Explicit formulae for one-loop weak corrections
in the On-Shell scheme

Gerrit Burgers *)
CERN, Geneva, Switzerland

Wolfgang Hollik
II. Institut f. Theoretische Physik, Universität Hamburg, F.R.G.

Abstract

Explicit expressions are listed for those Standard Model one-loop weak corrections which are relevant for experiments at LEP100. These are the one-particle irreducible self-energies and the Z and y fermion vertex corrections. The On-Shell renormalisation scheme is used. Also given are formulae for the renormalisation scheme invariant quantities $\Delta \alpha(M_Z^2)$, which is the change in the electromagnetic coupling from zero momentum transfer till the Z scale, $\Delta r$, which relates the $M_W/M_Z$ ratio to the Fermi coupling constant $G_F$, and $\Delta \rho(0)$, which governs the ratio of the charged to neutral current couplings. The fact that $(\Delta r - \Delta \alpha)$ does not depend on the quark masses of the light quarks, enables us to use the dispersion result for $\Delta \alpha$ to calculate $\Delta r$ unambiguously.

The parameters of the On-Shell (OS) renormalisation scheme [1] are the physical vector boson masses $M_W$ and $M_Z$ (the poles of the W and Z propagators) and the fermion masses $m_i$ and the fine structure constant $\alpha$. By definition, $\cos \theta_W = M_W/M_Z$. An advantage of this scheme is that the basic parameters have a clear physical meaning, whose values can be found e.g. in the particle properties data booklet. Another advantage is that the QED corrections decouple and can be dealt with in the usual way. A disadvantage is that the Z and W masses are known with little precision, while in the ideal case the basic parameters should be known to a better precision than derived ones. Another disadvantage is that in most cases the radiative corrections are rather large, $\alpha$ being a low-energy parameter and the W and Z masses being high-energy parameters. These properties of the OS scheme are in marked contrast with the "starred" scheme of Lynn et al. [2], which has opposite characteristics. However, physical observables calculated with the same physical input data, are of course independent of the renormalisation scheme used, once one has included all radiative corrections. Actually, already when one-loop corrections have been included, the differences between the various renormalisation schemes become negligible compared to the experimental accuracy.

First we will give the expressions for the self-energies in the OS scheme. These expression are quite long and not too transparent, but straightforward to evaluate. Next we indicate what to do with the Z-y mixing, and what are the vertex corrections in the OS scheme. Finally we discuss the renormalisation scheme invariant quantities $\Delta \alpha$, $\Delta \rho$ and $\Delta r$. It is pointed out how the light-quark mass dependence $\Delta \alpha$ can be related to that of $\Delta \rho$.

The tree couplings $eg_l^-$ and $eg_l^+$, ($-e$ is the charge of the electron) of the negative and positive helicity fermions to the Z boson are given by

*) Address after October 1st, 1988: NIKHEF-H, P.O.B. 41882, 1009 DB Amsterdam

136
$$g_f^- = (I_f^3 - Q_f \sin^2 \theta_W)/(\sin \theta_W \cos \theta_W) \quad \text{and} \quad g_f^+ = -Q_f \sin^2 \theta_W/(\sin \theta_W \cos \theta_W).$$

The vector and axial couplings are the following linear combinations of $g_f^-$ and $g_f^+$:

$$a_f = \frac{1}{2} (g_f^- - g_f^+) \quad \text{and} \quad v_f = \frac{1}{2} (g_f^+ + g_f^-).$$

The charges of the various fermions are $Q = -1$ for $e, \mu, \tau$, $Q = 0$ for $\nu_e, \nu_\mu, \nu_\tau$, and $Q = 2/3$ for the $u,c,t$ quarks, and $Q = -1/3$ for the $d,s,b$ quarks. The third component of the isospin is $1/2$ for the neutrinos and the $u, c$ and $t$ quarks, and $-1/2$ for the charged leptons and the $d,s,b$ quarks.

The one-loop renormalisation of the electroweak Standard Model has been treated in detail in [1]. There also one-loop renormalized self-energies, vertices and box diagrams have been calculated. Here we list some of the results.

The one-loop corrected $W, Z$ and $\gamma$ propagators are functions of the renormalized one-particle irreducible (1PI) self-energies [2,3]

$$G_W(s) = \frac{1}{s - M_W^2 + \Sigma_W(s)},$$

$$G_Z(s) = \frac{1}{s - M_Z^2 + \Sigma_Z(s) - \frac{s \Pi^Z_{\nu\nu}(s)}{1 - \Pi_{\nu\nu}(t)}},$$

$$G_\gamma(s) = \frac{1}{s \left(1 - \Pi_{\nu\nu}(s) - \frac{s \Pi^Z_{\nu\nu}(s)}{s - M_Z^2 + \Sigma_Z(s)}\right)}.$$

The renormalized 1PI self-energies can be expressed in the finite parts of the non-renormalized ones\(^4\). Some abbreviations which will be used in the following are:

$$z = M_Z^2, \quad w = M_H^2, \quad h = M_H^2 \quad \text{and} \quad s_w = \sin \theta_W, \quad c_w = \cos \theta_W.$$

The finite parts of the non-renormalized 1PI self-energies are given by

$$\Pi^{NR}_{\nu\nu}(s) = \frac{\pi}{3} \left\{ \sum_f N_f Q_f^2 P(s,m_f) + \left(\frac{3}{4} + \frac{w}{s}\right) F(s,M_W,M_W) \right\},$$

$$\Pi^{NR}_{\nu Z}(s) = \frac{\pi}{12 c_w s_w} \left[ \sum_f N_f Q_f P(s,m_f) - \frac{1}{m_u - m_d + m_s - m_b} \right]$$

$$+ \frac{1}{c_w s_w} \left[ \left(\frac{3}{4} c_w^2 + \frac{1}{24} + \frac{w}{s}(c_w^2 + \frac{1}{3}) \right) F(s,M_W,M_W) + \frac{1}{36} \right].$$

\(^4\) The $Z\nu$ mixing in the propagators had not been included in ref. [1]. Moreover, some terms have been shifted from the renormalisation constants to the finite parts of the non-renormalized self-energies in order simplify the relationships between the renormalized quantities and the finite parts of the non-renormalized ones.
\[
\Sigma^{NR}_{zz}(s) = \frac{\alpha}{n} \left\{ \frac{1}{3} \sum_{l=1} N_z \left[ 2a_z^2 s \left( F(s,m_l,m_l) - \frac{s}{3} \right) \right] + \frac{s-z}{6s} \left[ \log \frac{m_u}{m_d} + \log \frac{m_s}{m_s} + \log \frac{m_t}{m_b} + s_w^2 \right] \right. \\
+ \frac{1}{3} \sum_{j=r^+} \left[ \left( (s+2m_j^2)F(s,m_j,m_j) - \frac{s}{3} \right) - 6a_j^2 m_j^2 F(s,m_j,m_j) \right] \\
+ \frac{1}{48s_w^2} \left[ \left( c_w^2 (-40s - 80w) + (c_w^2 - s_w^2)^2 (8w + s) + 12w \right) F(s,M_w,M_w) \right] \\
+ \left( 10z - 2h + s + \frac{(h-z)^2}{s} \right) F(s,M_{H},M_{Z}) - 2h \log \frac{h}{w} - 2z \log \frac{z}{w} \\
+ \left( 10z - 2h + s \right) \left( 1 - \frac{h+z}{h-z} \log \frac{M_{H}}{M_{Z}} - \log \frac{M_{H}M_{Z}}{w} \right) \\
+ \left( 2 \frac{2}{3} + \frac{8}{3} \right) \left( c_w^2 - s_w^2 \right) \left( s + \frac{8}{3} \right) \right\}, \\
\Sigma^{NR}_{ww}(s) = \frac{\alpha}{4\pi s_w^2} \left\{ \frac{1}{3} \sum_{l=1} N_z \left[ (s - \frac{1}{2} m_l^2 - \frac{1}{2} m_l^2) F(s,0,m_l) + \frac{2}{3} s - \frac{1}{2} m_l^2 \right] \right. \\
+ \left[ \left( s - \frac{m_e^2 + m_e^2}{2} - \frac{(m_e^2 - m_e^2)^2}{2s} \right) F(s,m_e^2, m_e^2) \right] \\
+ \left( \frac{2m_e^2 m_e^2 - s(m_e^2 + m_e^2)}{m_e^2 - m_e^2} \right) \log \frac{m_e^2 - m_e^2}{m_e^2} + \frac{2}{3} s - \frac{1}{2} (m_e^2 + m_e^2) \left\} \\
+ \frac{2(s-w)}{3} \left[ \log \frac{m_u}{m_d} + \log \frac{m_s}{m_s} + \log \frac{m_t}{m_t} + s_w^2 \right] \\
+ \left( s_w^2 - \frac{1}{3} c_w^2 (7z + 7w + 10s - 2(z-w)^2) - \frac{1}{6} (w+z-\frac{s}{2} - (z-w)^2) \right) F(s,M_{Z},M_{W}) \\
+ \frac{1}{3} s_w^2 (-4w - 10s + \frac{2w^2}{s}) F(s,0,M_{W}) + \frac{1}{6} (5w - h + \frac{s}{2} + \frac{(h-w)^2}{2s}) F(s,M_{H},M_{W}) \\
+ \left( \frac{1}{3} c_w^2 (7z + 7w + 10s - 4(z-w)) - s_w^2 z + \frac{1}{6} (2w - \frac{s}{2}) \right) \log \frac{z}{z-w} \\
- \frac{2}{3} w + \frac{1}{12} s \frac{h}{w} \log \frac{h}{w} \\
- \frac{1}{3} c_w^2 (7z + 7w + \frac{32}{3} s) + s_w^4 z + \frac{1}{6} (\frac{5}{3} s + 4w - z - h) - s_w^2 \left( \frac{4}{3} w + \frac{32}{9} s \right) \right\}.
The colour factor $N_c$ is 3(1) for quarks (leptons). Here the functions $F$ and $P$ are given by

\[
F(s,m_1,m_2) = 1 + \left( \frac{m_1^2 - m_2^2}{s} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \right) \log \frac{m_2}{m_1} + \\
+ \frac{1}{s} \sqrt{(m_1^2 + m_2^2) - s} \sqrt{(m_1^2 - m_2^2) - s} \log \frac{\sqrt{(m_1^2 + m_2^2) - s} + \sqrt{(m_1^2 - m_2^2) - s}}{\sqrt{(m_1^2 + m_2^2) - s} - \sqrt{(m_1^2 - m_2^2) - s}} \\
s < (m_1 - m_2)^2
\]

\[
\left\{ \begin{array}{ll}
- \frac{2}{s} \sqrt{(m_1^2 + m_2^2) - s} \sqrt{(m_1^2 - m_2^2) - s} \arctan \left( \frac{s - (m_1 - m_2)^2}{\sqrt{(m_1^2 + m_2^2) - s}} \right) \\
(m_1 - m_2)^2 < s < (m_1 + m_2)^2
\end{array} \right.
\]

\[
- \frac{1}{s} \sqrt{s - (m_1^2 + m_2^2) - s} \left( \log \frac{\sqrt{s - (m_1^2 + m_2^2) - s} + \sqrt{s - (m_1^2 - m_2^2)}}{\sqrt{s - (m_1^2 + m_2^2) - s} - \sqrt{s - (m_1^2 - m_2^2)}} - \text{in} \right) \\
s > (m_1 + m_2)^2
\]

and

\[
P(s,m) = 1/3 - (1 + 2m^2/s)F(s,m,m).
\]

If one of the masses is zero, $F$ reduces to

\[
F(s,m,0) = \left\{ \begin{array}{ll}
1 + \left( \frac{m^2}{s} - 1 \right) \log \left( 1 - \frac{s}{m^2} \right) & s < m^2 \\
1 + \left( 1 - \frac{m^2}{s} \right) \left( \log \left( \frac{s}{m^2} - 1 \right) - \text{in} \right) & s > m^2.
\end{array} \right.
\]

In the limit $s \to 0$, $F$ and $P$ are given by

\[
\lim_{s \to 0} \frac{(m_2^2 - m_2^2)^2}{s} F(s,m_1,m_2) = \frac{m_1^2 + m_2^2}{2} - \frac{m_1^2 m_2^2}{m_1^2 - m_2^2} \log \frac{m_1}{m_2},
\]

\[
\lim_{s \to 0} P(s,m) = -\frac{1}{s} \frac{s}{m^2},
\]

while for large $s$

\[
\lim_{s \to \infty} F(s,m_1,m_2) = 1 - \log \frac{s}{m_1 m_2} + \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \log \frac{m_1}{m_2}.
\]
The renormalized self-energies in terms of the above finite parts of the non-renormalized one are then [3]:

\[
\Pi_{rr}(s) = \Pi_{rr}^{NR}(s),
\]

\[
\Pi_{rz}(s) = \Pi_{rz}^{NR}(s) - \frac{\cos\theta_w}{\sin\theta_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right),
\]

\[
\Sigma_{zz}(s) = \Sigma_{zz}^{NR}(s) - \delta M_Z^2 + (s-M_Z^2) \frac{\cos^2\theta_w - \sin^2\theta_w}{\sin^2\theta_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right),
\]

\[
\Sigma_{ww}(s) = \Sigma_{ww}^{NR}(s) - \delta M_w^2 + (s-M_w^2) \frac{\cos^2\theta_w}{\sin^2\theta_w} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_w^2}{M_w^2} \right),
\]

where

\[
\delta M_Z^2 = \text{Re} \left[ \Sigma_{zz}^{NR}(M_Z^2) - M_Z^2 \frac{(\Pi_{rz}(M_Z^2))^2}{1 - \Pi_{rr}(M_Z^2)} \right],
\]

\[
\delta M_w^2 = \text{Re} \Sigma_{ww}^{NR}(M_w^2).
\]

Here we have used that the finite parts satisfy \( \Pi_{rr}^{NR}(0) = \lim_{r \to 0} \Pi_{rr}^{NR}(s) = 0 \). The above equations are implicit equations for the renormalized quantities since \( \delta M_Z^2 \) contains the renormalized \( \Pi_{rz} \). However, it is straightforward to solve for the renormalized quantities in terms of the non-renormalized ones.

The \( \gamma Z \) mixing not only affects the \( Z \) and photon propagator, but also can make that a boson leaves a fermion line as a \( Z \) and arrives at another one as a photon. Thus one needs the following correction in a left–left helicity amplitude:

\[
(Q_{j1} Q_{j2} G_r + g_{j1}^+ g_{j2}^+ G_z) \to
\]

\[
(Q_{j1} Q_{j2} G_r + g_{j1}^- g_{j2}^- G_Z) + \left( g_{j1}^- Q_{j2} + Q_{j1} g_{j2}^- \right) G_Z \frac{\Pi_{rZ}}{1 - \Pi_{rr}}
\]

and similarly for the other helicity amplitudes. This can be either viewed as a vertex correction to the \( Z \) fermion vertex, or a kind of "\( Z - \gamma \) propagator". Note this correction is symmetric in \( (s-M_Z^2 + \Sigma_{zz}) \) and \( (1 - \Pi_{rr}) \).

The 1PI vertex corrections to the \( Z \) and photon fermion vertices are simple compared to the self-energy corrections. One has to make the following substitutions for left-handed fermions:

\[
\text{Z vertex:} \quad g_f^- \to g_f^- \left( 1 + \frac{\alpha}{4\pi} \left\{ (g_f^-)^2 \Lambda_2(z/s) + \frac{1}{2s}(2s^2 - 1) \Lambda_2(w/s) - \frac{3s}{s^2(2s^2 - 1)} \Lambda_2(w/s) \right\} \right),
\]

\[
\text{\gamma vertex:} \quad Q_f \to Q_f \left( 1 + \frac{\alpha}{4\pi} \left\{ (g_f^-)^2 \Lambda_2(z/s) + \frac{3}{2s^2} \Lambda_2(w/s) \right\} \right),
\]

140
and for right-handed fermions:

\[ Z \text{ vertex: } g^+_f \rightarrow g^+_f \left( 1 + \frac{\alpha}{4\pi} \left( (g^+_f)^2 \Lambda_2(x/s) \right) \right), \]

\[ \gamma \text{ vertex: } Q_f \rightarrow Q_f \left( 1 + \frac{\alpha}{4\pi} \left( (g^+_f)^2 \Lambda_3(x/s) \right) \right). \]

\[ \Lambda_2 = \Lambda_3 = 0 \text{ at } s = 0, \text{ and for } |s|, |z, w| > m^2 \text{ the functions } \Lambda_2 \text{ and } \Lambda_3 \text{ are given by} \]

\[ \Lambda_2(x) = -\frac{7}{2} - 2x - (2x + 3) \ln(-x + i \epsilon) + 2(1 + x)^2 \left( \frac{1}{x} \left( 1 + \frac{1}{x} \right) + \frac{\pi^2}{6} \right), \]

with \[ Li_2(x) = \int_0^1 \frac{dt}{t} \log(1 - xt), \]

and

\[ \Lambda_3(x) = \frac{5}{6} - \frac{2}{3} x + \left( \frac{2}{3} x + \frac{1}{3} \right) \log\left( \frac{\sqrt{1 - 4x} + 1 - i \epsilon}{\sqrt{1 - 4x} - 1 - i \epsilon} \right) + \frac{2}{3} x(x + 2) \left( \log\left( \frac{\sqrt{1 - 4x} + 1 - i \epsilon}{\sqrt{1 - 4x} - 1 - i \epsilon} \right)^2. \]

For the bottom [4,5] and top [6] vertices there are important top-mass dependent corrections to these formulae which are not listed here.

In the last part of this note we discuss how to arrive at explicit expressions for some renormalisation scheme independent quantities. The first one is \( \Delta \rho \), which is related to the ratio of the neutral to charged current. Its definition is

\[ \Delta \rho(0) = \frac{\Sigma_{ZZ}(0)}{M_Z^2} - \frac{\Sigma_{WW}(0)}{M_W^2} = \frac{\Sigma^{NR}_{ZZ}(0)}{M_Z^2} - \frac{\Sigma^{NR}_{WW}(0)}{M_W^2}. \]

The last equality, which can be verified by substitution, shows this definition of \( \Delta \rho \) is renormalisation scheme invariant. This, however, does not imply there is an universal adopted definition of \( \Delta \rho \). In the more usual definition,

\[ \overline{\Delta \rho}(0) = (c^2 \Sigma^{NR}_{Zz}(0) - \Sigma_{ww}(0)) \frac{\sqrt{2} \cdot m^2}{\pi \alpha}, \]

which is equal to \( \Delta \rho(0)/(1 - \Delta r) \) (for \( \Delta r \), see below). Substituting for \( \Sigma^{NR}_{Zz}(0) \) and \( \Sigma^{NR}_{ww}(0) \) the expressions given before, one finds

\[ \Delta \rho^{Zb}(0) = \frac{\alpha}{4\pi s_w^2 c_w^2 M_Z^2} \left\{ \frac{3}{4} (m_t^2 + m_b^2) - \frac{3}{2} m_t^2 m_b^2 \log m_b^2 \right\}, \]

\[ + \frac{3}{4} \frac{h w}{h - w} \log \frac{h}{w} - \frac{3}{4} \frac{h z}{h - z} \log \frac{h}{z} + \frac{17}{4} w - \frac{8w^2 + 9zw}{4(z - w)} \log \frac{z}{w}. \]
Note that $\Delta \rho = 0$ if the masses are degenerate, $m_\tau = m_b$ and $M_Z = M_W$. The contributions of the other fermion doublets can be obtained from the top-bottom contribution by replacing the fermion masses, and omitting the colour factor 3 in the leptonic case. The dependence of $\Delta \rho(0)$ on the masses of the light fermions ($e, \mu, \tau, u, d, s, c$, and $b$) is negligible, of the order of $m_l^2 / M_Z^2$.

The next quantity is $\Delta \alpha$, which gives the change in the electromagnetic coupling constant. It is defined as

$$
\Delta \alpha(s) = \text{Re} \left[ \Pi_{rr}^{NR}(s) - \Pi_{rr}^{NR}(0) \right].
$$

So in the OS scheme one has simply $\Delta \alpha(M_Z^2) = \text{Re} \Pi_{rr}(M_Z^2)$. However, we cannot directly use this formula because $\Pi_{rr}$ contains quark masses, and we do not know which values to use for these masses. Usually one solves this problem as follows. The contribution of the $u, d, s, c$ and $b$ quark to $\Delta \alpha$ is calculated separately by a dispersion relation. This is done by Burkharit et al. in this report [7]. They find

$$
\Delta \alpha^{(u, d, s, c, b)}(92\text{GeV}^2) = 0.0288 \pm 0.0009.
$$

(They use the notation $\Delta \rho^{(u, d, s, c, b)}$ for this quantity). The difference $\Delta \alpha(s) - \Delta \alpha((92\text{GeV}^2)^2)$ can be calculated perturbatively as long $s \gg m_l^2$. So a formula for $\Delta \alpha$, which does not depend on the masses of the $u, d, s, c$ and $b$ quark but uses the above result is

$$
\Delta \alpha(s) = \frac{\pi}{\alpha} \left\{ \frac{1}{3} \sum_{f=\ell,\nu,\nu} N_{e, f} O_f^2 \text{Re} P(s, m_f) + \left( \frac{3}{4} + \frac{w}{s} \right) \text{Re} F(s, M_W, M_W) + \frac{11}{9} \log \left( \frac{s}{(92\text{GeV})^2} \right) \right\} + 0.0288.
$$

Strong corrections are already contained in the dispersion result 0.0288 and if $s$ is within an order of magnitude of (92GeV)$^2$, than their influence on the logarithmic term is negligible.

As already mentioned before, $M_W$ is not very well known. On the other hand, the Fermi coupling constant $G_F$ is rather precisely known. If one wants to use the the Fermi coupling constant $G_F$ as an input parameter for numerical computations instead of $M_W$, one has to solve the implicit equation

$$
M_W^2 (1 - M_W^2 / M_Z^2) = A(1 - \Delta r) \quad \text{where} \quad A = \frac{\pi \alpha}{\sqrt{2} G_F} = (37.281 \text{GeV})^2.
$$

This is the defining relation of $\Delta r$, the quantity which specifies the relation between $M_W$, $\sin^2 \theta_W$ and $G_F$. It obviously is renormalisation-scheme independent. The one-loop expression for $\Delta r$ in the OS scheme is

$$
\Delta r = \frac{\text{Re} \Sigma_{WW}(0)}{M_W^2} + \frac{\alpha}{4 \pi s_W^2} \left\{ s_W^2 + \frac{7 - 4 s_W^2}{2 s_W^2} \log(\frac{c_W^2}{s_W^2}) \right\}.
$$

Again, one has here the problem of the light quark masses. However, it happens that the combination $(\Delta r - \Delta \alpha)$ is almost independent of the the light-quark masses [8]. Working through our formulae, one can check that the following combinations of non-renormalized self-energies

$$
\Delta_1 = \text{Re} \Pi_{rr}^{NR}(z) + \frac{c_w^2 \Sigma_{WW}(w)}{s_w^2} - \frac{c_w^2 \Sigma_{ZZ}(z)}{s_w^2} - \text{Re} \Pi_{rr}^{NR}(z),
$$

$$
\Delta_2 = \frac{\Sigma_{WW}(0)}{w} + \frac{c_w^2 - s_w^2}{s_w^2} \text{Re} \Sigma_{WW}(w) - \frac{c_w^2}{s_w^2} \text{Re} \Sigma_{ZZ}(z) - \text{Re} \Pi_{rr}^{NR}(z),
$$

142
do not depend on the masses of the light fermions:

\[
\Delta_1 = \frac{\alpha}{4\pi f_w^3 c_w} \left\{ \left( \frac{1}{6} - \frac{2}{3} \frac{w}{z} + \frac{11}{6} \frac{m_l^2}{z} - \frac{4}{3} \frac{w}{z} \frac{m_l^2}{z} \right) F(z,m_l,m_l) + \left( \frac{1}{3} \frac{w}{z} + \frac{1}{6} \right) \log \frac{z}{m_l^2} + \right.
\]
\[
\left. \left( \frac{w}{z} - \frac{1}{2} \frac{m_l^2}{z} - \frac{1}{2} \frac{h^2}{wz} \right) F(w,m_l,0) + \frac{3}{2} \frac{w}{z} \log \frac{z}{w} - \frac{1}{3} + \frac{1}{3} \frac{w}{z} - \frac{1}{2} \frac{m_l^2}{z} + \right.
\]
\[
\left. \left( \frac{1}{12} + \frac{17}{6} \frac{w}{z} + \frac{16}{3} \frac{w^2}{z^2} \right) F(z,M_w,M_w) + \right.
\]
\[
\left. \left( \frac{1}{12} \frac{z}{w} + \frac{4}{3} - \frac{17}{3} \frac{w}{z} - \frac{4}{3} \frac{w^2}{z^2} \right) F(w,M_z,M_w) + \right.
\]
\[
\left. \left( \frac{1}{3} \frac{h}{z} - \frac{1}{12} \frac{h^2}{z^2} \right) F(z,M_H,M_Z) + \left( \frac{w}{z} - \frac{1}{3} \frac{h}{z} + \frac{1}{12} \frac{h^2}{wz} \right) F(w,M_H,M_w) + \right.
\]
\[
\left. \left( \frac{11}{24} - \frac{1}{12} \frac{h}{z} \right) \frac{h+z}{h-z} \log \frac{h}{z} + \left( \frac{11}{24} + \frac{1}{12} \frac{h}{z} - \frac{3}{4} \frac{w}{z} \frac{h}{h-w} \right) \log \frac{h}{w} + \right.
\]
\[
\left. \left( - \frac{1}{12} \frac{h}{z} + \frac{33}{4} \frac{z}{z-w} - \frac{69}{8} - \frac{6}{3} \frac{w}{z} \right) \log \frac{z}{w} - \frac{1}{12} - \frac{73}{6} \frac{w}{z} + \frac{4}{3} \frac{w^2}{z^2} \right\},
\]

and

\[
\Delta_2 = \frac{\alpha}{4\pi f_w^3 c_w} \left\{ \left( \frac{5}{6} - \frac{4}{3} \frac{w}{z} + \frac{19}{6} \frac{m_l^2}{z} - \frac{8}{3} \frac{w}{z} \frac{m_l^2}{z} \right) F(z,m_l,m_l) + \left( 2 \frac{w}{z} - \frac{1}{6} \right) \log \frac{z}{m_l^2} + \right.
\]
\[
\left. \left( \frac{2}{3} \frac{w}{z} - 1 \right) \left( \frac{1}{2} - \frac{1}{2} \frac{m_l^2}{w} - \frac{1}{2} \frac{h^2}{w^2} \right) F(w,m_l,0) + \right.
\]
\[
\left. \left( \frac{6}{3} \frac{w}{z} - 3 \right) \log \frac{z}{w} - \frac{2}{3} + \frac{2}{3} \frac{w}{z} - \frac{1}{4} \frac{m_l^2}{w} - \frac{1}{4} \frac{m_l^2}{z} + \right.
\]
\[
\left. \left( - \frac{37}{12} + \frac{2}{3} \frac{w}{z} + \frac{32}{3} \frac{w^2}{z^2} \right) F(z,M_w,M_w) + \right.
\]
\[
\left. \left( \frac{2}{3} \frac{w}{z} \right) \left( \frac{1}{12} \frac{z}{w} + \frac{4}{3} - \frac{17}{3} \frac{w}{z} - \frac{4}{3} \frac{w^2}{z^2} \right) F(w,M_z,M_w) + \right.
\]
\[
\left. \left( \frac{1}{3} \frac{h}{z} - \frac{1}{12} \frac{h^2}{z^2} \right) F(z,M_H,M_Z) + \left( \frac{2}{3} \frac{w}{z} - \frac{1}{6} \frac{h}{z} + \frac{1}{12} \frac{h^2}{wz} \right) F(z,M_H,M_w) + \right.
\]
\[
\left. \left( \frac{11}{24} - \frac{1}{12} \frac{h}{z} \right) \frac{h+z}{h-z} \log \frac{h}{z} + \left( \frac{11}{24} + \frac{1}{12} \frac{h}{z} - \frac{3}{4} \frac{w}{z} \frac{h}{h-w} \right) \log \frac{h}{w} + \right.
\]
\[
\left. \left( - \frac{1}{12} \frac{h}{z} + \frac{33}{4} \frac{z}{z-w} - \frac{69}{8} - \frac{10}{3} \frac{w}{z} \right) \log \frac{z}{w} + \right.
\]
\[
\left. \left( \frac{1}{24} \frac{h}{w} - \frac{1}{24} \frac{h}{z} + \frac{1}{24} \frac{z}{w} + \frac{57}{8} - \frac{91}{4} \frac{w}{z} + \frac{22}{3} \frac{w^2}{z^2} \right) \right\}.
\]

The recipe to calculate \( \Delta r \) is now as follows. We neglect the small imaginary parts of \( \Pi_r \) and \( \Pi_{r_z} \). First one solves
\[ \text{Re}\Pi_{rz}(z) = \Delta_1 + \frac{c_w}{s_w} \frac{(\text{Re}\Pi_{rz}(z))^2}{1 - \Delta\alpha(z)} \]

for \( \text{Re}\Pi_{rz}(z) \). Then

\[ \Delta r = \Delta\alpha(z) + \Delta_2 + \frac{c_w^2}{s_w^2} \frac{(\text{Re}\Pi_{rz}(z))^2}{1 - \Delta\alpha(z)} + \frac{\alpha}{4\pi s_w^2} \left\{ 6 + \frac{7 - 4s_w^2}{2s_w^2} \log(c_w^2) \right\} . \]

If in these formulae \( \Delta\alpha \) is calculated using the dispersion result, 0.0288, for the hadronic contribution, then there is no dependence on the masses of the light fermions.

References