The β -function for pure gauge SU(3)*

Rajan Gupta[§]

Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545

Gregory W. Kilcup

Physics Department, Brown University, Providence, RI 02912

and

Boston University, Boston, MA 02215

Apoorva Patel

Theory Group, CERN, Genève 23

Stephen R. Sharpe

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

ABSTRACT

We present detailed results for the non-perturbative β -function of the SU(3) lattice gauge theory using $b = \sqrt{3}$ renormalization group transformation (RGT). The results are compared to those obtained from $b = 2$ RGT, and from measurements of the deconfinement temperature, the string tension and the scalar glueball mass. Above $6/g^2 \approx 6.75$ our results are consistent with asymptotic scaling, while for $6.0 < 6/g^2 < 6.75$ we find results 1 – 2 standard deviations below the 2-loop value. For larger g^2 , in the region where the β -function shows a dip, the glueball mass shows significant deviations from scaling.

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1. Introduction

The β -function in QCD measures the rate of change of the coupling constant g with a change in the renormalization scale a

$$a \frac{\partial g(a)}{\partial a} \equiv -\beta(g(a)) . \quad (1)$$

At small g we know the behavior of the β -function from perturbation theory

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \dots \quad (2)$$

where only the first two coefficients, β_0 and β_1 , are regularization scheme independent. They are

$$\beta_0 = \frac{11N - 2n_f}{3 \times 16\pi^2} \quad (3a)$$

$$\beta_1 = \frac{34N^2 - 10Nn_f - 3n_f(N^2 - 1)/N}{3 \times (16\pi^2)^2} \quad (3b)$$

where N is the number of colors and n_f the number of quark flavors. Thus the β -function is negative for the anticipated 6 quark flavors of the standard model. Because of this important property, $g \rightarrow 0$ as $a \rightarrow 0$ and the fixed point at $g_{bare} = 0$ is ultraviolet stable and is the point about which the continuum limit of the lattice theory is defined.

On the lattice we measure the discrete β -function, $\Delta\beta$, for a given scale change $b = a_2/a_1$. We define the lattice coupling for the simple plaquette action (Wilson action) in terms of $K \equiv 2N/g^2$. The expected asymptotic behavior for the pure gauge $SU(3)$ theory ($n_f = 0$) is obtained by integrating Eqn. (1) between K_1 and $K_2 = K_1 + \Delta\beta$ (corresponding to scales a_1 and a_2 respectively)

$$\Delta\beta = \left(\frac{33}{4\pi^2} + \frac{459}{16\pi^4} \frac{1}{K} \right) \log b . \quad (4)$$

It is crucial that at weak couplings numerical values of $\Delta\beta$ match onto this asymptotic formula. The smallest value of K at which this occurs determines the asymptotic scaling region in which one can reliably take the continuum limit.

The discrete non-perturbative β -function of QCD can be measured in a number of ways by lattice calculations. In this letter we consider only two: Wilson's 2-lattice Monte Carlo renormalization group (MCRG) method [1] and scaling analyses of physical observables. In the asymptotic scaling regime all methods must agree with the perturbative result, Eq. 4, and thus with each other¹. As one moves away from the continuum limit,

¹ We shall consistently use the term "agreement" to mean that the difference in results is smaller than some tolerance criterion.

dimensionless physical quantities (like mass ratios) differ from their fixed point values by corrections that start at $\mathcal{O}(a^2)$ (modified by $\ln a$ factors) with coefficients that depend on the lattice action and the particular observable being measured. In addition there can be non-perturbative effects due to unphysical phase transitions in the generalized lattice theory². These corrections cause $\Delta\beta$ to deviate from the perturbative result, but there may still be a region of couplings in which all non-perturbative methods give the same $\Delta\beta$. In this scaling region the β -function is universal but non-perturbative. For yet larger couplings lattice artifacts become important and no universal β -function can be defined. In this paper we attempt to improve the demarcation of these different regions. We work entirely on the Wilson axis of pure gauge SU(3).

2. The MCRG method and $\Delta\beta$ Results

Consider any one-parameter trajectory in the coupling constant space. The particular example we have in mind is the Wilson axis. For any value of the coupling K_1 there exists another point at stronger coupling, K_2 , at which a certain correlation length measured in lattice units is smaller by a prescribed scale factor b , i.e. $\xi_1 = b\xi_2$. We then define $\Delta\beta = K_1 - K_2$ as the discrete β -function for a scale change b . The scaling region is that neighborhood of the ultraviolet stable fixed point at $g_{bare} = 0$, where $\Delta\beta$ is independent of the particular correlation ξ used.

In the MCRG calculation of $\Delta\beta$, one picks K_2 to have the same long distance action as K_1 but with correlation lengths (in lattice units) smaller by a factor b . The purported advantage of MCRG is that in this method one compares the actions directly instead of long distance correlation lengths. This is achieved by simulating two lattices, L_1 of linear dimension b^n at coupling K_1 , and L_2 of linear dimension b^{n-1} at coupling K_2 . One then successively blocks both lattices, $L_1 \rightarrow L_1^1$ with size b^{n-1} etc., and compares the actions on lattices of the same size (to minimize finite size effects), i.e. L_1^m with L_2^{m-1} for $m = 1$ up to $n - 1$. The coupling K_2 is tuned until a matching is achieved. Since at each comparison the theory with starting size L_1 is blocked once more, at matching the two original actions are related by a scale factor b .

The block actions are matched by comparing Wilson loops. If all Wilson loops agree at some block level, then the two actions are identical (including redundant operators). Once matching is achieved, it will persist on all further blocked lattices. In practice, only a few Wilson loops are used, and only a few blocking steps can be done. Thus one needs to make consistency checks: $\Delta\beta$ should not depend on the Wilson loop used to extract it, nor on the blocking level once convergence has been achieved. For $g \approx 1$, it is reasonable

² An important part of the MCRG program, which we do not address here, is to find improved actions that minimize $\mathcal{O}(a^2)$ corrections. This issue is discussed in refs. [2].

to use only a few loops because the correlation length in lattice units is small, and the range of interactions in the action is even smaller. Of course, the only way to really test whether matching has occurred is to extend the analysis to larger loops and larger initial lattices.

MCRG calculations for $SU(3)$ use two different scale factors: $b = \sqrt{3}$ and $b = 2$. Limitations of computer time have allowed the use of only 4 blocking steps, i.e. the starting lattices have sizes 9^4 and 16^4 respectively. In the $\sqrt{3}$ calculation only the plaquette (in the fundamental and five higher representations) and the three 6-link loops are calculated on all levels. In the $b = 2$ calculation [3] larger loops are calculated, but the final answer is extracted with maximum weight given to the same four loops.

In our previous publications, we have presented estimates for $\Delta\beta$ in the range $K_1 \in [6.0, 7.0]$, and it was only at $K_1 = 7.0$ that we found agreement with asymptotic scaling [4]. The statistics at that point were, in retrospect, poor (650 measurements). Recently, using the $b = 2$ method, Bowler *et al.* [3] have measured $\Delta\beta$ for $K_1 = 6.9$ and 7.2 . They find that $\Delta\beta$ lies 1.5 – 2 standard deviations below the perturbative value at both values of the coupling. To check this result, we have extended our work using the $\sqrt{3}$ RGT in a number of directions. First, we have measured $\Delta\beta$ out to $K_1 = 7.5$. Second, we have improved the statistics of our evaluation of $\Delta\beta$. Third, we have made a single run on a $(9\sqrt{3})^4$ lattice, which allows us to do a fifth matching. And, finally, we have measured $\Delta\beta$ at smaller K_1 , $K_1 = 5.7 - 5.9$, in order to map out the dip in the β -function.

The complete results are collected in Table 1 as a function of the blocking level and K_1 . The data shows that $\Delta\beta$ differs by up to 15% at the last two levels. The trend is always to get a smaller value at the last level. This effect is sufficiently large to cause worry about possible lack of convergence. However, it appears that the convergence of $\Delta\beta$ towards its asymptotic value is oscillatory for the $\sqrt{3}$ RGT. This hypothesis is confirmed by the result at $K_1 = 6.75$, starting on the $(9\sqrt{3})^4$ lattice. In this case, the $\Delta\beta$ values agree with the 9^4 calculation as a function of the blocking step, and the result on the additional level is roughly the mean of the previous two. We therefore take the mean of the $\Delta\beta$ values on the last two levels as our best estimate of the asymptotic value, and quote the spread as an estimate of the error. The statistical errors, and the spread in $\Delta\beta$ from using different loops, are smaller. We assign this value of $\Delta\beta$ to the midpoint of the K -interval. Fig. 1 shows these results, along with the rescaled (as explained below) Edinburgh data using the $b = 2$ RGT [3] for comparison.

Both calculations agree on the large dip in $\Delta\beta$ near $K = 5.7$. However, there is some discrepancy on the approach to the 2-loop perturbative value. The $b = \sqrt{3}$ calculation gives a value smaller than the 2-loop result up to $K_1 = 7.0$, with $\approx 10\%$ deviations from the 2-loop value between $K_1 = 6.35$ and 7.0 . The smallest value of K_1 for which $\Delta\beta$ differs from the two-loop value by less than two standard deviations is 6.35. Since this

Starting K_1	Statistics	Matching Level 2	Matching Level 3	Matching Level 4	2-loop $\Delta\beta$
5.7	1000	.302(9)	.291(7)	.267(6)	.488
5.8	1000	.272(8)	.261(8)	.237(6)	.487
5.9	3000	.291(7)	.282(7)	.266(6)	.487
6.0	806	.337(5)	.323(5)	.308(6)	.486
6.125	817	.387(5)	.376(5)	.351(6)	.486
6.25	1000	.421(4)	.424(5)	.401(5)	.485
6.35	938	.431(4)	.452(5)	.445(9)	.485
6.45	1300	.432(4)	.464(6)	.423(12)	.484
6.5	3000	.431(4)	.456(6)	.433(10)	.484
6.75	1048	.430(4)	.485(5)	.443(9)	.483
6.75*	500	.42(2)	.47(1)	.42(1)	.44(2)*
7.0	4250	.42(2)	.485(9)	.42(1)	.482
7.25	5000	.41(2)	.50(2)	.46(1)	.481
7.50	5000	.38(3)	.50(2)	.44(2)	.481

Table 1: The values of $\Delta\beta$ for $b = \sqrt{3}$ RGT from matching at different levels of blocking. The couplings K_1 are for the starting 9^4 lattices along the Wilson axis. The $(3\sqrt{3})^4$ lattices were generated with K_2 spaced by 0.03, and the statistics on all of them are 3000-4000 measurements separated by 10 update sweeps. The matching value of K_2 is determined by linear interpolation. The value quoted is the mean from all the loops weighted by 1σ error. For $K_1 \geq 6.75$, the systematic errors in $\Delta\beta$ may be larger than the estimates. Also shown are the values of $\Delta\beta$ corresponding to asymptotic scaling. The results at 6.75* are from a $(9\sqrt{3})^4$ starting lattice, with the result of the extra blocking level shown in the last column.

result comes from an integral of the continuous β -function over the range of couplings [5.9-6.35], our conservative estimate for the beginning of 2σ deviation from asymptotic scaling is $K = 6.0$. The results at $K_1 = 7.25$ and 7.5 are consistent with asymptotic scaling. We performed 5000 measurements (50,000 sweeps) at these two points in order to estimate the statistical errors reliably. The result at $K_1 = 7.25$ corresponds to the range of couplings [6.75-7.25], and thus our estimate for the onset of asymptotic scaling is $K \approx 6.75$.

The $b = 2$ calculation shows $\approx 15\%$ deviations from the perturbative result even at $K_1 = 7.2$, i.e. over the range [6.7-7.2]. Even though the two MCRG results are consistent within errors, the difference between them appears to be systematic. Thus it is worthwhile considering the differences between the two methods in some detail.

The simplest way to preserve gauge invariance in the blocking procedure is to construct

the block link from the sum, Σ , of paths between the block sites. The block link U' can then be selected with probability $\exp[-P \text{ReTr}(\Sigma^\dagger U')]$, with P a free parameter to be tuned to obtain matching after as few blocking steps as possible. For $P = \infty$, this RGT reduces to the δ -function construction implemented by finding the U' that maximizes $\text{ReTr}(\Sigma^\dagger U')$. A simple way to check whether a finite P will lead to an improvement is to compare the behavior of different size loops on the first few levels. If the small loops need a much larger K_2 than large loops to match, then decreasing P will be useful. Physically, by decreasing P one is trying to bring the behavior of short and long distance physics closer together for RGT's that unduly order the short distance (non-universal) correlations at initial blocking steps. For the $\sqrt{3}$ RGT, the short distance physics gets relatively disordered, so $P = \infty$ turns out to be optimal. The Edinburgh group finds that for the $b = 2$ RGT $P \approx 30$ is optimal and gives matching one level earlier when $K_1 \in [6.0, 6.6]$. However, they also find that at weaker coupling their results show a strong P and loop size dependence even after 3 blocking steps. This is especially true at $K_1 = 7.2$. Thus a simple one parameter optimization seems to be insufficient.

Recently Lang and Salmhofer [5] have compared different implementations of the $b = 2$ RGT for $SU(2)$. They examine matching as a function of the strengths with which paths of different length and topology are summed to produce Σ . They find that for the construction used by the Edinburgh group one needs more than 3 blocking steps to get a converged $\Delta\beta$ for $K > 2.8$. Note that, for $SU(3)$, $K \approx 6.6$ corresponds to the same correlation length.

The previous two arguments suggest that the systematic errors for the $b = 2$ RGT are increasing at weak coupling. This could be an explanation for the deviation from asymptotic scaling observed at $K_1 = 6.9$ and 7.2 .

The important question is to what extent does the $\sqrt{3}$ RGT suffer from such systematic errors? Certainly, it has advantages over the $b = 2$ RGT: the paths lie symmetrically about the block link; a larger fraction of the degrees of freedom are incorporated into the block link; and the scale factor is smaller. In particular, there is no need for an optimization along the lines of Ref. [5]. Thus we think it is plausible that the $\sqrt{3}$ RGT is more reliable. However, this is an assertion that can ultimately be justified only by further tests.

One possible problem with the $\sqrt{3}$ data is that the initial lattices are smaller than those needed for $b = 2$. This possibility has been studied for $SU(2)$ [6], by comparing results for $\Delta\beta$ in the range $K_1 \in [2.5, 3.5]$ using a starting 18^4 lattice with those from a 9^4 lattice. Both lattices allow four blocking steps, but with a factor of 2^4 difference in size at each level. The conclusion is that if finite size effects are present, they are much smaller than the systematic error due to incomplete convergence of $\Delta\beta$ as a function of blocking level. Also, for both lattice sizes the systematic error due to lack of convergence between the last two levels is $< 10\%$ even at $K_1 = 3.5$. For $SU(3)$, the data at $K_1 = 6.75$ again

shows that finite size effects are small between $L_1 = 9$ and $9\sqrt{3}$.

From our analysis, we infer that simulations on lattices which allow one more blocking level will significantly decrease the systematic errors. For the $\sqrt{3}$ RGT, tests on the $9\sqrt{3}$ lattice to support our assertion that the convergence of $\Delta\beta$ is oscillatory need to be repeated at more values of K_1 . The test can also be made by using larger blocked loops (more than 6 links) in the matching because we expect the amplitude of oscillations to decrease with loop size. Lastly, we need to think of additional ways to improve convergence for both RGT.

3. $\Delta\beta$ from Physical Observables

We now turn to the region $K \approx 6.0$ (near the dip), where we can include the data for physical observables and test scaling. To calculate $\Delta\beta$ from a physical observable with the dimension of mass (e.g. hadron masses and the deconfinement transition temperature), one measures ma at many values of K . Then from every pair of points we get $\Delta\beta = K_1 - K_2$ for a scale change $b = ma_2/ma_1$ with m held constant. In general, each observable produces a different β -function. The condition for scaling is that all observables give the same $\Delta\beta$ up to some error criterion.

The best measured physical observable is the pure gauge deconfinement transition temperature T_c [7]. In the talk at Seillac, the Columbia group presented evidence that there are significant finite volume corrections in the determination of the critical coupling. Their conclusion is that using the 3/4 deconfined criterion of Gottlieb *et al.* gives a better approximation to the infinite volume result. We therefore use the 3/4 criterion results in our analysis and estimate a systematic uncertainty in $\Delta\beta$ of at least 5% from residual finite size effects. The string tension, σ , is particularly hard to measure, and it has not yet been shown that the estimate of σ is independent of the loop size used in the measurement. Also, there is a significant difference between σ extracted from large Wilson loops and from correlators of Polyakov lines. Similarly, estimates of the scalar glueball mass have large finite volume errors. These limitations could lead to large systematic errors in $\Delta\beta$ and should be kept in mind when interpreting the results. The global data set used for calculating $\Delta\beta$ from T_c , σ and 0^{++} glueball mass is catalogued in Ref. [8].

To compare $\Delta\beta$ from physical observables with the $\sqrt{3}$ MCRG results we have used a naive rescaling of all data: $\Delta\beta(\sqrt{3}) = \Delta\beta(b) \times (\log \sqrt{3} / \log b)$. To reduce the systematic error due to rescaling we only use pairs of data points for which the scale change is already close to $b = \sqrt{3}$. By comparing $\Delta\beta$ from neglected pairs of points with $b > 2.0$ and $b < 1.5$ with those having $b \approx \sqrt{3}$, we estimate that there is at most a 5% error introduced by the rescaling.

Fig. 2 compares the $\sqrt{3}$ MCRG and T_c data. Both show a similar profile in the region of the dip, but are shifted relative to one another. The minimum in $\Delta\beta$ occurs at $K \approx 5.65$

for the $\sqrt{3}$ MCRG data and at $K \approx 5.8$ for the T_c data. The $b = 2$ MCRG data agree marginally better with the T_c data.

At couplings where scaling is violated, we expect the MCRG answer to be closest to that obtained from the string tension, since in both cases the basic observables are Wilson loops. In Fig. 3 we compare the string tension data with the $\sqrt{3}$ MCRG results. Even though the data have large errors, the overall picture is consistent with the dip.

We have also added a selection of the 0^{++} glueball data to the plot in Fig. 3. There are disagreements of up to 20% between different groups on the scalar glueball mass, so we have calculated $\Delta\beta$ only for pairs of points from the same group. The three data points are from de Forcrand *et al.* [9], APE collaboration [10], and Michael and Teper [11]. The figure clearly indicates that in spite of the large uncertainty in the glueball mass estimate, the data does not agree with the scaling behavior seen in T_c and σ .

The presence of the dip in $\Delta\beta$ has been attributed to the phase structure in the fundamental-adjoint coupling constant space. A dip in $\Delta\beta$ corresponds to a scale change that is faster than asymptotic scaling. Thus lines of constant σ and T_c are bunched closer together in the neighborhood of the dip. However, these observables are unlikely to vanish at the end point of the phase transition line. The much larger $\Delta\beta$ seen in the 0^{++} glueball data is consistent the 0^{++} glueball mass vanishing at the end point of the phase transition line and leaving a signature on the Wilson axis. This breakdown of scaling makes it imperative that physical observables be calculated outside the region of influence of the end point singularity in the extended coupling space.

The approach to asymptotic scaling can also be tested by plotting $\frac{m}{\Lambda}$, where m is any observable with the dimension of mass. This ratio should asymptotically approach a constant. Presented in this way, the T_c data appears to show smaller deviations from asymptotic scaling than if one uses $\Delta\beta$. For example T_c/Λ only changes by 3% between $K = 6.18$ ($N_t = 10$) and $K = 6.45$ ($N_t = 14$), while the same data gives a $\Delta\beta$ that differs by 9% from the 2-loop value. We would like to stress that this has nothing to do with the rescaling necessary to compare with the $\sqrt{3}$ data. The difference is simply due to the use of a different variable to measure the approach to the asymptotic value.

4. Conclusions

The reason for measuring $\Delta\beta$ is to demarcate the regions of scaling and asymptotic scaling. We expect asymptotic scaling to be valid for $K > K_{asy}$, where K_{asy} is the value at which the non-perturbative result for $\Delta\beta$ first agrees with the 2-loop perturbative result. We have shown that this occurs only for $K > 6.75$ while there are $\approx 10\%$ deviations in the interval $6.0 < K < 6.75$. As for scaling, the different measurements of $\Delta\beta$, with the exception of the scalar glueball data, are qualitatively consistent with each other in the region of the dip. Quantitatively, the $b = \sqrt{3}$ MCRG results show a small deviation from

the $b = 2$ MCRG and T_c data, while the string tension data is too inaccurate to merit a conclusion. Despite the large errors, the significant deviation seen in the scalar glueball data points to a lack of scaling for $K < 6.0$.

This still leaves a large window between $K = 6.0$ and 6.75 , where scaling may exist. It is important to ascertain whether the differences we have observed in this region are real effects or are simply due to systematic errors in the various methods. The prime source of unknown systematic errors in the present MCRG results is the small number of blocking steps. One would also like to know whether there are additional unphysical phase transitions for a generalized lattice action, and what their effect is on physical observables for $K > 6.0$. These questions can be settled by future MCRG calculations.

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Figures

1. Comparison of $\Delta\beta$ results of the $b = \sqrt{3}$ and $b = 2$ MCRG methods. The latter data have been rescaled to $b = \sqrt{3}$. The lone point from a $(9\sqrt{3})^4$ lattice is shown with a different symbol. The solid line is the two loop perturbative prediction of Eqn. 4.
2. Comparison of $\Delta\beta$ for the $\sqrt{3}$ MCRG with that from T_c measurements. The format is as in Fig. 1.
3. Comparison of $\Delta\beta$ for the $\sqrt{3}$ MCRG with that from string tension data. The format is as in Fig. 1. The three glueball points are from [9] ($K = 5.7$), [10] ($K = 5.9$) and [11] ($K = 6.05$).

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