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AN INVESTIGATION
OF THE PHASE STRUCTURE
OF THE TWO-DIMENSIONAL $O(2)$ AND $O(4)$ SYMMETRIC
NONLINEAR SIGMA-MODELS

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Abstract: We investigate the phase structure of the two-dimensional $O(2)$ and $O(4)$ lattice σ -models by means of a Monte Carlo simulation using Binder's phenomenological renormalization group. For the $O(2)$ model the transition temperature β_c^{-1} is estimated and the expected line of critical points is found. For the $O(4)$ model no signal of a phase transition is found in the range of couplings considered. This is in contradiction to a recent claim that the $O(4)$ model has a critical point at finite β .

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For many years the two-dimensional ($2D$) $O(N)$ -symmetric nonlinear σ -models have served as a testing ground [1, 2] for ideas about four-dimensional gauge theories, because they share several important properties; in particular asymptotic freedom [1, 3] in the case of nonabelian symmetry groups. (For other results on $2D$ spin systems and arguments suggesting analogies between $2D$ spin systems and $4D$ gauge theories see [4].)

In a recent paper Patrascioiu et al. [5] question the conventional picture one has about the phase structure of the $2D$ $O(N)$ nonlinear σ -models. In a companion paper Seiler et al. [6] present results of Monte Carlo (MC) calculations for the $O(2)$ and $O(4)$ σ -models. From fits to their data they conclude that both models have ordinary critical points (i.e. with power singularities) at finite couplings. According to their claims there would be no qualitative difference between the abelian and the nonabelian model — contrary to the conventional picture that in the $O(2)$ model there is the Kosterlitz-Thouless (KT) transition* [7] whereas the $O(4)$ model is asymptotically free [1, 3] and is expected to have only one (massive) phase for all positive couplings. In view of the above-mentioned analogies between $2D$ spin systems and $4D$ gauge theories Patrascioiu et al. [5] speculate that a first-order deconfining phase transition takes place in four-dimensional QCD.

The claims of [6] were, however, contradicted by a recent Monte Carlo Renormalization Group (MCRG) analysis of the $O(4)$ model by Heller [9]. His results do not show any critical behaviour at finite coupling but are consistent with asymptotic freedom. In turn mainly two objections were raised against these results by Patrascioiu et al. [10]: Heller's measurements might suffer from critical slowing down and the MCRG methods might be unreliable due to the introduction of uncontrollable approximations, e.g. truncations, and due to finite-size effects.

In this paper we apply a method for locating critical points, originally proposed by Binder [11] for the Ising model, in a MC simulation of the $2D$ $O(2)$ and $O(4)$ nonlinear σ -models. The procedure is a variant of the powerful phenomenological renormalization group [12, 13, 14] and can also be viewed as a sophisticated finite-size scaling method. We showed [15] (in the context of the Euclidean $O(N)$ -symmetric lattice $(\phi^2)^2$ theory) that it can be derived from the field-theoretical formulation of the renormalization group (RG). Applying Binder's procedure in a MC study of the $O(4)$ -invariant $(\phi^2)^2$ theory in $D = 3, 4$ dimensions we found [16] that it allows, in particular, to locate critical points quite accurately even on moderately sized lattices. It is to be emphasized that in Binder's procedure, like in other phenomenological RG methods, but unlike conventional MCRG, RG transformations are implemented exactly in the sense that no truncation of the renormalized Hamiltonian has to be made. (However, phenomenological RG methods do not allow to determine the flow lines in the space of coupling constants.) Furthermore in this method finite-size effects are exploited to get information about critical behaviour and no fits are involved to detect and locate a critical point. It is known from various applications (for reviews see [17]) that reliable information about the existence or non-existence of phase transitions can be obtained using this approach in MC simulations on rather small lattices.

* For the abelian case the transition from a massive high-temperature phase to a massless spin-wave phase is generally accepted on the basis of a proof by Fröhlich and Spencer [8], but ref. [5, 6] question the KT nature of the transition.

Binder's method is therefore ideally suited for searching for a phase transition in the $O(4)$ model whose existence is claimed in [6].

Although we are mainly interested in the $O(4)$ model, we have simulated the $O(2)$ model (the planar model) as well, in order to demonstrate that Binder's method is capable of detecting the KT transition. In fact this was shown before for the xy -model (with spin modulus $|S| \leq 1$) in [18]. A different phenomenological RG method was used in [13] for studying the KT transition in the $O(2)$ model. A standard MC analysis was performed in [19]. Many other numerical results on the 2D $O(2)$ model can be found in the literature – see [14] for a review and [6] for a compilation of some results.

We study the $O(N)$ ($N = 2, 4$) nonlinear σ -model on a two-dimensional square lattice of length L with periodic boundary conditions. The model is defined by the action

$$A = -\beta \sum_n \sum_{\mu=1}^2 S(n) \cdot S(n + \hat{\mu}) \quad (1)$$

where S denotes an N -dimensional vector of unit length, $\beta = 1/g$ is the inverse coupling (temperature), n denotes the sites of the finite lattice, and $\hat{\mu}$ is a unit vector in the positive μ -direction. The lattice spacing is set equal to one.

Binder's method is based upon the L -dependence of an $O(N)$ -invariant quantity U_L , called cumulant in abuse of terminology. It is constructed from the lattice average

$$\bar{S} = L^{-2} \sum_n S(n) \quad (2)$$

of the spin field S in the following way:

$$U_L(\beta) = 1 + \frac{2}{N} - \frac{\langle (\bar{S}^2)^2 \rangle}{\langle \bar{S}^2 \rangle^2} = - \frac{\langle (\bar{S}^2)^2 \rangle - (1 + 2/N) \langle \bar{S}^2 \rangle^2}{\langle \bar{S}^2 \rangle^2}. \quad (3)$$

Note that the numerator of the last expression is proportional to the connected $O(N)$ -invariant four-point function at zero momenta. If there is a critical point β_c the cumulant U_L becomes a RG invariant in the critical region $|(\beta - \beta_c)/\beta_c| \ll 1$. (The RG analysis of [15] can be easily adapted to the σ -model.) As a function of β , U_L behaves as follows: At $\beta = 0$ one gets $U_L(0) = O(L^{-2})$. For $L \rightarrow \infty$ one obtains $U_L(\infty) \rightarrow 2/N$, and one expects [11] that for fixed L the cumulant U_L increases monotonically with β .

Consider now the transformation $\beta \rightarrow \beta'$ associated with a change $L \rightarrow L'$ which is implicitly defined by

$$U_L(\beta) = U_{L'}(\beta'). \quad (4)$$

This is Binder's version of a phenomenological RG transformation. It has two "trivial" fixed points at $\beta = 0$ and $\beta = \infty$, respectively, when L, L' tend to infinity. A nontrivial fixed point would coincide with an inverse critical temperature β_c [11, 15].

The measured values of $\bar{S}^2$ at a critical point may be used to determine the critical exponent η by a standard finite-size scaling analysis. At an ordinary critical point it is known that the susceptibility

$$\chi(L, \beta) = L^2 \langle \bar{S}^2 \rangle \quad (5)$$

behaves to leading order in L like

$$\chi(L, \beta_c) = cL^{2-\eta} \quad (6)$$

if Wegner's correction-to-scaling exponent $\omega > 0$.

In the case of a KT transition the correlation length and the susceptibility develop an essential singularity when the KT temperature β_c^{-1} is approached from above, and the low-temperature phase $\beta \geq \beta_c$ is critical. In a finite system one expects the susceptibility in the low-temperature phase to behave similarly to (6):

$$\chi(L, \beta) \sim L^{2-\eta(\beta)}, \quad \beta > \beta_c, \quad L \rightarrow \infty \quad (7)$$

with a temperature-dependent exponent $\eta(\beta)$ [14, 20].

We have simulated the $N = 2, 4$ models by heat bath algorithms implementing the continuous symmetry groups. (Note that in [6] discrete subgroups of $O(2)$ and $O(4)$ were used in the MC calculations.) For $N = 2$ we performed calculations on lattices of length $L=8, 16, 24, 48$ and for $N = 4$ lattices of length $L = 10, 20, 30, 50, 70$ were used. For given β and L we performed 10^5 MC steps per site (MCS) for $N = 2$, 10^5 ($L = 10$) and $2 \cdot 10^5$ to $4 \cdot 10^5$ MCS ($L = 20, \dots, 70$) for $N = 4$. Most of the data were obtained using two independent start configurations. For each configuration we computed $\bar{S}^2$ according to (2). Averaging over successive configurations we obtained estimates for $\langle \bar{S}^2 \rangle$ and $\langle (\bar{S}^2)^2 \rangle$ and thus also for U_L . Typically the first 30 000 (80 000) configurations were discarded from these averages for $N = 2$ (4). As in [16] we estimated statistical errors by a binning procedure.

In the $O(2)$ model one expects a line of critical points starting at some β_c . Therefore, $U_L(\beta)$ should become L -independent for $\beta > \beta_c$. Our results for U_L (see Table 1) shown in Fig. 1 for $\beta \geq 1$ match (within the errors) this expectation for $L=16, 24, 48$. Yet comparison of U_8 and U_L ($L = 16, 24, 48$) seems to indicate closer resemblance to the behaviour at an isolated critical point. Presumably the lattice volume 8^2 is too small to allow for a definitive conclusion about the nature of the transition. A similar phenomenon was observed in the 6-state vector Potts model ($Z(6)$ model) in [21]. Still, including our $L = 8$ data in the determination of β_c yields consistent estimates. We obtain $\beta_c = 1.06(2)$.

From the measured values of $\bar{S}^2$ we calculated $\eta(\beta)$ by fitting χ according to

$$\ln \chi = a + (2 - \eta) \ln L \quad (8)$$

(cf. (7)). The results are given in Table 2. The susceptibilities and the exponent estimate for $\beta = 1$ given in [6] can be compared with the corresponding values of Table 2 and they agree. However, according to the behaviour of the cumulants U_L , $\beta = 1$ is probably below β_c . In the L -dependence of χ this shows up by the fact that our data for $\ln \chi(L, 1)$ plotted versus $\ln L$ do not follow a straight line. (For a compilation of other estimates of η published in the literature, see [6]. To the extent that these estimates can be compared with ours, there is agreement.) A clear β -dependence of η is borne out by our estimates. We have made no effort to generate sufficiently precise data for χ and the correlation length

ξ for various $\beta \leq \beta_c$, which would allow for a distinction between KT fits and standard critical point fits as was done in [6].

Our results for U_L in the $O(4)$ model are given in Table 3 and are displayed for $\beta \geq 2.4$ in Fig. 2. The data show the behaviour which is expected in the high-temperature phase of the model — extending down to zero temperature ($\beta = \infty$) according to conventional wisdom. In particular, no sign of criticality, i.e., L -independence of U_L , is seen near $\beta = 2.58$ where according to [6] a line of critical points should start. Note that we do not have to rely on our measurements on larger lattices ($L > 20$) alone. The absence of a critical point in the range of β -values studied can be deduced already from our results for $L = 10$ and $L = 20$. As a check we compared our measured susceptibilities at $\beta = 2.8$ and $\beta = 3.0$ with Hasenfratz's one-loop formula for χ on a periodic square lattice [22]. Evaluating this formula numerically we find agreement within 1 to 2 percent.

Finally a remark concerning the $2D$ $O(N)$ σ -model in the continuum. Its RG beta-function $B(g)$ is known to four-loop order [23] in a minimal subtraction scheme. For large β — more precisely: for $g/2\pi < 1$ — one may use this approximate form and check whether there is a hint for a critical point g_c , which must satisfy $B(g_c) = 0$. Using the four-loop approximation to B no positive solution exists for $N = 4$.

In conclusion, our numerical results imply that the phase structure of the $2D$ abelian $O(2)$ model and the nonabelian $O(4)$ model is clearly different — contrary to the results of Seiler et al. [6]. In particular we conclude from our analysis that there is no phase transition in the $2D$ $O(4)$ model at or in the vicinity of the coupling given in [6].

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Table Captions

Table 1: MC results for U_L in the $O(2)$ model.

Table 2: MC results for χ in the $O(2)$ model and values for η resulting from fits (cf. (8)).

Table 3: MC results for U_L in the $O(4)$ model.

Figure Captions

Fig. 1: MC results for U_L in the $O(2)$ model displayed for $\beta \geq 1$. Error bars are omitted. Curves are drawn to guide the eye. Full squares indicate equal locations of several U_L .

Fig. 2: MC results for U_L in the $O(4)$ model displayed for $\beta \geq 2.4$. Curves are drawn to guide the eye.

β	$L = 8$	$L = 16$	$L = 24$	$L = 48$
0.8	0.7444(42)	0.4022(11)	0.188(26)	0.051(28)
1.0	0.9541(10)	0.9436(29)	0.9350(62)	0.879(13)
1.02	0.9588(9)	0.9546(21)	0.9508(38)	0.924(14)
1.04	0.9645(9)	0.9617(19)	0.9624(20)	0.9619(30)
1.06	0.9687(7)	0.9693(9)	0.9672(30)	0.9681(21)
1.1	0.9748(5)	0.9771(6)	0.9773(12)	0.9797(21)
1.2	0.9836(3)	0.9866(3)	0.9875(4)	0.9883(5)
1.4	0.9908(1)	0.9924(3)	0.9929(4)	0.9936 (2)

Table 1

β	L = 8	L = 16	L = 24	L = 48	η
0.8	19.41 $\pm$.11	27.8 $\pm$.5	29.3 $\pm$.9	30.0 $\pm$.7	
1.0	34.38 $\pm$.07	106.3 $\pm$.6	202.5 $\pm$ 2.3	548 $\pm$ 10	0.391 $\pm$.010
1.02	35.28 $\pm$.06	112.0 $\pm$.5	217.1 $\pm$ 1.7	648 $\pm$ 12	0.347 $\pm$.010
1.04	36.22 $\pm$.09	116.8 $\pm$.4	231.3 $\pm$ 1.1	737 $\pm$ 6	0.315 $\pm$.004
1.06	37.09 $\pm$.07	121.9 $\pm$.3	242.4 $\pm$ 1.1	784 $\pm$ 7	0.291 $\pm$.003
1.10	38.48 $\pm$.07	129.3 $\pm$.4	263.4 $\pm$ 1.1	889 $\pm$ 6	0.250 $\pm$.003
1.2	41.40 $\pm$.05	144.2 $\pm$.2	299.9 $\pm$ 1.1	1048 $\pm$ 4	0.198 $\pm$.003
1.4	45.28 $\pm$.03	163.2 $\pm$.2	345.8 $\pm$.7	1248 $\pm$ 5	0.150 $\pm$.003

Table 2

β	L = 10	L = 20	L = 30	L = 50	L = 70
1.6		0.098(8)	0.045(12)		
2.0	0.4503(13)	0.376(5)	0.270(14)	0.131(18)	
2.4	0.4814(3)	0.4734(11)	0.4599(27)	0.424(11)	0.379(9)
2.537	0.4853(2)	0.4803(2)	0.4731(6)	0.4590(21)	0.442(4)
2.6	0.4867(2)	0.4826(2)	0.4765(7)	0.4656(24)	0.453(2)
2.8	0.4900(1)	0.4874(4)	0.4847(9)	0.4788(18)	0.4746(10)
3.0	0.4922(1)	0.4906(3)	0.4892(4)	0.4862(10)	0.4828(10)

Table 3

Fig. 1

Fig. 2