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GHOST TRIALITY AND SUPERSTRING PARTITION FUNCTIONS

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ABSTRACT

In three or more space-time dimensions a heterotic string can have just four kinds of space-time supersymmetry, corresponding to the exceptional algebras E_6 , E_7 , E_8 and F_4 . We give a simple argument why this correspondence, first observed in the context of the covariant lattice construction, holds for any heterotic superstring, no matter how it is constructed. Using the fact that $SO(8)$ triality is an inner automorphism of the exceptional algebras, we derive a set of identities for character-valued partition functions, valid for any superstring in any space-time dimension and at any genus. These identities have exactly the structure of the Riemann identities, and are identical to the Riemann identities in the case of ten-dimensional strings. Their relevance for proving vanishing of the cosmological constant and non-renormalization theorems is discussed.

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It was observed in [1] that there is an intimate relation between space-time supersymmetry and exceptional groups. Although this observation was made initially in the context of covariant lattice string theories, it has become clear that it applies to all space-time supersymmetric theories. This follows indirectly from the observation in [2] that the vertex operator algebra of space-time supersymmetry is the same for any string theory, no matter how it is constructed. In this paper we will present an elementary argument establishing the relationship between supersymmetry and exceptional groups for any superstring, and we will demonstrate that it is a powerful tool in the analysis of multi-loop superstring partition functions.

Consider a general heterotic string theory in d dimensions. Its world sheet field content consists of d left- and right-moving bosons X^μ , d right-moving NSR-fermions ψ^μ , left- and right-moving reparametrization ghosts b and c , and right-moving superghosts β and γ . In addition there are internal degrees of freedom that are determined by some unspecified conformal field theory $\mathcal{C}$ with $c_{\text{left}} = 26 - d$, $c_{\text{right}} = 15 - \frac{3}{2}d$. We will assume only that the complete theory satisfies the appropriate consistency conditions for a string theory, such as multi-loop modular invariance, world sheet supersymmetry and positivity of the spectrum of L_0 and $\tilde{L}_0$ on $\mathcal{C}$.

A crucial ingredient in our construction is the bosonic string map, which takes the superconformal sector to a bosonic string sector, preserving conformal and modular invariance. This map works as follows [1][3-5]. The NSR fermions generate a level 1 $SO(d)$ Kac-Moody algebra that is the affine extension of the euclideanized Lorentz group. Under the map, its $SO(d-2)$ transverse part is preserved, while the longitudinal $SO(2)$ part together with the superconformal ghost system is replaced by $SO(8) \times E_8$. As the correlation between the conjugacy classes of $SO(d-2) \times SO(2)$ is preserved in going to $SO(d-2) \times SO(8) \times E_8$, this amounts to replacing $SO(d-2)$ by $SO(d+6) \times E_8$.

Under this map, vertex operators in the canonical pictures (ghost charge $-\frac{1}{2}$ or -1) transform into a finite set of operators of the same conformal weight. In particular, the right-moving part of the gravitino vertex operator (*i.e.* the supercharge) becomes a set of operators $\{Q_a\}$ of weight one transforming as a spinor of $SO(d+6)$.

It is critical to realize that the bosonic string map has replaced an indefinite ‘‘Hilbert’’ space by a positive definite Hilbert space whose states have non-negative energy. As a result it can be shown (see *e.g.* [6]) that $\{Q_a\}$ must extend the $SO(d+6)$ Kac-Moody algebra

to a larger level 1 Kac-Moody algebra. The possibilities are completely determined by classification of Lie algebras: we need all index 1 embeddings of $SO(d+6)$ in an algebra $\mathcal{G}$ such that the adjoint of $\mathcal{G}$ contains at least one $SO(d+6)$ spinor. The number of such spinors dictates the amount of supersymmetry, and the possibilities are summarized in table 1. It is a remarkable fact that (for $d \geq 3$) the entire supermultiplet structure of any supersymmetric theory coming from strings is encoded entirely in the representation theory of exceptional groups. In view of the results of [7,8] this is likely to extend to auxiliary fields. It is interesting to observe that in some cases $SO(d+6)$ is extended to a group of higher rank, which implies the existence of extra free bosons in the internal space [2].

Although the result of the bosonic string map is of course a bosonic string and not a superstring, it still contains all the information about the spectrum of the original superstring. To extract the light-cone spectrum one first drops the extra E_8 factor introduced above, and decomposes all the states of the theory with respect to the $SO(8)$ factor. Let S_x denote the set of states associated with (and orthogonal to) the conjugacy class (x) of $SO(8)$. Then the bosonic theory can be decomposed in the following way

$$(S_0, (0)) \oplus (S_v, (v)) \oplus (S_c, (c)) \oplus (S_s, (s)) \quad .$$

To isolate the physical states of the superstring in the light-cone gauge one keeps only S_v and S_s , the former corresponding to space-time bosons and the latter to space-time fermions [1].

If the theory is space-time supersymmetric, then the $SO(d+6)$ group is extended to an exceptional group^{*}. This has the important consequence that the triality outer automorphism of the aforementioned $SO(8)$ sub-algebra acts within an exceptional algebra, and hence it becomes an inner automorphism of the theory. An obvious consequence of this is that the number of space-time bosons and fermions is the same at every level, as one would expect in a theory with a gravitino in its spectrum. Consequently the one-loop partition function vanishes.

Under the usual assumptions [9] about fixing superdiffeomorphism invariance and inte-

^{*} From here on we exclude the $Spin(N)$ case

grating over supermoduli, the multi-loop partition function can be decomposed to [10]

$$P = \sum_{\alpha} P_{\alpha}(\Omega, \bar{\Omega}) Y_{\alpha}(\Omega) \ , \quad (1)$$

where the sum is over all spin-structures α at genus g . Each function Y_{α} contains a superghost determinant, the picture changing insertions and the contribution of two longitudinal world-sheet fermions. The function P_{α} contains the contributions to the partition function of the matter fields and reparametrization ghost determinants. The precise form of the functions Y_{α} depends in a complicated way on a number of gauge choices [9], but the Y_{α} are universal functions in the sense that they are essentially the same for all superstring theories, in any dimension. This is obvious for the superghost path integrals, but perhaps not immediately obvious for the picture changing insertion, which appears to depend on the choice of supercurrent and hence on the details of the internal sector. However, the conditions for world sheet supersymmetry in any string theory require the supercurrents, irrespective of how obtained, to satisfy the same operator products, and to have the same periodicity along the homology cycles as the NSR-fermions with space-time indices. This means that the picture changing operator correlations will indeed be theory-independent.

Our primary interest here will be in the properties of the functions P_{α} which contain all the information about the structure of distinct theories. Even though one knows little about the detailed properties of these functions, we will show that space-time supersymmetry guarantees that they satisfy certain generalizations of the Riemann-identities (indeed, we will also give a trivial proof of the Riemann-identities themselves).

It can be shown [11] that the functions Y_{α} introduced here[†] have exactly the same modular properties as the functions X_{α} defined by

$$X_{\alpha} = [\det(D_{1/2})_{\alpha}]^4 \sum_{\beta} [\det(D_{1/2})_{\beta}]^8 \ .$$

Replacing Y_{α} by X_{α} in (1) converts P into the partition function P' of a modular invariant bosonic string. The simplest way to see that Y_{α} and X_{α} have the same modular properties

[†] The functions Y_{α} considered here are not exactly those of [11], since in [11] the supercurrent insertions were omitted for simplicity. Their absence should not affect discussions of modular invariance, but is certainly not allowed in calculations of physical quantities.

is to consider the function P for the ten-dimensional $E_8 \times E_8$ heterotic string, and observe that the corresponding P' is the manifestly modular invariant partition of the 26-dimensional bosonic string compactified on the lattice $(E_8 \times E_8)_L \times (E_8 \times E_8)_R$. If we assume modular invariance of the $E_8 \times E_8$ heterotic string theory, then the result follows from universality of Y_α .

The factors $[\det(D_{1/2})_\alpha]^4$ and $\sum_\beta [\det(D_{1/2})_\beta]^8$ in X_α correspond precisely to the addition of $SO(8) \times E_8$ in the bosonic lattice representation. To understand how triality acts on the partition function P' we first convert to a conjugacy class basis of this $SO(8)$. This was described in [11] and amounts simply to performing an orthogonal transformation on the θ -functions. At one loop this works as follows

$$\begin{aligned}\theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \frac{1}{2}(\theta_0^{(4)} + \theta_v^{(4)}) \\ \theta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{2}(\theta_0^{(4)} - \theta_v^{(4)}) \\ \theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{1}{2}(\theta_s^{(4)} + \theta_c^{(4)}) \\ \theta^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{2}(\theta_s^{(4)} - \theta_c^{(4)}) \quad ,\end{aligned}$$

where $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ are standard θ -functions and

$$\theta_x^{(n)} = \sum_{\tilde{\lambda} \in (x)} e^{i\pi\tau\tilde{\lambda}^2}$$

and (x) denotes a conjugacy class of the D_n weight lattice.

At higher genus one makes this replacement in the obvious way, handle by handle. For example at two loops we have

$$\theta_{0s}^{(4)} = \frac{1}{4} \left(\theta^4 \begin{bmatrix} 01 \\ 00 \end{bmatrix} + \theta^4 \begin{bmatrix} 01 \\ 10 \end{bmatrix} + \theta^4 \begin{bmatrix} 01 \\ 01 \end{bmatrix} + \theta^4 \begin{bmatrix} 01 \\ 11 \end{bmatrix} \right) \quad ,$$

This change of basis in the spin structure labelling is (up to an overall normalization)

an orthogonal transformation, and so we have

$$\sum_{\alpha} P_{\alpha} X_{\alpha} = 2^g \sum_{[x]} P_{[x]} X_{[x]} \quad ,$$

where $[x]$ is a collection of conjugacy class labels. It is now a trivial consequence of the foregoing observations that in a space-time supersymmetric string theory

$$P_{[x]} = P_{t[x]} \quad ,$$

where t is a triality rotation that acts simultaneously on all handles.

It is elementary to generalize this observation to character-valued partition functions. If one has an E_n supersymmetry we consider a set of vectors $\vec{z}_i$, $i = 1, \dots, g$, in the n -dimensional weight space of E_n , whose components vanish in the $SO(8)$ directions, but which are otherwise arbitrary. Since the bosonic string contains an E_n factor, we can write the character valued partition function in terms of E_n conjugacy class partition functions, whose labels we will denote by $[y]$, *i.e.*,

$$P'(\vec{z}_i) = \left\{ \sum_{[y]} \tilde{P}_{[y]} E_{[y]}(\vec{z}_i) \right\} \left\{ \sum_{\beta} [\det(D_{1/2})_{\beta}]^8 \right\} \quad ,$$

where

$$E_{[y]}(\vec{z}_i) = (\mathcal{R}(\Omega))^{-\frac{n}{2}} \sum_{\vec{\lambda}_i \in [y]} e^{i\pi \vec{\lambda}_i \Omega_{ij} \vec{\lambda}_j} e^{2\pi i \vec{\lambda}_i \cdot \vec{z}_i} \quad ,$$

where $\mathcal{R}(\Omega)$ is the holomorphic square root of $\det'(-\Delta)$.

Let K_{n-4} be the sub-algebra of E_n commuting with D_4 (*i.e.* $K_4 = D_4$, $K_3 = A_1 D_2$, $K_2 = D_1 U(1)$), and decompose the functions $E_{[y]}$ into partition functions of the conjugacy classes of $D_4 \times K_{n-4}$. One obtains

$$E_{[y]}(\vec{z}_i) = (\mathcal{R}(\Omega))^{-\frac{n}{2}} \sum_{[x]} K_{[y]}^{[x]}(\vec{z}_i) \theta_{[x]}^{(4)} \quad .$$

Here $K_{[y]}^{[x]}$ is the character-valued partition function of K_{n-4} that is associated with the

D_4 -class $[x]$ in the E_n class $[y]$. Therefore the character valued extension of $P_{[x]}$ is simply given by

$$P_{[x]}(\vec{z}_i) = (\mathcal{R}(\Omega))^{-\frac{n-4}{2}} \sum_{[y]} \tilde{P}_{[y]} K_{[y]}^{[x]}(\vec{z}_i) .$$

Under a triality rotation the E_n conjugacy classes are, of course, unchanged, while the conjugacy class labels $[x]$ and the character parameters $\vec{z}_i$ rotate, so that

$$P_{[x]}(\vec{z}_i) = P_{t[x]}(T\vec{z}_i) , \quad (2)$$

where T is the matrix of the triality transformation acting on the Cartan sub-algebra of E_n ^{*}. This equation is the main result of this paper.

The matrices T can be constructed explicitly. Let $\vec{\lambda}$ be a vector of length 1 belonging to conjugacy class (x) of $SO(8)$. There is a pair of E_n roots of the form $\vec{\beta}_1 = (\vec{\lambda}, \vec{\mu})$ and $\vec{\beta}_2 = (\vec{\lambda}, -\vec{\mu})$. Note that $\vec{\beta}_1 \cdot \vec{\beta}_2 = 0$. Define t to be the composition of Weyl reflections generated by $\vec{\beta}_1$ and $\vec{\beta}_2$, *i.e.* the matrix T is given by

$$T\vec{z} = \vec{z} - (\vec{\beta}_1 \cdot \vec{z})\vec{\beta}_1 - (\vec{\beta}_2 \cdot \vec{z})\vec{\beta}_2$$

This transformation fixes the conjugacy class (x) and interchanges the other two. Moreover, if $\vec{z}$ is orthogonal to the D_4 factor then

$$\begin{aligned} T\vec{z} &= \vec{z} - 2(\vec{z} \cdot \vec{\mu})\vec{\mu} \\ &= (\mathbf{1} - 2\vec{\mu} \otimes \vec{\mu})\vec{z} , \end{aligned}$$

where $\mathbf{1}$ is the identity matrix. From this we can read off the rotation matrix T . For example to obtain the triality rotation that interchanges (v) and (s) we take $\vec{\lambda} \in (c)$, and for E_6 we

^{*} Note that (2) holds with $P_{[x]}$ replaced by $K_{[x]}^{[y]}$ for any choice of $[y]$, and that therefore we have established non-trivial identities for the characters of K_{n-4} . For the one-loop case, such identities have been presented in [12].

can take $\vec{\mu} = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, for E_7 $\vec{\mu} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2})$ and for E_8 $\vec{\mu} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence, restricting to the part K_{n-4} of E_n orthogonal to D_4 one obtains the following matrices

$$\begin{aligned} T_{E_6} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix} \\ T_{E_7} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{2} & 0 \end{pmatrix} \\ T_{E_8} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

The transformation that interchanges (s) and (c) can be chosen so that T is given by the identity matrix with one sign reversed on the diagonal.

For the space-time supersymmetric heterotic strings in ten-dimensions (i.e. $n = 8$) (2) gives nothing other than the classical Riemann-identities for the θ^4 factor in the partition function. To get these identities in their usual form one starts with any odd spin structure, writes it in conjugacy class basis, applies (2) and then converts back to spin-structure basis. For the triality rotation given above this procedure corresponds to replacing $\vec{z}$ by $T\vec{z}$ and transforming the spin-structures in each handle according to

$$\begin{aligned} \begin{bmatrix} 0 \\ \epsilon \end{bmatrix} &= (0) + \epsilon(v) \rightarrow (0) + \epsilon(s) \rightarrow \frac{1}{2} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \epsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\ \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} &= (s) + \epsilon(c) \rightarrow (v) + \epsilon(c) \rightarrow \frac{1}{2} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \epsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \epsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \quad , \end{aligned}$$

where $\epsilon = \pm 1$. The result of this transformation of the spin-structures can be summarized by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \rightarrow \frac{1}{2} \sum_{\delta_1, \delta_2=0}^1 e^{i\pi(\alpha_1\delta_2 + \alpha_2\delta_1)} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

and hence

$$2^g \prod_{a=1}^4 \theta \left[\begin{smallmatrix} \vec{\alpha}_1 \\ \vec{\alpha}_2 \end{smallmatrix} \right] (z_i^a | \tau) \equiv \sum_{\vec{\delta}_1, \vec{\delta}_2} e^{i\pi(\vec{\alpha}_1 \cdot \vec{\delta}_2 + \vec{\alpha}_2 \cdot \vec{\delta}_1)} \prod_{a=1}^4 \theta \left[\begin{smallmatrix} \vec{\delta}_1 \\ \vec{\delta}_2 \end{smallmatrix} \right] (T_b^a z_i^b | \tau) ,$$

where z_i^a are the components of $\vec{z}_i$, and $\vec{\delta}_1$ and $\vec{\delta}_2$ run over all the spin-structures. In string loop computations, one often uses this Riemann identity to express the complete sum over spin-structures in terms of one odd spin-structure. Note, however, that one need not restrict the spin-structure $\left[\begin{smallmatrix} \vec{\alpha}_1 \\ \vec{\alpha}_2 \end{smallmatrix} \right]$ to be odd.

Returning now to the multi-loop partition function, we consider the properties of the functions $Y_\alpha(\bar{\Omega})$. At one loop they are constants, and their modular transformation properties fix them to be

$$Y \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -Y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -Y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \pm Y \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \quad (3)$$

In conjugacy class basis this equation implies

$$Y_0 = Y_v + Y_s + Y_c \equiv 0 .$$

Note that these combinations are precisely the triality singlets.

The modular transformations for arbitrary genus are generated by the following transformations^{*}

$$\begin{aligned} T_i : Y \begin{bmatrix} \alpha_1 \dots \alpha_g \\ \beta_1 \dots \beta_g \end{bmatrix} &\longrightarrow (-1)^{\alpha_i+1} Y \begin{bmatrix} \alpha_1 \dots \alpha_i & & \dots \alpha_g \\ \beta_1, \dots \beta_i + \alpha_i + 1, \dots \beta_g \end{bmatrix} \\ S_i : Y \begin{bmatrix} \alpha_1 \dots \alpha_g \\ \beta_1 \dots \beta_g \end{bmatrix} &\longrightarrow Y \begin{bmatrix} \alpha_1 \dots \alpha_{i-1}, \beta_i, \alpha_{i+1} \dots \alpha_g \\ \beta_1 \dots \beta_{i-1}, \alpha_i, \beta_{i+1} \dots \beta_g \end{bmatrix} \\ U_{ij} : Y \begin{bmatrix} \alpha_1 \dots \alpha_g \\ \beta_1 \dots \beta_g \end{bmatrix} &\longrightarrow Y \begin{bmatrix} \alpha_1, \dots, \alpha_i & & \dots, \alpha_j & & \dots \beta_g \\ \beta_1, \dots, \beta_i + \alpha_j, \dots, \beta_j + \alpha_i, \dots \beta_g \end{bmatrix} . \end{aligned}$$

The transformations T_i and S_i correspond to the Dehn twists on handle i , and U_{ij} to

^{*} For comparison with [13] we note that our functions $Y \begin{bmatrix} \epsilon_i \\ \epsilon'_i \end{bmatrix}$ are related to the functions $X \begin{bmatrix} \epsilon_i \\ \epsilon'_i \end{bmatrix}$ of [13] by a factor $(-1)^\eta$, $\eta = \sum_i (\epsilon_i + \epsilon'_i)$.

the Dehn twist on the cycle linking the i^{th} with the j^{th} handle. Passing to the conjugacy class basis we have

$$\begin{aligned}
T_i &: Y_{[x_1, \dots, x_g]} \longrightarrow \epsilon(x_i) Y_{[x_1, \dots, x_g]} \\
S_i &: Y_{[x_1, \dots, x_g]} \longrightarrow \frac{1}{4} \sum_{x'_i = (0), (v), (s), (c)} (-1)^{2\vec{x}_i \cdot \vec{x}'_i} Y_{[x_1, \dots, x'_i, \dots, x_g]} \\
U_{ij} &: Y_{[x_1, \dots, x_g]} \longrightarrow \eta(x_i, x_j) Y_{[x_1, \dots, x_g]} ,
\end{aligned} \tag{4}$$

where $\epsilon(x_i) = -1$ if $x_i = (0)$ and $+1$ otherwise, $\vec{x}_i \cdot \vec{x}'_i$ is the dot product of two arbitrary representatives of the corresponding $SO(8)$ conjugacy classes, and $\eta(x_i, x_j) = +1$ if $i = j$ and -1 otherwise. It is manifest from (4) that these transformations, including phases, commute with triality.

In [13] a higher loop generalization of (3) was suggested. The naive generalization in which the $Y \begin{bmatrix} \epsilon_i \\ \epsilon'_i \end{bmatrix}$ are constant cannot hold since, for example, $T_2 S_2 U_{12} S_2 T_2 T_1$ sends $Y \begin{bmatrix} 00 \\ 00 \end{bmatrix} \rightarrow -Y \begin{bmatrix} 00 \\ 00 \end{bmatrix}$, and these functions are not allowed to be identically zero. Instead of this it was proposed in [13] that the Y 's should satisfy certain linear identities, which in our notation can be written as

$$Y_{[000\dots 0]} \equiv 0 \tag{5}$$

plus all modular transformations thereof.

Using (4) it is straightforward to see that the modular orbits of the identity (5) at genus g give precisely $2^{g-1} + \frac{1}{3}(2^{2g-1} + 1)$ identities corresponding to all the independent triality singlets amongst the $Y_{[x]}$. One picks a subset $\{P_{[r]}\}$ of $\{P_{[x]}\}$, such that one $P_{[r]}$ belongs to each of the triality orbits in $\{P_{[x]}\}$. Then, using (2), the partition function, P , can be reduced to the form

$$\begin{aligned}
P &= \sum_r P_{[r]} \cdot \{ \text{triality singlets of } Y_{[x]} \} \\
&\equiv 0 .
\end{aligned}$$

If the assumption in [13] were correct, the foregoing would constitute a proof that the cosmological constant vanishes to all loops for any space-time supersymmetric string theory, independent of any particular construction. Although the conjecture (5) is consistent with

modular invariance and factorization, and appears to have been explicitly checked at two loops [14,15], it is by no means clear that this conjecture holds for higher loops. Moreover, there appears to be a difficulty with the two-loop result. That is, if $Y_{[00]}$ vanishes point by point in moduli space, it is hard to see how one could generate the non-zero cosmological constant of [16] that appears through the one-loop D -term tadpole in some four-dimensional supersymmetric theories. It is possible that a δ -function contribution at the boundary of moduli space has been overlooked. Even if this issue were to be resolved, and if the cosmological constant in fact vanishes, it is also not at all clear that one can choose a gauge in which the partition function itself vanishes pointwise in moduli space.

These difficulties do not however have anything to do with the main purpose of this paper, which is to emphasize the important role of ghost triality and to use it to establish (2). These identities can be viewed as supersymmetric Ward identities, whereas (5) and its modular orbits are in no way related to supersymmetry.

The character-valued identities (2) will undoubtedly be crucial for establishing non-renormalization theorems, and “Ward identities” between non-vanishing amplitudes such as discussed in [12]. It is clear that the form of these theorems will be dictated only by the exceptional group structure. Thus, for example, the theorems will be the same for $N = 4$ supersymmetry in four dimensions and for $N = 1$ supersymmetry in ten dimensions. The number of character parameters will obviously dictate the maximum number of external lines for which non-renormalization theorems hold.

Finally we would like to point out that ghost triality is not the same as the well-known $SO(8)$ light-cone triality transformation that underlies the relationship between Green-Schwarz and Neveu-Schwarz-Ramond formalism. In E_8 , for example, ghost triality and light-cone triality act in different, commuting $SO(8)$ subgroups. The beauty of ghost triality is that it remains equally powerful in any supersymmetric string theory below ten dimensions.

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Type	Space-Time Dimension								
	10	9	8	7	6	5	4	3	2
E_8	1	1	1	2	2	4	4	8	(8, 8)
E_7	—	—	—	—	1	2	2	4	(4, 4)
E_6	—	—	—	—	—	—	1	2	(2, 2)
F_4	—	—	—	—	—	—	—	1	(1, 1)
$Spin(N)$	—	—	—	—	—	—	—	—	$(N - 8, 0)$ or $(0, N - 8)$

Table 1: *Complete classification of extended supersymmetry in heterotic string theories.*

REFERENCES

- [1]. W. Lerche, D. Lüst and A.N. Schellekens, Nucl. Phys. B287 (1987) 477.
- [2]. W. Boucher, D. Friedan and A. Kent, Phys. Lett. 172B (1986) 316; A. Sen, Nucl. Phys. B278 (1986) 289; Nucl. Phys. B284 (1987) 423; T. Banks, L.J. Dixon, D. Friedan and E. Martinec, Nucl. Phys. B299 (1988) 613; T. Banks and L. Dixon, *Constraints on String Vacua with Space-time Supersymmetry*, Princeton preprint PUPT-1086
- [3]. A. Casher, F. Englert, H. Nicolai and A. Taormina, Phys. Lett. 162B (1985) 121.
- [4]. F. Englert, H. Nicolai and A.N. Schellekens, Nucl. Phys. B274 (1986) 315.
- [5]. H. Nicolai and A.N. Schellekens, in *Proceedings of the 5th Adriatic Meeting on Superstrings, Anomalies and Unification*, Dubrovnik (1986).
- [6]. P. Goddard, *Notes on conformal field theory*, unpublished.
- [7]. J. Atick, L. Dixon, and A. Sen, Nucl. Phys. B292 (1987) 109; M. Dine, I. Ichinose and N. Seiberg, Nucl. Phys. B293 (1987) 253; S. Cecotti, S. Ferrara and L. Girardello, Phys. Lett. 198B (1987) 336.
- [8]. J. Lauer, D. Lüst and S. Theisen, *Four-dimensional Supergravity from Four-dimensional Strings*, preprint MPI-PAE/Pth 86/87; *Supersymmetric String Theories, Superconformal Field Theory and Exceptional Groups* preprint MPI-PAE/Pth 19/88.

- [9]. see e.g. E. Verlinde and H. Verlinde, Phys. Lett. 192B (1987) 97; J. Atick, J. Rabin and A. Sen, Nucl. Phys. B299 (1988) 279; J. Atick, G. Moore and A. Sen, *Some Global Issues in String Perturbation Theory and Catoptric Tadpoles*, Princeton preprints IASSNS-HEP-87/67 and 87/66; O. Lechtenfeld and A. Parkes, Phys. Lett. 202B (1988) 75, and references in these papers.
- [10]. A. Parkes, Phys. Lett. 184B (1987) 19.
- [11]. A.N. Schellekens, Phys. Lett. 199B (1987) 427.
- [12]. W. Lerche, *Elliptic Index and Superstring Effective Actions*, CERN preprint CERN-TH.5011/88, to appear in Nucl. Phys. B.
- [13]. D. Arnaudon, C. Bachas, V. Rivasseau and P. Vegreville, Phys. Lett. 195B (1987) 167.
- [14]. D. Arnaudon, *Non-renormalization theorems and cosmological constant in four-dimensional superstring models*, Ecole Polytechnique preprint 88-0267.
- [15]. E. Gava, R. Iengo and G. Sotkov, *Modular Invariance and the two-loop vanishing of the cosmological constant*, Trieste preprint IAEA-ICTP 88-51; R. Iengo, G. Sotkov and C. Zhu, *The two-loop amplitude in four-dimensional heterotic strings*, Trieste preprint SISSA 69-88 EP.
- [16]. M. Dine, N. Seiberg and E. Witten, Nucl. Phys. B289 (1987) 589; J.J. Atick and A. Sen, Nucl. Phys. B296B (1988) 157.