Comparing the expected accuracies of the future data on s_W^2 to be obtained at LEP (Table 3) with the effects (2.26) induced by the bosonic-vacuum polarization, we conclude that a quantitative measurement of the bosonic-vacuum polarization is only possible in the limit of extreme values of m_H . Almost needless to say, future precision data may reveal other new effects beyond the known vector-boson fermion interactions, which are unrelated to the limit of $m_H \gtrsim 1$ TeV.

In Fig. 3 we show the radiative correction, Δr , as a function of the top-quark mass, m_t , as obtained in our calculation, and, for comparison, the results of the full one-loop calculation of Jegerlehner²³⁾ (compare also Ref. 24) for $m_H = 10$ GeV and $m_H = 1$ TeV*. For comparison we also indicate in Fig. 3 the present value^{25,26)} of Δr and the future experimental errors in Δr calculated from the expected future errors in measuring M_Z and s_W^2 (compare Table 3). Again, we conclude that deviations from the value of Δr obtained by only taking into account the vacuum polarization due to leptons and quarks (hadrons) within the $SU(2)_L \times U(1)_Y$ theory will be unobservable, unless m_H assumes extreme values.

* The results for Δr by Jegerlehner²³⁾ show a slightly steeper m_t -dependence than ours. This is due to QCD corrections to the top-quark contribution.

3. Radiative Corrections to Neutrino Scattering and Atomic Parity Violation Experiments

The treatment of the radiative corrections relevant for neutrino scattering²⁷⁾ largely parallels the one given in (I), the only difference resulting from taking into account the effects due to the top-quark for all values of m_t precisely. In addition, we will give the radiatively corrected values for the standard parameters used in model-independent fits to the neutral-current neutrino scattering data, as well as the radiative corrections appropriate for atomic parity-violation experiments not considered in (I).

In the leading-log approximation for the light quarks the low-energy effective weak interaction including radiative corrections is given by²⁸⁾

$$\begin{aligned} H = & \frac{G_\mu}{\sqrt{2}} \bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \mu \bar{e} \gamma_\lambda (1 - \gamma_5) \nu_e + \text{h.c.} \\ & + \left(1 + \frac{\alpha(M^2)}{2\pi} \ln \left(\frac{M^2}{v_\mu^2}\right)\right) (\bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \mu \bar{u} \gamma_\lambda (1 - \gamma_5) u + \text{h.c.}) \\ & + 2\rho \bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \nu_\mu (j_{L\lambda}^{(3)} - \hat{S}_W^2(q^2) j_\lambda^{\text{em}}) \\ & - \bar{e} \gamma^\lambda \gamma_5 e (c_{1u} \bar{u} \gamma_\lambda u + c_{1d} \bar{d} \gamma_\lambda d) \quad , \end{aligned} \quad (3.1)$$

where in distinction from (I), we have added the last term, which is relevant for the description of atomic parity-violation effects and deep-inelastic polarization asymmetries.

The effective Hamiltonian in (3.1) contains the charged-current neutrino-lepton interaction, the charged-current interaction of neutrinos with u and d quarks, the neutral-current interaction of neutrinos with lepton and quarks^{27,18)}, and a parity violating electron-quark interaction relevant for atomic parity violation and parity violating asymmetries²⁹⁾. The charged-current interaction of neutrinos with quarks contains the well-known photon-exchange correction^{8,28)},

$$f = 1 + \frac{\alpha(M^2)}{2\pi} \ln \left(\frac{M^2}{v_\mu^2}\right) \quad , \quad (3.2)$$

where $v_\mu^2 = \max(|q^2|, m_\mu^2)$ with q^2 denoting the average neutrino momentum transfer in the scattering process. The normalization of the neutral-current interaction is given by the ρ -parameter (compare (A51), (A19))

$$\rho = 1 + \frac{\sqrt{2} G_\mu m_t^2}{8\pi^2} \left\{ \frac{1}{\beta} + \left(\frac{1}{\beta^2} - 1\right) \ln |\beta - 1| + 3F_R(\beta) - 4 - \frac{2q^2}{M_Z^2} (F_R(\gamma_Z) - 2) \right\} \quad , \quad (3.3)$$

where

$$\beta \equiv \frac{q^2}{m_t^2} \quad , \quad (3.4)$$

and γ_Z and $F_R(\gamma)$ have already been defined in (2.7), (2.8) and (A25). One finds that the ρ -parameter in (3.3) is essentially independent of the momentum transfer q^2 in the region of

$$10^{-6} \lesssim |q^2| \lesssim 25 \text{ GeV}^2 \quad , \quad (3.5)$$

which is relevant for the low-energy neutral-current measurements. Moreover, using (2.8), one finds from (3.3)

$$\rho = \begin{cases} 1 & \text{for } m_t \ll M_W \\ 1 + \frac{3\sqrt{2} G_\mu m_t^2}{16\pi^2} & \text{for } m_t \gg M_W \end{cases} \quad , \quad (3.6)$$

in agreement with (2.37) in (I).

The effective weak angle for neutrino processes, $\hat{s}_W^2(q^2)$, in (3.1) is given by

$$\hat{s}_W^2(q^2) = \bar{s}_W^2(q^2) - \frac{\alpha(q^2)}{6\pi\rho} \ln \left(\frac{M^2}{v_\mu^2} \right) \quad , \quad (3.7)$$

and the neutral-current parameters relevant for atomic parity violation experiments by

$$C_{1i} = -\rho \left[\frac{\tau(i)}{2} - 2Q_i \bar{s}_W^2(q^2) \right] - \frac{\rho\alpha(q^2)}{12\pi} (1 - 4\bar{s}_W^2(q^2)) Q_i [9\tau_3^{(i)} \ln \left(\frac{M^2}{v_i^2} \right) - 4 \ln \left(\frac{M^2}{v_e^2} \right)] \quad , \quad (3.8)$$

where $i = u, d$ refers to the quark flavor, while $v_e^2 = \max(m_e^2, |q^2|)$ and $v_i^2 = \max(m_i^2, |q^2|)$. The q^2 -dependent ("running") weak angle, $\bar{s}_W^2(q^2)$ in (3.7) is related to the conventional one, $s_W^2 \equiv 1 - M_W^2/M_Z^2$, via (see (A47) and (A45))

$$\bar{s}_W^2(q^2) = \frac{\alpha(q^2)}{\alpha(M^2)} \{s_W^2 + (1 - s_W^2)(\bar{\rho} - 1) - \alpha(M^2) b_2 A(q^2)\} \quad , \quad (3.9)$$

where $\alpha(q^2)$, the running electromagnetic coupling at the scale $|q^2|$, is given by (see (A46), (A48))

$$\alpha(q^2) = \frac{\alpha(M^2)}{1 - \alpha(M^2) b_Q A(q^2)} \quad (3.10)$$

with

$$b_Q A(q^2) = \frac{1}{3\pi} \sum_{i \neq t} Q_i^2 \left\{ \ln \left| \frac{q^2}{m_i^2} \right| \theta(|q^2| - m_i^2) - \ln \left| \frac{M^2}{m_i^2} \right| \right\} + \frac{9}{4\pi} T(q^2, \gamma_Z) \quad , \quad (3.11)$$

while $b_2 A(q^2)$ in (3.9) reads (compare (A49))

$$b_2 A(q^2) = \frac{1}{12\pi} \sum_{i \neq t} Q_i \tau_3^{(i)} \left\{ \ln \left| \frac{q^2}{m_i^2} \right| \theta(|q^2| - m_i^2) - \ln \left| \frac{M^2}{m_i^2} \right| \right\} + \frac{1}{6\pi} T(q^2, \gamma_Z) \quad (3.12)$$

The function $T(q^2, \gamma_Z)$ is defined by (A32) combined with (A25) and describes the top-quark contribution. Finally, $\bar{\rho}$ in (3.9), obtained to first order in G_μ from (A43) and (A19), is given by

$$\begin{aligned} \bar{\rho} = 1 &+ \frac{\sqrt{2} G_\mu}{8\pi^2} m_t^2 \left\{ \frac{1}{\gamma_W} + \left(\frac{1}{\gamma_W^2} - 1 \right) \ln |\gamma_W - 1| + F_R(\gamma_Z) \right\} \\ &+ \frac{\sqrt{2} G_\mu}{12\pi^2} M_W^2 \left\{ \left(\frac{1}{2c_W^2} + 1 \right) \ln \left(\frac{m_t^2}{M^2} \right) + \frac{1}{2c_W^2} \left[\left(1 + \frac{8}{\gamma_Z} \right) F_R(\gamma_Z) - \frac{16}{\gamma_Z} \right] \right. \\ &\left. + 3 \left(1 - \frac{1}{\gamma_W} \right) \ln |\gamma_W - 1| + \frac{8}{\gamma_Z} - 2 \left(1 + \frac{2}{\gamma_Z} \right) F_R(\gamma_Z) \right\} \quad (3.13) \end{aligned}$$

Using the limiting expression for $F_R(\gamma)$ given by (2.8), one finds

$$\bar{\rho} = \begin{cases} 1 & \text{for } m_t \ll M_W, \\ 1 + \frac{\sqrt{2} G_\mu}{8\pi^2} \left\{ \frac{3}{2} m_t^2 + \frac{M_W^2}{3} \left(\frac{1}{c_W^2} + 2 \right) \ln \frac{m_t^2}{M^2} \right\} & \text{for } m_t \gg M_W. \end{cases} \quad (3.14)$$

in agreement with (2.41) of (I).

The physical meaning of the other terms in (3.7) and (3.8) is as follows. The second term in (3.7) gives the neutrino charge radius contribution to neutrino scattering (Fig. 4a)³⁰⁾, the term proportional to $-4 \ln(M^2/\nu_e^2)$ in (3.8) is due to the axial electric charge radius of the electron (Fig. 4b), and, finally, the term proportional to $9T_3^{(i)} \ln(M^2/\nu_i^2)$ supplies the photon-exchange correction to the parity violating electron-quark interaction (Fig. 4c)²⁸⁾.

Before turning to the discussion of the numerical evaluation of the radiative corrections, we note that in model-independent fits the low-energy neutrino neutral current data, it has become standard to use the effective interaction

$$\begin{aligned} H_{NC} = \frac{G_\mu}{\sqrt{2}} \bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \nu_\mu \left\{ \epsilon_L(u) \bar{u} \gamma_\lambda (1 - \gamma_5) u + \epsilon_R(u) \bar{u} \gamma_\lambda (1 + \gamma_5) u \right. \\ \left. + (u \rightarrow d) + g_V^e \bar{e} \gamma_\lambda e - g_A^e \bar{e} \gamma_\lambda \gamma_5 e \right\} \quad (3.15) \end{aligned}$$

A comparison with the radiatively corrected effective interaction (3.1) yields

$$\epsilon_L(i) = \rho \left\{ \frac{\tau_3^{(i)}}{2} - Q_i \hat{S}_W^2(q^2) \right\} \quad (3.16)$$

$$\epsilon_R(i) = -\rho Q_i \hat{s}_W^2(q^2) \quad , \quad (3.17)$$

$$g_V^e = \rho(-\frac{1}{2} + 2\hat{s}_W^2(q^2)) \quad , \quad (3.18)$$

$$g_A^e = -\frac{\rho}{2} \quad , \quad (3.19)$$

where $i = u$ or d . Besides (3.16) to (3.19) one uses

$$g_L^2 = \epsilon_L^2(u) + \epsilon_L^2(d) \quad , \quad (3.20)$$

$$g_R^2 = \epsilon_R^2(u) + \epsilon_R^2(d) \quad , \quad (3.21)$$

and

$$\theta_L = \text{arctg}(\epsilon_L(u)/\epsilon_L(d)) \quad , \quad \theta_R = \text{arctg}(\epsilon_R(u)/\epsilon_R(d)) \quad . \quad (3.22)$$

Finally, the Born approximation in the various low-energy interactions is obviously obtained by substituting $\rho = f = 1$ and $\hat{s}_W^2(q^2) = \bar{s}_W^2(q^2) = s_B^2$ in (3.1), (3.8) and (3.15-22), where s_B^2 denotes the Weinberg angle in the Born approximation.

Based on the Hamiltonian (3.1), in (I) we have given simple formulae for the radiative correction

$$(\delta s_W^2)_{RC} \equiv s_W^2 - s_B^2 \quad (3.23)$$

to be applied upon extracting the Weinberg angle from neutrino scattering in the Born approximation. We refer to (4.14) and (4.19) in (I) for neutrino-hadron and neutrino-electron scattering, respectively. We have reevaluated these formulae using the above expressions for ρ , $\bar{\rho}$ and $b_2 A(q^2)$, which are now valid for any value of m_t , in contrast to (I), where an interpolation between the limiting cases of $m_t \rightarrow 0$ and $m_t \rightarrow \infty$ was used. The resulting corrections $(\delta s_W^2)_{RC}$ to be applied, when extracting s_W^2 from the well-known ratios R_ν , $R^\pm$ and $\nu_\mu e/\bar{\nu}_\mu e$, experimentally determined in neutrino scattering, are given in Table 4.

In the case of R_ν , complete one-loop calculations by Bardin and Dokuchaeva³¹⁾ and also by Stuart³²⁾ are available. The results of ref. 31) are also shown in Table 4 for comparison. The discrepancy between these results and ours is always less than 0.001 in $(\delta s_W^2)_{RC}$, and thus, our simple formulae are sufficiently accurate for all practical purposes, as experimental errors are typically of the order of 0.005 in s_W^2 . In distinction from (I), our results for R_ν are now also reliable in the threshold region of $40 \text{ GeV} \lesssim m_t \lesssim 100 \text{ GeV}$. As regards $R^\pm$, R^- and the $\nu_\mu e/\bar{\nu}_\mu e$ ratio, the only complete one-loop results available in the literature concern the m_t -dependence studied by Sirlin³³⁾, who gives the difference of $(\delta s_W^2)_{RC}$

evaluated at $m_t = 45$ GeV and at certain other values of m_t . Our results for these differences agree with Sirlin's, as expected, since the t -quark mass has been taken into account exactly in both calculations.

Finally, in Table 4 we also give the numerical results for the radiative corrections to be applied to atomic parity violation experiments

$$\begin{aligned}
 (\delta s_W^2)_{RC} = & \alpha(M^2) \left\{ b_2 A(q^2) + \left[\frac{1}{\alpha(q^2)} - \frac{1}{\alpha(M^2)} \right] s_W^2 \right. \\
 & + \frac{(1 - 4\bar{s}_W^2(q^2))}{24\pi Z} \left[(12A + 3Z) \ln\left(\frac{M^2}{V_1^2}\right) - 4Z \ln\left(\frac{M^2}{V_e^2}\right) \right] \Big\} \\
 & + (\rho - 1) \left(\frac{1}{Z} - \frac{A}{4Z} - s_W^2 \right) - (\bar{\rho} - 1) (1 - s_W^2) . \quad (3.24)
 \end{aligned}$$

The numerical results in Table 4 are given for Cesium (Cs, with $A = 133$ and $Z = 55^*$) and agree with those of Sirlin³³⁾.

Recently, two new model-independent fits^{25,26)} to the low-energy neutral-current data have been presented, which determine the experimental values of the parameters (3.8) describing the parity violating electron-quark interaction, as well as the experimental values of the parameters (3.15) to (3.19) for neutrino interactions with quarks and leptons. The results of the fits are given in Table 5 together with the complete one-loop predictions²⁵⁾ in the $SU(2)_L \times U(1)_Y$ theory based on $s_W^2 = 0.23$, $m_t = 45$ GeV and $m_H = 100$ GeV. In Table 5, we also present our predictions, based on equations (3.8) and (3.16-22), for various values of m_t and for the present experimental value³⁴⁾ of the Z -mass, $M_Z = 91.8 \pm 0.9$ GeV.

On the basis of Table 5, we first observe, that our results for $m_t = 45$ GeV agree with the corresponding ones of ref. 25), derived from a complete one-loop calculation. The results agree with the fits to the experimental data. The results in Tables 4 and 5 thus explicitly show that our approximations are completely adequate for these low-energy processes also. Furthermore, Table 5 elucidates the way, in which fits to the low-energy data provide constraints like^{25,26)}

$$m_t \lesssim 200 \text{ GeV} . \quad (3.25)$$

It seems that the most important role in obtaining the constraint (3.25) is played by the parameter g_L^2 , which can be determined quite accurately from the low-energy data,

* Z is not to be confused with the Z boson, nor with the wave-function renormalization constant, Z , in the Appendix!

while at the same time it also quite sensitively depends on m_t for $m_t \gtrsim 100$ GeV. Thus, the present experimental knowledge on g_L^2 implies $m_t \lesssim 180$ GeV, if $M_Z = 91.8 \pm 0.9$ GeV. A smaller uncertainty in M_Z might reduce this upper bound on m_t .

In conclusion, we reaffirm that a simple treatment of the radiative corrections to neutrino scattering and atomic parity violation experiments based upon the effective Hamiltonian (3.1), which takes into account fermion-loop corrections to the propagators in terms of "running" coupling constants ($\alpha(q^2), \bar{s}^2(q^2)$), as well as photon-exchange corrections, is sufficiently accurate for all practical purposes.

4. Summarizing Conclusions

Within the present paper we reconsidered the radiative corrections to the electroweak parameters, emphasizing the necessity of a distinction between the radiative effects due to the empirically known vector-boson fermion (lepton and quark) interactions and the radiative effects due to the "new physics" of the VBSI and the Higgs scalar. We stress that this distinction is an essential one for the interpretation of future precision tests of the electroweak theory.

Our results may be summarized as follows:

i) Due to the widely different scales set by the masses of the W and Z on the one hand, and the energies relevant in the measurement of α and G_{μ} on the other hand, the radiative corrections to the electroweak parameters are dominated by vacuum-polarization effects in the vector-boson propagators, apart from additional photon-exchange corrections. Vacuum-polarization is dominated by the leptons and quarks. In our calculation we treated all leptons and quarks in the leading-log approximation, apart from the top-quark, which was taken into account exactly. A detailed error analysis and a comparison with calculations taking into account hadronic effects, and correcting for such effects, shows and improves the reliability of our procedure. Within the $SU(2)_L \times U(1)_Y$ theory, additional radiative corrections can only be due to vacuum-polarization effects of the vector-bosons and the Higgs scalar. Our detailed comparison with the full one-loop results (Tables 1 and 2 and Fig. 3) shows that such effects are completely negligible as long as $m_H \approx 100$ GeV. For $m_H \approx 1$ TeV, bosonic vacuum-polarization leads to a shift in the prediction for M_W of the order of -200 MeV (corresponding to a shift of 0.004 in s_W^2), to be compared with an estimated uncertainty in our results, as well as in complete one-loop calculations, of the order of less than 60 MeV ($\lesssim 0.0012$ in s_W^2).

ii) The above-mentioned possible magnitude of the bosonic vacuum-polarization sets the scale for the accuracy needed in order to test the electroweak theory beyond the known vector-boson fermion interactions. Depending on the magnitude of m_H , the precision experiments at LEP may be able to quantitatively establish a difference between the theoretical results obtained without and with the inclusion of the bosonic vacuum-polarization. Nevertheless, as regards precision tests of the electroweak theory, the situation is very much different from the situation in low-energy QED precision experiments, such as measurements of the Lamb-shift, where the high accuracy allows for a quantitative isolation of the lepton vacuum-polarization contribution of -27 Mc to the total shift of 1058 Mc.

Future precision experiments at LEP may nevertheless establish discrepancies from the results of the present calculation which originate from entirely new phenomena, not incorporated within the $SU(2)_L \times U(1)_Y$ electroweak theory.

iii) On the basis of the simple effective Hamiltonian (3.1), which takes into account photon-exchange corrections besides vacuum-polarization effects in the propagators, we also derived simple and intuitively satisfactory formulae for the radiatively corrected parameters relevant for neutrino scattering and atomic-physics parity-violation experiments. The effect of varying m_t has been explicitly displayed (Table 4). A comparison with complete one-loop calculations shows that our approximate treatment is sufficiently accurate for all practical purposes.

iv) Technically, as in (I), we differ from previous work in our treatment of the γ -Z propagator matrix, which is diagonalized for any q^2 , in conjunction with the use of "running" coupling constants and masses. Moreover, using the notion of "running" coupling constants in an exact treatment of the vacuum-polarization of the top-quark, we showed that the use of "running" coupling constants is by no means tied in with the leading-log approximation. The concept of a "running" coupling constant is meaningful within a more general context.

Appendix: The W and γZ Propagators

In this Appendix we calculate the corrections to the W and γZ propagators arising from lepton and quark loops (Fig. 1). Compared with the scale, M, set by the W- and Z-masses, $M \approx M_W \approx M_Z$, all leptons and quarks can be considered as very light, except for the top-quark. The top-quark will thus be treated exactly^{*}, when evaluating loop-corrections to the propagator, while the leading-log approximation as in (I), will be employed for all the other fermions.

Within the present context only the transverse parts of the W- and Z-propagators are of relevance. Dropping the transverse polarization tensor,

$$T_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \quad (A1)$$

the inverse of the unrenormalized W-propagator may be written as

$$-i(\Delta_W^{(0)}(q^2))^{-1} = q^2 (Z_W^{-1} + \Pi_W^{(0)}(q^2)) - \tilde{M}_W^{(0)2} - \mu_W^{(0)2}(q^2)_y + i(\text{imag. part}). \quad (A2)$$

Here, as well as below, the index (o) denotes unrenormalized quantities. The functions $\Pi_W^{(0)}(q^2)$ and $\mu_W^{(0)}(q^2)$ are real, as their imaginary parts have been separated off in (A2). The choice of the (physical) W-mass, M_W , as renormalization scale^{**}, M, yields

$$\Pi_W^{(0)}(M_W^2) = \mu_W^{(0)2}(M_W^2) = 0. \quad (A3a)$$

Moreover, $\Pi_W^{(0)}(q^2)$ is chosen such that

$$\lim_{q^2 \rightarrow 0} q^2 \Pi_W^{(0)}(q^2) = 0, \quad (A3b)$$

while

$$\mu^{(0)2}(q^2 \rightarrow 0) \neq 0. \quad (A3c)$$

The structure of the propagator in (A2) differs^{***} from the expression given in (I) by the m_t -dependent additive term $\mu_W^{(0)2}(q^2)$. In the limiting case of

^{*}) The mass of the b quark will nevertheless be neglected in the $t\bar{b}(\bar{t}b)$ -loop contributing to the W propagator.

^{**}) Actually, we will use a common scale, $M \approx (M_W + M_Z)/2$, for both the W- and Z-propagators. Different choices of M, e.g. $M \equiv M_W^Z$ or $M = M_Z$, amount to higher order corrections (compare the discussion in the main text).

^{***}) By subtraction and addition of $\mu_W^{(0)2}(q^2=0)$ in (A2) and appropriate rearrangement of the propagator in (A2) may be cast into the form given in (I), i.e. without an additive q^2 dependent contribution. This procedure amounts to a different choice for the renormalization scheme. We find it convenient to work with the form (A2).

$m_t \rightarrow 0$, the function $\mu_W^{(0)^2}(q^2)$ vanishes in accordance with (I), while for $m_t \gg M$ the function $\mu_W^{(0)^2}(q^2)$ approaches an m_t -independent function which becomes negligible (also in accordance with (I)) compared with the leading m_t -dependent contribution to $\tilde{M}_W^{(0)^2}$, which rises as m_t^2 (compare (A13b) below).

In terms of the variables

$$\beta = \frac{q^2}{m_t^2} \quad (A4a)$$

and

$$\gamma_W = \frac{M_W^2}{m_t^2} \quad (A4b)$$

the wave-function renormalization-constant for the W in (A2) is given by

$$Z_W^{-1} = 1 + \frac{\alpha_W^{(0)}}{3\pi} \left\{ b_1' \left[\ln \frac{\Lambda^2}{M^2} + \frac{5}{3} \right] + \frac{3}{4} \left[\ln \left(\frac{\Lambda^2}{m_t^2} \right) + \frac{5}{3} - \left(1 - \frac{1}{\gamma_W} \right) \ln |\gamma_W - 1| \right] \right\} \quad (A5)$$

where $\alpha_W^{(0)}$ is the unrenormalized coupling of the W, i.e.

$$\alpha_W^{(0)} = g_W^{(0)^2}/4\pi,$$

$$b_1' = \frac{n}{4} = \frac{9}{4}, \quad (A6)$$

n being the number of fermion doublets other than (t,b) and Λ denotes the ultraviolet cut-off. The first square bracket in (A5) contains the contribution of the light fermions, while the second one originates from the (t,b)-loop. The potentially large constant, 5/3, is kept in (A5), in order to explicitly verify that it does not yield any contribution to physically measurable quantities (see below).

From (A2) we obtain the renormalized propagator

$$\begin{aligned} -i(\Delta_W(q^2))^{-1} &= -iZ_W(\Delta_W^{(0)}(q^2))^{-1} \\ &= q^2(1 + \Pi_W(q^2)) - M_W^2 - \mu_W^2(q^2) + i(\text{imag. part}). \end{aligned} \quad (A7)$$

It is to be used in conjunction with the renormalized W-coupling at the scale M ,

$$\alpha_W(M^2) = Z_W \alpha_W^{(0)}. \quad (A8)$$

The (physical) W mass, M_W , (defined by the pole of the W propagator) is related to the unrenormalized mass in (A2) via

$$M_W^2 = Z_W \tilde{M}_W^{(0)2} \equiv Z_W M_W^{(0)2} + m_t^2 t_W, \quad (A9)$$

where

$$\begin{aligned} m_t^2 t_W &\equiv m_t^2 Z_W t_W^{(0)} = \\ &= m_t^2 \frac{3\alpha_W(M^2)}{8\pi} \left\{ \ln \frac{\Lambda^2}{m_t^2} + \frac{1}{3} \left[2 + \frac{1}{\gamma_W} + \left(\frac{1}{\gamma_W^2} - 1 \right) \ln |\gamma_W - 1| \right] \right\} \end{aligned} \quad (A10)$$

In agreement with (A5) in (I), $m_t^2 t_W$ vanishes for $m_t \rightarrow 0$, while

$$m_t^2 t_W = \frac{3\alpha_W(M^2)}{8\pi} m_t^2 \left(\ln \left(\frac{\Lambda^2}{m_t^2} \right) + \frac{1}{2} \right) \quad \text{for } m_t \gg M_W \quad (A11)$$

Finally, the renormalized vacuum-polarization functions in (A7) are given by

$$\begin{aligned} \Pi_W(q^2) &= Z_W \Pi_W^{(0)}(q^2) \\ &= - \frac{\alpha_W(M^2)}{3\pi} \left\{ \bar{b}_1' \ln \left| \frac{q^2}{M^2} \right| + \frac{3}{4} \left[\left(1 - \frac{1}{\beta} \right) \ln |\beta - 1| - \left(1 - \frac{1}{\gamma_W} \right) \ln |\gamma_W - 1| \right] \right\} \end{aligned} \quad (A12)$$

and

$$\begin{aligned} \mu_W^2(q^2) &= Z_W \mu_W^{(0)2}(q^2) \\ &= \frac{\alpha_W(M^2)}{8\pi} m_t^2 \left\{ \frac{1}{\beta} + \left(\frac{1}{\beta^2} - 1 \right) \ln |\beta - 1| - \frac{1}{\gamma_W} - \left(\frac{1}{\gamma_W^2} - 1 \right) \ln |\gamma_W - 1| \right\} \end{aligned} \quad (A13a)$$

The second term within the curly bracket in the expression (A12) for $\Pi_W(q^2)$ is due to the t-quark vacuum-polarization. In accordance with (I), this term vanishes for $m_t \rightarrow \infty$ and turns into the usual leading-log form for $m_t \rightarrow 0$. From (A13a) one finds the above-mentioned dependence of $\mu_W^2(q^2)$ on the t-quark

mass, m_t , namely

$$\mu_W^2(q^2) \rightarrow \begin{cases} 0 & \text{for } m_t \rightarrow 0, \\ \frac{\alpha_W(M^2)}{24\pi} (q^2 - M_W^2) & \text{for } m_t \rightarrow \infty. \end{cases} \quad (\text{A13b})$$

Both, $\Pi_W(q^2)$ as well as $\mu_W^2(q^2)$, vanish for $q^2 = M^2$ (compare (A3a)).

We add a brief remark on the relation between $\alpha_W(M^2)$, which is the W-coupling to be used in conjunction with the W-propagator (A7), and the coupling of the W-boson experimentally measured as the residue of the pole of the W-propagator multiplied by $\alpha_W(M^2)$. As the residue of the pole of the W-propagator deviates from 1, the physical coupling^{*} defined by the residue of the pole of the propagator, $\alpha_{W,\text{res.}}(M_W^2)$, deviates from $\alpha_W(M_W^2)$ by a correction factor given by

$$\alpha_{W,\text{res.}}(M_W^2) = \frac{\alpha_W(M_W^2)}{1 + M_W^2 \Pi_W'(M_W^2) - (\mu_W^2)'(M_W^2)}, \quad (\text{A14})$$

where Π_W' and $(\mu_W^2)'$ denote the derivatives of the corresponding functions evaluated at $q^2 = M_W^2$.

The propagator (A7) may now be rewritten by introducing a "running" coupling constant,

$$\alpha_W(q^2) = \frac{\alpha_W(M^2)}{1 + \Pi_W(q^2)}, \quad (\text{A15})$$

as well as a "running" W-mass,

$$M_W^2(q^2) = (M_W^2 + \mu_W^2(q^2)) \frac{\alpha_W(q^2)}{\alpha_W(M^2)}, \quad (\text{A16})$$

where, of course, $M_W^2(M^2) \equiv M_W^2$. One obtains

$$\alpha_W(M_W^2) \Delta_W(q^2) = \frac{-i\alpha_W(q^2)}{q^2 - M_W^2(q^2) + i(\text{imag. part})} \quad (\text{A16a})$$

The running coupling constant $\alpha_W(q^2)$ in (A15) acquires the usual leading-log form, if one evaluates $\Pi_W(q^2)$ for the limiting cases of $m_t \rightarrow 0$ and $m_t \rightarrow \infty$. As it stands, (A15) generalizes the notion of a "running coupling constant" from a concept valid within the leading-log approximation to a concept of more general significance^{**}).

^{*}) We disregard vertex corrections, which have to be taken into account, however, if higher order corrections are to be taken into account when extracting the coupling constant from data.

^{**}) Similar ideas were recently presented by Kennedy and Lynn²¹⁾.

Using (A16), we now may relate the Fermi-coupling, G_μ , measured in μ -decay, to the mass of the W. Noting that^{*})

$$\frac{\sqrt{2}}{\pi} G_\mu = \frac{\alpha_W(0)}{M_W^2(0)}, \quad (A18)$$

one finds from (A16) that

$$\frac{\sqrt{2}}{\pi} G_\mu = \frac{\alpha_W(M^2)}{M_W^2} \left(1 - \frac{\sqrt{2}G_\mu}{\pi} f_W\right)^{-1}, \quad (A19)$$

where

$$f_W \equiv \frac{\mu_W^2(0)}{\alpha_W(M^2)} = -\frac{m_t^2}{8\pi} \left(\frac{1}{2} + \frac{1}{\gamma_W} + \left(\frac{1}{\gamma_W^2} - 1\right) \ln|\gamma_W - 1|\right). \quad (A20)$$

The factor proportional to f_W in (A19) provides a small correction (not present in the approximation given in (I)) to $\alpha_W(M^2)$ for given G_μ and M_W . One finds $f_W = 0$ for $m_t = 0$, while $f_W = -M_W^2/12\pi$ for $m_t \rightarrow \infty$ and $\sqrt{2}G_\mu f_W/\pi = -(3/16)\alpha_W$ for $m_t = M_W$. According to (A16) and (A20), this correction may be traced back to the term $\mu_W^2(q^2)$ in the propagator (A2) to start with. As mentioned, this additional term in the propagator may be avoided by choosing a different renormalization scheme which leads to changes in Z entering (A40) and thus to a small change in Δr , in agreement with the general result that physical observables in different renormalization schemes differ by terms of the magnitude of higher order corrections^{**}).

The q^2 -dependent width of the W may be obtained from

$$\text{im}(-i \Delta_W^{-1}(q^2)) \equiv M_W(q^2) \Gamma_W(q^2). \quad (A21)$$

The imaginary part of the W-propagator, not explicitly shown in (A7), gives

$$\Gamma_W(q^2) = \frac{\alpha_W(M^2)}{M_W} \left\{ b' q^2 + \theta(q^2 - m_t^2) \left[\frac{3}{4} q^2 + \frac{3}{8} m_t^2 \left(\frac{m_t^4}{q^4} - 3 \right) \right] \right\} \quad (A22)$$

where all fermion masses, except m_t , were neglected. Notice that $\Gamma_W(q^2)$ is approximately proportional to q^2 .

Turning to the γZ propagator, we introduce the variable

^{*}) QED corrections are taken into account, when extracting G_μ from the data, and other corrections are negligible (compare the discussion in the main text).

^{**}) See ref. 35 for a discussion of cases where physical results may be affected by changes in the renormalization scheme.

$$\gamma_Z = \frac{M_Z^2}{m_t^2}, \quad (A23)$$

as well as the constants

$$\begin{aligned} \bar{b}_1 &= \frac{1}{8} \sum_{i \neq t} (\tau_3^i)^2 \\ \bar{b}_2 &= \frac{1}{4} \sum_{i \neq t} Q_i \tau_3^i \\ \bar{b}_Q &= \sum_{i \neq t} Q_i^2 \end{aligned} \quad (A24)$$

where τ_3^i is equal to +1 and -1 for a fermion of weak isospin +1/2 and -1/2, respectively, and Q_i is the charge of the i'th fermion. Finally, we also need the function

$$F(\beta) = \begin{cases} 2 \left(\frac{4}{\beta} - 1\right)^{1/2} \operatorname{artg} \left(\frac{4}{\beta} - 1\right)^{-1/2} & \text{for } 0 \leq \beta \leq 4 \\ \left(1 - \frac{4}{\beta}\right)^{1/2} \left\{ -i\pi \theta(\beta-4) + \ln \left| \frac{1+\sqrt{1-4/\beta}}{1-\sqrt{1-4/\beta}} \right| \right\} & \text{for } \beta < 0 \text{ or } \beta > 4, \end{cases} \quad (A25a)$$

the real part of which,

$$F_R(\beta) = \operatorname{Re} F(\beta), \quad (A25b)$$

satisfies

$$F_R(\beta) \approx 2 - \frac{\beta}{6} \quad \text{for } 0 \leq \beta \ll 4, \quad (A26a)$$

and

$$F_R(\beta) \approx \ln \beta \quad \text{for } \beta \gg 4. \quad (A26b)$$

The unrenormalized γZ propagator, in analogy to (A2), is written as

$$\begin{aligned} -i(\Delta_{\gamma Z}^{(0)})^{-1} &= q^2(Z^{-1} + \Pi_{\gamma Z}^{(0)}(q^2)) - M_{\gamma Z}^{(0)^2} - \mu_{\gamma Z}^{(0)}(q^2) \\ &+ i(\operatorname{imag. part}), \end{aligned} \quad (A27)$$

where the 2×2 matrices $M_{\gamma Z}^{(0)^2}$ and $\mu_{\gamma Z}^{(0)}(q^2)$ have non-vanishing elements

only for the Z-Z transition. As in (I), and in analogy with (A3), we have

$$\Pi_{\gamma Z}^{(0)}(M_Z^2) = \mu_{\gamma Z}^{(0)^2}(M_Z^2) = 0 \quad (A28)$$

As in (I), $\Pi_{\gamma Z}^{(0)}(q^2)$ is chosen to include the leading log terms, while the dominant contribution of a very massive top-quark is absorbed in $M_{\gamma Z}^{(0)^2}$. The limiting behaviour of $\Pi_{\gamma Z}^{(0)}(q^2)$ and $M_{\gamma Z}^{(0)^2}$ for $m_t \rightarrow 0$ and $m_t \rightarrow \infty$ does not uniquely determine the distribution of the various contributions to the propagator (A27) among the additive terms. Thus we may choose for $\Pi_{\gamma Z}^{(0)}(q^2)$ an expression, which is formally identical to the one given in (I), allowing us to use many results of (I) also within the present analysis.

We write

$$\Pi_{\gamma Z}^{(0)}(q^2) = -\alpha^{(0)} A(q^2) \begin{pmatrix} b_Q & \frac{b_2 - s_W^{(0)^2} b_Q}{s_W^{(0)} c_W^{(0)}} \\ \frac{b_2 - s_W^{(0)^2} b_Q}{s_W^{(0)} c_W^{(0)}} & \frac{b_1 + s_W^{(0)^4} b_Q - 2s_W^{(0)^2} b_2}{(s_W^{(0)} c_W^{(0)})^2} \end{pmatrix} \quad (A29)$$

where $s_W^{(0)}$ and $c_W^{(0)}$ denote the unrenormalized values for $\sin\theta_W$ and $\cos\theta_W$, and

$$A(q^2) = \frac{1}{3\pi} \ln \left| \frac{q^2}{M^2} \right|. \quad (A30)$$

The quantities b_1 , b_Q and b_2 differ from the corresponding barred quantities defined in (A24) by the t-quark contribution,

$$\begin{aligned} b_1 &= \bar{b}_1 + \frac{3}{8} \frac{T(q^2, \gamma_Z)}{3\pi A(q^2)}, \\ b_2 &= \bar{b}_2 + \frac{1}{2} \frac{T(q^2, \gamma_Z)}{3\pi A(q^2)}, \\ b_Q &= \bar{b}_Q + \frac{4}{3} \frac{T(q^2, \gamma_Z)}{3\pi A(q^2)}, \end{aligned} \quad (A31)$$

with

$$T(q^2, \gamma_Z) = -\frac{4}{\beta} + \frac{4}{\gamma_Z} + \left(1 + \frac{2}{\beta}\right) F_R(\beta) - \left(1 + \frac{2}{\gamma_Z}\right) F_R(\gamma_Z), \quad (A32)$$

where the definitions (A4), (A23) and (A25) were used. As expected, $T(q^2, \gamma_Z)/(3\pi A(q^2))$ becomes equal to 1 in the limit of $m_t \rightarrow 0$ and vanishes for $m_t \rightarrow \infty$ (compare (A26)), thus implying that $\pi_{YZ}^{(0)}(q^2)$ in (A29) for these limiting cases agrees with the expression in (I). The wave-function renormalization matrix Z also retains its previous form ¹⁾

$$Z^{-1} = \begin{pmatrix} Z_Y^{-1} & \frac{s_W^{(0)}}{c_W^{(0)}} \left(\frac{1}{Z_{W2}} - \frac{1}{Z_Y} \right) \\ \frac{s_W^{(0)}}{c_W^{(0)}} \left(\frac{1}{Z_{W2}} - \frac{1}{Z_Y} \right) & \frac{s_W^{(0)^2}}{c_W^{(0)^2} \left(\frac{1}{Z_Y} - \frac{2}{Z_{W2}} \right) + \frac{1}{c_W^{(0)^2} Z_{W1}}} \end{pmatrix} \quad (A33)$$

where we now have

$$Z_Y^{-1} = 1 + \frac{\alpha_W^{(0)}}{3\pi} \left\{ \bar{b}_Q \left[\ln\left(\frac{\Lambda^2}{M^2}\right) + \frac{5}{3} \right] + \frac{4}{3} \left[\ln\left(\frac{\Lambda^2}{m_t^2}\right) + \frac{5}{3} + \frac{4}{\gamma_Z} - \left(1 + \frac{2}{\gamma_Z}\right) F_R(\gamma_Z) \right] \right\} \quad (A34)$$

$$Z_W/Z_{W2} = 1 + \frac{\alpha_W(M^2)}{12\pi} \left\{ \ln\left(\frac{m_t^2}{M^2}\right) + 3\left(1 - \frac{1}{\gamma_W}\right) \ln|\gamma_W - 1| + \frac{8}{\gamma_Z} - 2\left(1 + \frac{2}{\gamma_Z}\right) F_R(\gamma_Z) \right\} \quad (A35)$$

$$Z_W/Z_{W1} = 1 + \frac{\alpha_W(M^2)}{8\pi} \left\{ \ln\left(\frac{m_t^2}{M^2}\right) - F_R(\gamma_Z) + 2\left(1 - \frac{1}{\gamma_W}\right) \ln|\gamma_W - 1| \right\} \quad (A36)$$

In (A34), the term proportional to $\bar{b}_Q$ gives the contribution due to the light quarks, while the other term originates from the t-quark. The structure of Z^{-1} is similar to the structure of Z_W^{-1} given in (A5). As only the ratios (A35) and (A36) enter observable quantities, the potentially large terms proportional to 5/3 in (A34) become irrelevant, as they drop out in the ratios (A35) and (A36). In the limit of $m_t \rightarrow 0$, these ratios become 1, while for $m_t \gg M_W$ they are dominated by $\ln(m_t^2/M^2)$.

Returning to (A27), we note that

$$M_{YZ}^{(0)^2} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}_Z^{(0)^2} \end{pmatrix} \quad (A37a)$$

$$\mu_{YZ}^{(0)^2}(q^2) = \frac{1}{Z_W c_W^{(0)^2}} \begin{pmatrix} 0 & 0 \\ 0 & \mu_Z^2(q^2) \end{pmatrix} \quad (A38a)$$

where

$$\tilde{M}_Z^{(0)2} = \frac{1}{c_W^{(0)2}} \{M_W^{(0)2} + m_t^2 \frac{t_Z}{Z_W}\} \quad (A37b)$$

with

$$m_t^2 t_Z = m_t^2 t_W - \frac{\alpha_W(M^2)}{8\pi} m_t^2 \left\{ F_R(\gamma_Z) + \frac{1}{\gamma_W} + \left(\frac{1}{\gamma_W^2} - 1 \right) \ln |\gamma_W - 1| \right\} \quad (A37c)$$

and^{*})

$$\mu_Z^2(q^2) = \frac{\alpha_W(M^2)}{8\pi} m_t^2 \left\{ F_R(\gamma_Z) \left(1 + \frac{2q^2}{M_Z^2} \right) - 3 F_R(\beta) + 4 - \frac{4q^2}{M_Z^2} \right\}. \quad (A38b)$$

We remark that $\Pi_{\gamma Z}^{(0)}(q^2)$ in (A29) as well as $\mu_Z^2(q^2)$ in (A38) correctly vanish for $q^2 = M_Z^2$, as required by (A28).

The renormalization of the γ -Z propagator proceeds by diagonalizing the matrix Z in (A35), making use of the non-orthogonal transformation given in Appendix B of (I). Thus, the electromagnetic coupling at the scale M is given by

$$\alpha(M^2) = Z_\gamma \alpha^{(0)}, \quad (A39)$$

while the weak angle, $\bar{s}_W^{-2}$, defined by the coupling of the Z to the electromagnetic current, j_μ^{em} , at the scale M is given by

$$\bar{s}_W^2 \equiv \bar{s}_W^2(M^2) = \frac{\alpha(M^2)}{\alpha_W(M^2)} \frac{Z_W}{Z_{W2}}. \quad (A40)$$

As before¹⁾, the term Z_W/Z_{W2} generates a breaking of the tree-level relation among $\bar{s}_W^2$, $\alpha(M^2)$ and $\alpha_W(M^2)$ which is induced by the t-quark. The renormalized Z-coupling satisfies

$$\alpha_Z(M^2) = \frac{\alpha(M^2)}{\bar{s}_W^2 \bar{c}_W^2 d} \quad (A41)$$

with¹⁾

$$d = 1 + \left(\frac{Z_{W2}}{Z_{W1}} - 1 \right) / \bar{c}_W^2 \quad (A42)$$

^{*}) As expected, $\mu_Z^2(q^2)$ vanishes for $m_t \rightarrow 0$ and remains small for all m_t and q^2 , thus having the same behaviour as $\mu_W^2(q^2)$.

describing, through (A35) and (A36), the t-quark induced breaking of the tree-level formula.

Finally, we find

$$\begin{aligned}
 \bar{\rho} &\equiv \frac{M_W^2}{\bar{c}_W^2 M_W^2} = \frac{m_t^2}{M_W^2} (t_W - t_Z) + \frac{Z_W}{Z_{W2}} + \frac{1}{\bar{c}_W^2} \left(\frac{Z_W}{Z_{W1}} - \frac{Z_W}{Z_{W2}} \right) = \\
 &= 1 + \frac{\alpha_W(M^2)}{8\pi} \frac{m_t^2}{M_W^2} \left\{ \frac{1}{\gamma_W} + \left(\frac{1}{\gamma_W^2} - 1 \right) \ln|\gamma_W - 1| + F_r(\gamma_Z) \right\} + \\
 &+ \frac{\alpha_W(M^2)}{12\pi} \left\{ \left(-\frac{1}{2\bar{c}_W^2} + 1 \right) \ln\left(\frac{m_t^2}{M^2}\right) + \frac{1}{2\bar{c}_W^2} \left[\left(1 + \frac{8}{\gamma_Z}\right) F_r(\gamma_Z) - \frac{16}{\gamma_Z} \right] + \right. \\
 &\left. + 3\left(1 - \frac{1}{\gamma_W}\right) \ln|\gamma_W - 1| + \frac{8}{\gamma_Z} - 2\left(1 + \frac{2}{\gamma_Z}\right) F_r(\gamma_Z) \right\} ,
 \end{aligned} \tag{A43}$$

where (A37c), (A35) and (A36) were used. For a light t-quark $\bar{\rho} \rightarrow 1$, as expected. The quantity $\bar{\rho}$ allows us to relate $\bar{s}_W^2$, defined by the coupling of the physical Z boson, to the quantity

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2} \tag{A44}$$

introduced by Sirlin¹⁰⁾. The relation between $\bar{s}_W^2$ and s_W^2 reads

$$s_W^2 = \bar{s}_W^2 - \bar{c}_W^2(\bar{\rho} - 1) . \tag{A45}$$

We now turn to the calculation of the running couplings and masses. For this purpose we need the renormalized expression for $\Pi_{\gamma Z}(q^2)$ which can be calculated from $\Pi_{\gamma Z}^{(0)}(q^2)$ given in (A29). As expected, $\Pi_{\gamma Z}(q^2)$ is formally also identical to the corresponding expression given in (2.15) of (I), provided we use (A31) for the b-parameters, and (A42) for the parameter d. This fact immediately implies that the running equations for $\alpha(q^2)$ and $\bar{s}_W^2(q^2)$ retain their form given in (I), namely^{*)}

$$\alpha(q^2) = \frac{\alpha(M^2)}{1 - \alpha(M^2) b_Q A(q^2)} , \tag{A46}$$

$$\bar{s}_W^2(q^2) = \frac{\alpha(q^2)}{\alpha(M^2)} \{ \bar{s}_W^2(M^2) - \alpha(M^2) b_2 A(q^2) \} . \tag{A47}$$

*) It is important to note here that the diagonalization of the γ -Z propagator at any q^2 assures that the (running) photon at $q^2=0$ does not mix with the Z. We have therefore the usual (intuitive) separation between the electromagnetic and weak interactions at low energies, in spite of the fact that we have renormalized the γ -Z propagator at M_Z through (A28).

The appropriate thresholds for the light fermions and the correct contribution of the t -quark may easily be taken into account in (A46), (A47) by substituting

$$b_Q A(q^2) \rightarrow \frac{1}{3\pi} \sum_{i \neq t} Q_i^2 \left\{ \ln \left| \frac{q^2}{m_t^2} \right| \theta(|q^2| - m_t^2) - \ln \left| \frac{M^2}{m_t^2} \right| \right\} +$$

$$+ \frac{4}{9\pi} T(q^2, \gamma_Z), \quad (A48)$$

$$b_2 A(q^2) \rightarrow \frac{1}{12\pi} \sum_{i \neq t} Q_i T_3^{(i)} \left\{ \ln \left| \frac{q^2}{m_t^2} \right| \theta(|q^2| - m_t^2) - \ln \left| \frac{M^2}{m_t^2} \right| \right\} +$$

$$+ \frac{1}{6\pi} T(q^2, \gamma_Z), \quad (A49)$$

where $T(q^2, \gamma_Z)$ has been defined by (A32) and (A25).

The ratio of $\alpha_Z(q^2)/M_Z^2(q^2)$, which in the approximation used in (I) is constant, is found to slightly vary with q^2 , in analogy to the corresponding result for the W given by (A16). Noting that the ratio of the neutral to charged current interactions at low energies is determined by the parameter $\rho(q^2)$ defined by

$$\frac{\alpha_Z(q^2)}{M_Z^2(q^2)} \equiv \rho(q^2) \frac{\alpha_W(q^2)}{M_W^2(q^2)}, \quad (A50)$$

we find

$$\rho(q^2) = 1 + \frac{\alpha_W(M^2)}{8\pi} \frac{m_t^2}{M_W^2} \left\{ \frac{1}{\beta} + \left(\frac{1}{\beta^2} - 1 \right) \ln |\beta - 1| + 3F_R(\beta) - \right.$$

$$\left. - 4 - \frac{2q^2}{M_Z^2} (F_R(\gamma_Z) - 2) \right\}, \quad (A51)$$

where β , γ_Z and the function F_R are defined by (A4a), (A23) and (A25), respectively.

The Z -width may be obtained from the imaginary part of the inverse propagator, in analogy to (A21,22). We find

$$\Gamma_Z(q^2) = \frac{\alpha_Z(M^2)}{3M_Z} \left\{ q^2 (b_1 + s_W^4 b_Q - 2s_W^2 b_2) + \left[\frac{3}{8} (q^2 - m_t^2) + q^2 s_W^2 \left(\frac{4}{3} s_W^2 - 1 \right) \right] \right.$$

$$\left. \left(\frac{4m_t^2}{q^2} - 1 \right)^{1/2} \theta(q^2 - 4m_t^2) \right\} \quad (A52)$$

where the first term within the curly bracket gives the contribution of the light fermions, while the second term gives the t-quark contribution. We see that the Z-width is approximately proportional to q^2 . This affects the Z-line shape and implies a correction³⁶⁾ of a few tens of MeV when extracting M_Z from the data.

Finally, (2.1) and (2.2) may be combined with (A19), (A40) and (A45), to give

$$1 - \Delta r = \frac{\alpha(0)}{\alpha(M^2)} + f_W \frac{\sqrt{2}G_\mu}{\Pi} + \left(\frac{Z_{W2}}{Z_W} - 1 \right) + \frac{c_W^2}{s_W^2} (\bar{\rho} - 1). \quad (A53)$$

Substituting (A20), (A35) and (A43) in (A53) yields the expression (2.4) for $1 - \Delta r$ in the main text.

Table 1: Our results for M_W and $s_W^2 \equiv \sin^2 \theta_W$ in the dominant fermion-loop approximation of the present paper compared with the results of the complete one-loop calculation of ref. 19).

m_t (GeV)	M_Z (GeV)	M_W (GeV) (Quark-loops)	s_W^2	M_W (GeV) (hadr. corr.)	s_W^2	M_W (GeV) ($m_H = 100$ GeV, ref. 19)	ΔM_W (MeV)
30	88	75.72	0.2596	75.67	0.2606		
	90	78.31	0.2428	78.27	0.2437	78.25(3)	-20 ± 30
	92	80.83	0.2281	80.79	0.2288	80.77(2)	-20 ± 20
	94	83.29	0.2149	83.25	0.2156	83.23(2)	-20 ± 20
	96	85.70	0.2031	85.67	0.2037	85.64(2)	-30 ± 20
60	88	75.71	0.2598	75.66	0.2607		
	90	78.29	0.2432	78.25	0.2441	78.26(4)	10 ± 40
	92	80.80	0.2287	80.76	0.2294	80.76(4)	0 ± 40
	94	83.24	0.2157	83.21	0.2164	83.20(4)	-10 ± 40
	96	85.64	0.2041	85.61	0.2047	85.60(4)	-10 ± 40
90	88	75.92	0.2557	75.87	0.2567		
	90	78.51	0.2391	78.46	0.2399	78.48(3)	20 ± 30
	92	81.02	0.2245	80.98	0.2253	80.99(3)	10 ± 30
	94	83.47	0.2116	83.43	0.2122	83.44(3)	10 ± 30
	96	85.87	0.1999	85.83	0.2006	85.84(3)	10 ± 30
120	88	76.09	0.2523	76.05	0.2532		
	90	78.69	0.2356	78.64	0.2364	78.67(4)	30 ± 40
	92	81.20	0.2209	81.17	0.2217	81.19(4)	20 ± 40
	94	83.66	0.2078	83.63	0.2085	83.64(4)	10 ± 40
	96	86.07	0.1961	86.04	0.1967	86.05(3)	10 ± 30
150	88	76.29	0.2485	76.24	0.2494		
	90	78.88	0.2318	78.84	0.2326	78.88(5)	40 ± 50
	92	81.40	0.2171	81.36	0.2178	81.40(5)	40 ± 50
	94	83.87	0.2040	83.83	0.2047	83.86(4)	30 ± 40
	96	86.28	0.1922	86.25	0.1928	86.27(4)	20 ± 40

Table 2: Our results for M_W and $s_W^2 \equiv \sin^2 \theta_W$ in the dominant fermion-loop approximation of the present paper compared with the results of the complete one-loop calculation of ref. 22).

m_t (GeV)	M_Z (GeV)	M_W (GeV)	s_W^2	M_W (GeV)	s_W^2	M_W (GeV)	s_W^2	ΔM_W (MeV)	Δs_W^2
		(Quark-loops)		(hadr. corr.)		($m_H = 100$ GeV, ref. 21)			
30	88	75.72	0.2596	75.67	0.2606				
	90	78.31	0.2428	78.27	0.2437	78.29	0.2434	+20	-0.0003
	92	80.83	0.2281	80.79	0.2288	80.80	0.2287	+10	-0.0001
	94	83.29	0.2149	83.25	0.2156	83.26	0.2155	+10	-0.0001
	96	85.70	0.2031	85.67	0.2037	85.66	0.2038	-10	+0.0001
60	88	75.71	0.2598	75.66	0.2607				
	90	78.29	0.2432	78.25	0.2441	78.27	0.2436	+20	-0.0005
	92	80.80	0.2287	80.76	0.2294	80.78	0.2291	+20	-0.0003
	94	83.24	0.2157	83.21	0.2164	83.22	0.2162	+10	-0.0002
	96	85.64	0.2041	85.61	0.2047	85.62	0.2046	+10	-0.0001
90	88	75.92	0.2557	75.87	0.2567				
	90	78.51	0.2391	78.46	0.2399	78.47	0.2398	+10	-0.0001
	92	81.02	0.2245	80.98	0.2253	80.98	0.2253	0	0.0000
	94	83.47	0.2116	83.43	0.2122	83.43	0.2123	0	+0.0001
	96	85.87	0.1999	85.83	0.2006	85.82	0.2008	-10	+0.0002
130	88	76.16	0.2511	76.11	0.2520				
	90	78.75	0.2344	78.71	0.2352	78.70	0.2353	-10	+0.0001
	92	81.27	0.2197	81.23	0.2204	81.22	0.2207	-10	+0.0003
	94	83.73	0.2066	83.69	0.2073	83.67	0.2076	-20	+0.0003
	96	86.14	0.1949	86.11	0.1955	86.08	0.1959	-30	+0.0004
180	88	76.51	0.2440	76.47	0.2449				
	90	79.11	0.2274	79.07	0.2282	79.06	0.2284	-10	+0.0002
	92	81.63	0.2127	81.59	0.2135	81.57	0.2138	-20	+0.0003
	94	84.09	0.1997	84.06	0.2003	84.04	0.2007	-20	+0.0004
	96	86.51	0.1879	86.48	0.1885	86.46	0.1889	-20	+0.0004
230	88	76.96	0.2353	76.91	0.2361				
	90	79.55	0.2187	79.51	0.2195	79.52	0.2194	+10	-0.0001
	92	82.07	0.2041	82.04	0.2048	82.04	0.2047	0	-0.0001
	94	84.54	0.1911	84.51	0.1917	84.52	0.1916	+10	-0.0001
	96	86.96	0.1793	86.93	0.1799	86.95	0.1797	+20	-0.0002

Table 3: The present and the expected future experimental errors on the Z-mass, M_Z , the Weinberg-angle, s_W^2 , as well as the radiative-correction factor Δr .

	δM_Z (GeV)	δs_W^2	$\delta \Delta r$
Present (ref. 25, 26)	± 1.80	± 0.0060	± 0.037
LEP, unpol ⁸⁾	± 0.05	± 0.0010	± 0.003
LEP, pol ⁸⁾	± 0.02	± 0.0004	± 0.001

Table 4: The radiative correction, $(\delta s_W^2)_{RC} = s_W^2 - s_B^2$, to be applied to s_B^2 , where s_B^2 is the weak angle extracted from neutrino scattering by using the Born approximation. The radiative correction is given for neutrino-hadron (R_ν , R^+ , R^-) and neutrino-electron scattering as a function of the top-quark mass, m_t . We assume $M_Z = 91.8$ GeV, which together with α and G_μ fixes s_W^2 as shown in the Table. As for the momentum transfer, we assumed $|q^2| = 20$ GeV² for neutrino-hadron and $0.01 \lesssim |q^2| \lesssim 0.5$ GeV² for neutrino-electron scattering. Results for $(\delta s_W^2)_{RC}$ appropriate for the atomic parity violation observed in Cs are also given ($|q^2| \simeq (2-3) \times 10^{-6}$ GeV²).

m_t (GeV)	s_W^2	R_ν ($B = 5/9 \times 1.39$)		R^+	R^-	$\nu_\mu e / \bar{\nu}_\mu e$	Cs
		Present Calculation	Ref. 31	$B = 10/9$	$B = 0$		$A = 133$ $Z = 55$
		$(\delta s_W^2)_{RC}$	$(\delta s_W^2)_{RC}$	$(\delta s_W^2)_{RC}$	$(\delta s_W^2)_{RC}$	$(\delta s_W^2)_{RC}$	$(\delta s_W^2)_{RC}$
45	0.2303	-0.008	-0.009	-0.010	-0.005	-0.002	-0.006
60	0.2301	-0.007	-0.008	-0.009	-0.004	-0.002	-0.006
90	0.2259	-0.009	-0.010	-0.011	-0.007	-0.005	-0.009
120	0.2223	-0.010	-0.011	-0.011	-0.009	-0.008	-0.013
180	0.2141	-0.010	-0.011	-0.009	-0.011	-0.014	-0.021
240	0.2036	-0.010	-0.011	-0.007	-0.014	-0.022	-0.030

Table 5: Model-independent fits to the neutral current parameters compared with our predictions (3.16) to (3.22) and (3.8). Upper and lower values correspond to the mass of the Z-boson, $M_Z = 91.8 \pm 0.9$ GeV.

Average $ q^2 $ (GeV ²)		Model Indep. Fits		$m_t = 45$ GeV $s_W^2 = 0.23$ (ref. 25)	Our predictions for various values of m_t (GeV) and $M_Z = 91.8 \pm 0.9$ GeV.							
		ref. 25	ref. 26		30	45	60	90	120	180	240	
20	$\epsilon_L(u)$	0.3390 ± 0.0170	0.3562 ± 0.0192	0.345	0.346 +0.004 -0.004	0.346 +0.004 -0.004	0.346 +0.004 -0.004	0.348 +0.004 -0.004	0.349 +0.004 -0.004	0.352 +0.004 -0.004	0.357 +0.004 -0.004	
	$\epsilon_L(d)$	-0.4290 ± 0.0140	-0.4162 ± 0.0172	-0.427	-0.423 +0.002 -0.002	-0.423 +0.002 -0.002	-0.424 +0.002 -0.002	-0.424 +0.002 -0.002	-0.426 +0.002 -0.002	-0.429 +0.002 -0.002	-0.433 +0.002 -0.002	
	$\epsilon_R(u)$	-0.1720 ± 0.0140	-0.1617 ± 0.0271	-0.152	-0.154 +0.004 -0.004	-0.154 +0.004 -0.004	-0.154 +0.004 -0.004	-0.154 +0.004 -0.004	-0.153 +0.004 -0.004	-0.153 +0.004 -0.004	-0.152 +0.004 -0.004	
	$\epsilon_R(d)$	-0.0110 ± 0.0810 -0.0510	0.0639 ± 0.0551	0.076	0.077 +0.002 -0.002	0.077 +0.002 -0.002	0.077 +0.002 -0.002	0.077 +0.002 -0.002	0.077 +0.002 -0.002	0.076 +0.002 -0.002	0.076 +0.002 -0.002	
	g_L^2	0.2996 ± 0.0044	0.3001 ± 0.0020	0.301	0.299 +0.004 -0.005	0.299 +0.004 -0.005	0.299 +0.004 -0.005	0.301 +0.004 -0.005	0.303 +0.004 -0.005	0.308 +0.004 -0.005	0.315 +0.004 -0.005	
	g_R^2	0.0298 ± 0.0038	0.0302 ± 0.0013	0.029	0.030 +0.002 -0.001	0.030 +0.002 -0.001	0.030 +0.002 -0.001	0.030 +0.002 -0.001	0.029 +0.001 -0.002	0.029 +0.001 -0.002	0.029 +0.001 -0.002	
	θ_L	2.4700 ± 0.0400		2.46	2.46 +0.01 -0.01	2.46 +0.01 -0.01	2.46 +0.01 -0.01	2.46 +0.01 -0.01	2.46 +0.01 -0.01	2.45 +0.01 -0.01	2.45 +0.01 -0.01	
	θ_R	4.6500 ± 0.3200		5.18	5.18	5.18	5.18	5.18	5.18	5.18	5.18	
0.01	g_A^e	-0.4980 ± 0.0270	-0.5010 ± 0.0290	-0.503	-0.500	-0.500	-0.501	-0.501	-0.502	-0.505	-0.509	
	g_V^e	-0.0440 ± 0.0360	-0.0460 ± 0.0290	-0.045	-0.038 +0.012 -0.012	-0.037 +0.012 -0.012	-0.038 +0.012 -0.012	-0.040 +0.011 -0.013	-0.041 +0.011 -0.013	-0.046 +0.012 -0.012	-0.053 +0.012 -0.012	
	C_{1u}	-0.2490 ± 0.0710	-0.1810 ± 0.0560	-0.191	-0.185 +0.008 -0.008	-0.184 +0.007 -0.009	-0.185 +0.008 -0.009	-0.186 +0.008 -0.008	-0.188 +0.007 -0.009	-0.192 +0.008 -0.008	-0.198 +0.008 -0.008	
	C_{1d}	0.3810 ± 0.0640	0.3270 ± 0.0500	0.340	0.342 +0.004 -0.003	0.341 +0.003 -0.004	0.342 +0.003 -0.004	0.343 +0.003 -0.004	0.344 +0.004 -0.004	0.348 +0.004 -0.003	0.352 +0.003 -0.004	

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Figure Captions

Figure 1: Fermion-loop diagrams contributing to the W and γ - Z propagators.

Figure 2: Diagrams contributing to μ -decay in the approximation of vanishing electron and muon masses. The blob in diagram 2e denotes the gauge-boson and Higgs contribution to the W -propagator.

Figure 3: The radiative-correction factor, Δr , as a function of the top-quark mass, m_t , in the dominant fermion-loop approximation of the present paper (full lines) compared with the complete one-loop calculation in the $SU(2)_L \times U(1)_Y$ theory for several values of the Higgs mass²³⁾ ($M_Z = 92$ GeV). The curve (a) is the result of taking into account a calculation of quark-loops, while (b) is based on the e^+e^- annihilation data and thus involves strong-interaction corrections. The present value^{25,26)} of Δr , (1), as well as the experimental errors in Δr expected for future precision experiments at LEP without polarization of the electrons, (2), and with polarization, (3), are also indicated (compare also Table 3).

Figure 4: (a) The neutrino charge-radius contribution to neutrino scattering from leptons and quarks.
 (b) The axial electron charge-radius contribution to electron scattering from leptons and quarks.
 (c) Photon-exchange correction to electron scattering.

Figure 1

(a)

(b)

(c)

(d)

(e)

Figure 2

Figure 3

Figure 4