CERN-TH.5134/88

INFINITE DIMENSIONAL FREE DIFFERENTIAL ALGEBRAS
AND STRING FIELD THEORY

L. Castellani^{*)}

CERN - Geneva

A B S T R A C T

We consider Kac-Moody extensions of free differential algebras, and their BRST quantization. These infinite dimensional algebras appear in the context of a recently proposed field theory of loops.

^{*)} On leave of absence from the Istituto Nazionale di Fisica Nucleare, Torino University.

1. Introduction

Nonperturbative information will probably be of crucial importance in making superstring theory predictive. For example, hints on the "true vacuum" and on gauge or supersymmetry breaking would be helpful in the present state of affairs.

Since closed string excitations do contain gravitons, it is natural to try and generalize the geometrical methods of gravity theories. This, and aesthetical reasons, have prompted several geometric approaches to string field theory¹⁾.

In ref.2) we have proposed a field theory of loops based on a generalized equivalence principle, i.e. invariance under loop diffeomorphisms $\delta X^\mu(\sigma) = \xi^\mu[X, \sigma]$. These reduce, in the constant loop limit (= field theory limit), to the usual general coordinate transformations in spacetime. Also, local loop reparametrizations $\delta X^\mu(\sigma) = \xi[X, \sigma] \frac{\partial}{\partial \sigma} X^\mu(\sigma)$ are incorporated in the formalism as gauge symmetries, much as the local Lorentz rotations of gravity.

In the group manifold construction of supergravity theories^{3),4)}, (super)-spacetime is interpreted as a coset space G/H where G is an appropriate extension of the (super)Poincaré group, and H includes the Lorentz group. H plays the rôle of gauge group, and the theory is invariant under diffeomorphisms on G/H . In ref. 2) we have applied these methods to construct a general relativity of loops. Loop space arises as the infinite dimensional coset space G/H , where G contains $(\widehat{\text{Poincaré}}) \ltimes \text{Diff}(S^1)$ and H contains $(\widehat{\text{Lorentz}}) \ltimes \text{Diff}(S^1)$ ($\ltimes$ indicates the semidirect product and the hat means the Kac-Moody extension), in the same way as ordinary spacetime can be seen as the coset manifold

$$\frac{G}{H} = \frac{\widehat{\text{Poincaré}}}{\widehat{\text{Lorentz}}}$$

Thus we gain an algebraic understanding of the geometric arena for string functionals, i.e. loop space. The algebras involved contain Kac-Moody extensions⁵⁾ of the Poincaré algebra, and the $\text{Diff}(S^1)$ algebra (the algebra of loop reparametrizations).

The field theory limit is retrieved by truncation of the $n \neq 0$ modes (= constant loop limit). For example, to construct a loop field theory reducing to a $D = 10$ supergravity in the 0-mode sector, one chooses G to contain the Kac-Moody extension of the corresponding free differential algebra. For a review on free differential algebras, henceforth denoted by FDA, see ref. 4). A brief account on FDA's is also included in Sect. 2 of the present paper.

We remind that FDA's are the algebraic structures relevant to (super)gravity

theories containing antisymmetric tensors^{6),7)}. To our knowledge, Kac-Moody extensions of FDA's have not so far been considered.

In this paper we study the Kac-Moody extension of the simplest FDA, containing an ordinary Lie algebra and a 2-form potential. This FDA is named FDA1 in the following. We construct the corresponding quantum BRST operator Q ^{8),9)}, and find under which conditions this operator is nilpotent, i.e. under which conditions the quantum $\widehat{\text{FDA}}$ is anomaly-free.

In BRST covariant quantization of infinite dimensional algebras, the ghosts¹⁰⁾, besides restoring unitarity, can also do the favour of cancelling anomalies. Indeed anomalies arise because of normal ordering of the field modes, and the ghosts, having opposite statistics, may cancel these anomalies in particular (critical) situations.

The best known example is the first quantized bosonic string, whose gauge-fixed action retains a residual invariance under the infinite dimensional Virasoro (=conformal) algebra. Nilpotency of the quantum Virasoro BRST operator, i.e. absence of conformal anomalies, is achieved only for the critical value of space-time dimensions = 26 .

Nilpotency of the quantum BRST operator has also been studied for the semi-direct sum of (super)Kac-Moody algebras with the (super)conformal algebra¹¹⁾. Our results for $\widehat{\text{FDA1}} \in \text{Diff}(S^1)$ reproduce those of refs. 11) when we ignore the contribution of the two-form field.

The paper is organized as follows. In Sect. 2 we give a short review on FDA's and discuss their BRST quantization. In Sect. 3 we deal with the Kac-Moody extension of FDA1, construct the corresponding (quantum) Q operator, and find the conditions for its nilpotency. Sect. 4 contains our conclusions.

2. Free differential algebras

Given a group element of G , $g(y) = \exp y^A T_A$ (T_A = generators of G), the left-invariant 1-form

$$\sigma(y) = g^{-1}(y) dg(y) \quad (2.1)$$

belongs to the Lie algebra of G , i.e.

$$\sigma(y) = \sigma^A(y) T_A \quad (2.2)$$

$\sigma^A(y)$ defines a cotangent basis on G . The Cartan-Maurer equations

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \quad (2.3)$$

follow immediately from the definition (2.1), and can be considered as the dual formulation of the Lie algebra

$$[T_A, T_B] = C^C_{AB} T_C \quad (2.4)$$

Jacobi identities on C^A_{BC} translate into $d^2 = 0$ in the dual formulation. Eq. (2.3) deals with forms, and is therefore better suited to the purpose of constructing integrands (lagrangians). Also, it naturally extends to p -forms ($p \geq 1$):

$$d\vartheta^i_{(p)} + \sum \frac{1}{m} C^i_{i_1 \dots i_m} \vartheta^{i_1}_{(p_1)} \wedge \dots \wedge \vartheta^{i_m}_{(p_m)} = 0 \quad (2.5)$$

($p+1 = p_1 + \dots + p_m$)

$p, p_1, \dots, p_m$ are respectively the degrees of the forms $\vartheta^i, \vartheta^{i_1}, \dots, \vartheta^{i_m}$, $i, i_1, \dots, i_m$ are indices in irreps of G , and $C^i_{i_1 i_2 \dots i_m}$ are generalized structure constants satisfying generalized Jacobi identities (from $d^2 = 0$). When $p = p_1 = p_2 = 1$ and i, i_1, i_2 belong to the adjoint representation, eqs. (2.5) reduce to the ordinary Cartan-Maurer equations. The (anti)symmetry properties of the indices $i_1, \dots, i_m$ depend on the bosonic or fermionic character of the forms $\vartheta^{i_1}, \dots, \vartheta^{i_m}$.

In general eqs. (2.5) define a free differential algebra ^{6,7)}(FDA). The possible FDA extensions G' of a Lie algebra G have been studied in refs. ¹²⁾, ⁷⁾, and rely on the existence of Chevalley cohomology classes in G ¹²⁾.

Suppose that, given an ordinary Lie algebra $\mathbb{G}$, there exists a p-form

$$\Omega_{(p)}^i(\sigma) = \Omega_{A_1 \dots A_p}^i \sigma^{A_1} \wedge \dots \wedge \sigma^{A_p} \quad \begin{array}{l} i: \text{runs on a } \mathbb{G}\text{-irrep} \\ \Omega_{A_1 \dots A_p}^i = \text{constants} \end{array} \quad (2.6)$$

which is covariantly closed but not covariantly exact, i.e.

$$\nabla \Omega_{(p)}^i \equiv d\Omega_{(p)}^i + \sigma^A \wedge \mathcal{D}(T_A)^i_j \Omega_{(p)}^j = 0 \quad (2.7a)$$

$$\Omega_{(p)}^i \neq \nabla \Phi_{(p-1)}^i \quad (2.7b)$$

Then $\Omega_{(p)}^i$ is said to be a representative of a Chevalley cohomology class in the D_j^i irrep of $\mathbb{G}$. ∇ is the boundary operator satisfying $\nabla^2 = 0$ (use eqs. (2.6) and (2.3)). $\Omega_{(p)}^i(\sigma)$ allows the extension of the Lie algebra (2.3) to the FDA $\mathbb{G}'$:

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \quad (2.8a)$$

$$\nabla \Sigma_{(p-1)}^i + \Omega_{(p)}^i(\sigma) = 0 \quad (2.8b)$$

where $\Sigma_{(p-1)}^i$ is a new (p-1)-form, not contained in $\mathbb{G}$. Closure of (2.8) is ensured because $\nabla \Omega_{(p)}^i(\sigma) = 0$. Note that (2.8a) can be cast in a form analogous to (2.8b):

$$\nabla \sigma^A - \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \quad (2.9)$$

(recall that $(T_A)^B_C = C_{AC}^B$ in the adjoint representation). Closure of (2.9) again implies Jacobi identities on the structure constants.

It is clear that $\Omega_{(p)}^i(\sigma)$ differing by exact pieces $\nabla \Phi_{(p-1)}^i$ lead to equivalent FDA's, via the redefinition $\Sigma_{(p-1)}^i \rightarrow \Sigma_{(p-1)}^i + \Phi_{(p-1)}^i$.

The whole game can be repeated on the free differential algebra $\mathbb{G}'$ which contains now $\sigma^A, \Sigma_{(p-1)}^i$. One looks for the existence of polynomials in $\sigma^A, \Sigma_{(p-1)}^i$:

$$\Omega_{(q)}^i(\sigma, \Sigma) = \Omega_{A_1 \dots A_k i_1 \dots i_s}^i \sigma^{A_1} \wedge \dots \wedge \sigma^{A_k} \Sigma_{(p-1)}^{i_1} \wedge \dots \wedge \Sigma_{(p-1)}^{i_s} \quad (2.10)$$

satisfying the cohomology conditions (2.7). If such a polynomial exists, the FDA $\mathcal{G}'$ of eqs. (2.8) can be further extended to $\mathcal{G}''$, and so on.

In constructing D-dimensional supergravity theories we can choose as starting point the super-Poincaré Lie algebra, whose dual formulation is:

$$d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} = 0 \quad (2.11a)$$

$$\mathcal{D}V^a - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi \equiv dV^a - \omega^{ab} \wedge V^b - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi = 0 \quad (2.11b)$$

$$\mathcal{D}\psi \equiv d\psi - \frac{1}{4} \omega^{ab} \wedge \Gamma_{ab} \psi = 0 \quad (2.11c)$$

ω^{ab} , V^a and ψ are the 1-forms associated to Lorentz rotations, translations and supersymmetry charge(s) respectively. The possible $\mathcal{G}'$ extensions to FDA's depend on D. For example in D = 11 there is a cohomology class on $\mathcal{G} : \{V^a, \omega^{ab}, \psi\}$ in the identity representation⁶⁾:

$$\Omega(V, \omega, \psi) = \frac{1}{2} \bar{\psi} \wedge \Gamma^{ab} \psi \wedge V^a \wedge V^b \quad (2.12)$$

$d\Omega = 0$ holds because of the D = 11 Fierz identity

$$\bar{\psi} \wedge \Gamma^{ab} \psi \wedge \bar{\psi} \wedge \Gamma^a \psi \wedge V^b = 0 \quad (2.13)$$

This allows the extension of (2.11) by means of a 3-form A :

$$dA - \Omega(V, \omega, \psi) = 0 \quad (2.14)$$

Note that only nonsemisimple algebras can have FDA extensions in a nontrivial $\mathcal{G}$ -irrep. Indeed a theorem by Chevalley and Eilenberg¹³⁾ states that there is no nontrivial cohomology class of $\mathcal{G}$ in nontrivial $\mathcal{G}$ -irreps when $\mathcal{G}$ is semisimple.

We proceed now to construct the BRST operator Q associated to FDA's. We will give its explicit form only for the case of FDA1; the same method used here can be applied to find Q for any FDA.

An elegant way to introduce ghosts and BRST transformations is provided by an algorithm (the "Russian formula") due to Stora¹⁴⁾. With this algorithm one essentially constructs an algebra isomorphic to the original FDA :

$$\tilde{d}\tilde{\sigma}^A + \frac{1}{2} C^A_{BC} \tilde{\sigma}^B \wedge \tilde{\sigma}^C = 0 \quad (2.15a)$$

$$\tilde{d}\tilde{\Sigma}^i_{(p)} + C^i_{A_j} \tilde{\Sigma}^j_{(p)} + \Omega^i_{(p+1)}(\tilde{\sigma}) = 0 \quad (2.15b)$$

where

$$\tilde{d} = d + s \quad (2.16a)$$

$$\tilde{\sigma}^A = \sigma^A + g^A, \quad \tilde{\Sigma}^i_{(p)} = \Sigma^i_{(p)} + g^i_{(p-1,1)} + \dots + g^i_{(0,p)} \quad (2.16b)$$

The closure of the original FDA (equivalent to the generalized Jacobi identities) implies $\tilde{d}^2 = (d + s)^2 = 0$, or

$$d^2 = s^2 = ds + sd = 0 \quad (2.17)$$

Eqs. (2.16) can be interpreted geometrically¹⁵⁾ by adding an extra Grassmann coordinate θ to the original group manifold directions x^α . Then d decomposes as

$$d = dx^\alpha \frac{\partial}{\partial x^\alpha} + d\theta \frac{\partial}{\partial \theta} = d + s \quad (2.18)$$

and $\sigma^A, \Sigma^i_{(p)}$ are expanded on the basis $(dx^\alpha, d\theta)$:

$$\begin{aligned} \sigma^A &= \sigma^A_\alpha dx^\alpha + g^A d\theta \\ \Sigma^i_{(p)} &= \sum_{(p,0) \alpha_1 \dots \alpha_p}^i dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \\ &\quad + g^i_{(p-1,1) \alpha_1 \dots \alpha_{p-1}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{p-1}} d\theta \\ &\quad + \dots \end{aligned} \quad (2.19)$$

The components along $d\theta$ differentials are interpreted as ghosts, ghosts for ghosts, etc. All fields are assigned a bidegree (p,q) where p is the form degree and q is the ghost degree.

Considering eqs. (2.15) for each bidegree we obtain the usual FDA, and the transformation rules of the fields under BRST (these are defined by the action of s).

We illustrate this in the case of FDA1, the first FDA extension of a Lie algebra, whose generalized Cartan-Maurer equations read*)

$$\begin{aligned} d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \sigma^C &= 0 \\ dB^i + C^i_{Aj} \sigma^A B^j + \frac{1}{6} C^i_{ABC} \sigma^A \sigma^B \sigma^C &\equiv \nabla B^i + \frac{1}{6} C^i_{ABC} \sigma^A \sigma^B \sigma^C = 0. \end{aligned} \quad (2.20)$$

where B^i is a two-form in a representation D^i_j of G . The generalized Jacobi identities ($d^2 = 0$) are

$$C^i_{Aj} C^j_{Bk} - C^i_{Bj} C^j_{Ak} = C^C_{AB} C^i_{Ck} \quad \text{representation condition} \quad (2.21a)$$

$$-\frac{1}{6} C^j_{[ABC} C^i_{D]j} + \frac{1}{4} C^E_{[DA} C^i_{BC]E} = 0 \quad \text{3-cocycle condition} \quad (2.21b)$$

Eq. (2.21a) implies that $(C^i_A)^j \equiv C^i_{Aj}$ is a matrix representation of G , while eq. (2.21b) is just the statement that $C^i \equiv C^i_{ABC} \sigma^A \sigma^B \sigma^C$ is a 3-cocycle, i.e. $\nabla C^i = 0$.

Expanding now d and the forms σ^A, B^i as in (2.16), and projecting on each bidegree, one finds^{15),16)}:

$$\begin{aligned} (1,1) \quad s\sigma^A &= -dg^A - C^A_{BC} g^B g^C \equiv -(\nabla g)^A \\ (0,2) \quad sg^A &= -\frac{1}{2} C^A_{BC} g^B g^C \\ (2,1) \quad sB^i &= -db^i - C^i_{Aj} g^A B^j - C^i_{Aj} \sigma^A b^j - \frac{1}{2} C^i_{ABC} \sigma^A \sigma^B g^C \\ (1,2) \quad sb^i &= -dg^i - C^i_{Aj} g^A g^j - C^i_{Aj} \sigma^A g^j - \frac{1}{2} C^i_{ABC} \sigma^A g^B g^C \\ (0,3) \quad sg^i &= -C^i_{Aj} g^A g^j - \frac{1}{6} C^i_{ABC} g^A g^B g^C \end{aligned} \quad (2.22)$$

To avoid explicit use of the bidegree in the field notation, we have renamed the ghosts of B^i as follows:

$$\begin{aligned} B^i_{(1,1)} &\longrightarrow b^i \\ B^i_{(0,2)} &\longrightarrow g^i \end{aligned} \quad (2.23)$$

while $B^i_{(2,0)}$ simply becomes B^i .

We introduce now the antighosts h_A, k_i, h_i and the momenta π_A, K_i satisfying the (anti)commutation relations:

*) we omit wedge symbols in the following.

$$\begin{aligned}
 \{g^A, h_B\} &= \delta_B^A & \{\sigma^A, \pi_B\} &= \delta_B^A \\
 [b^i, k_i] &= \delta_i^i & [B^i, K_j] &= \delta_j^i \\
 [g^i, h_i] &= \delta_i^i & \text{all other (anti)commutators} & \\
 & & \text{vanishing} &
 \end{aligned} \tag{2.24}$$

We use the shorthand notation $\{g^A, h_B\} = \delta_B^A$ for $\{g^A(x), h_B(y)\} = \delta_B^A \delta(x-y)$ etc. The bidegrees of conjugate fields are opposite to each other.

The BRST operator Q is defined via the action of s on the fields ϕ :

$$s\phi = [\phi, \phi] \tag{2.25}$$

$[\]$ means anticommutator when ϕ has odd total degree (= form degree + ghost degree), commutator otherwise.

It is easy to verify that the operator

$$\begin{aligned}
 \phi_{FDA1} = & g^A \pi_A + b^i \pi_i + g^i E_i \\
 & + \frac{1}{2} C^A_{BC} g^B g^C h_A + C^i_{Aj} g^A b^j k_i + C^i_{Aj} g^A g^j h_i \\
 & - \frac{1}{2} C^i_{ABC} \sigma^A \sigma^B g^C K_i - \frac{1}{2} C^i_{ABC} \sigma^A g^B g^C k_i \\
 & - \frac{1}{6} C^i_{ABC} g^A g^B g^C h_i
 \end{aligned} \tag{2.26}$$

with

$$g^A \pi_A = g^A T_A - C^B_{AC} g^A g^C h_B - C^i_{Aj} g^A b^j k_i - C^i_{Aj} g^A g^j h_i \tag{2.27a}$$

$$b^i \pi_i = b^i T_i - C^i_{Ai} b^i g^A k_j \tag{2.27b}$$

$$g^i E_i = g^i E_i - C^i_{Ai} g^i g^A h_j \tag{2.27c}$$

$$g^A T_A = -(dg^A + C^A_{BC} g^B g^C) \pi_A - C^i_{Aj} g^A B^j K_i \tag{2.27d}$$

$$b^i T_i = -(db^i + C^i_{Aj} \sigma^A b^j) K_i \tag{2.27e}$$

$$g^i E_i = -(dg^i + C^i_{Aj} \sigma^A g^j) k_i \tag{2.27f}$$

does indeed satisfy (2.25). Moreover, from (2.27) and (2.24) one deduces the

commutation relations:

$$\begin{aligned} [\mathbb{T}_A, \mathbb{T}_B] &= C^C_{AB} \mathbb{T}_C \\ [\mathbb{T}_A, \mathbb{T}_i] &= -C^j_{Ai} \mathbb{T}_j \\ [\mathbb{T}_A, E_i] &= -C^j_{Ai} E_j \end{aligned} \quad (2.28)$$

all other commutators vanishing.

Q_{FDA1} is nilpotent by construction, since $s^2 = 0$.

3. The Kac-Moody $\oplus \text{Diff}(S^1)$ extension of FDA's

Consider first the semidirect sum of an ordinary Kac-Moody algebra (KM) with the $\text{Diff}(S^1)$ algebra:

$$[T^m_A, T^m_B] = C^C_{AB} T^{m+m}_C + k \delta_{m,-m} \eta_{AB} \quad (\eta_{AB} = G\text{-group metric}) \quad (3.1a)$$

$$[L^m, T^m_A] = -m T^{m+m}_A \quad (3.1b)$$

$$[L^m, L^m] = (m-m) L^{m+m} + \frac{c}{12} (m^3 - m) \delta_{m,-m} \quad (3.1c)$$

k and c are the respective central extensions^{*)}. The Cartan-Maurer equations for the algebra (3.1) read:

$$d\sigma^A_m + \frac{1}{2} \sum_n C^A_{BC} \sigma^B_n \sigma^C_{m-n} - \sum_n (m-n) A_n \sigma^A_{m-n} = 0 \quad (3.2a)$$

$$d\sigma + k \sum_n n \sigma^A_n \sigma^B_{-n} \eta_{AB} = 0 \quad (3.2b)$$

$$dA_m + \sum_n \frac{2n-m}{2} A_n A_{m-n} = 0 \quad (3.2c)$$

$$dA + \sum_n c \frac{n^3 - n}{24} A_n A_{-n} = 0. \quad (3.2d)$$

^{*)} Since in general G will not be simple, we should consider different k 's corresponding to the simple subalgebras of G . However this will not be essential for our subsequent arguments.

where σ_m^A and A_n are the one-forms dual to the generators T_A^n, L^n , and the one-forms σ and A are dual to the central charges k and c .

We can extend this system to include p-forms, thereby obtaining a $\widehat{FDA} \in \text{Diff}(S^1)$ algebra. The first extension involves an infinite set of two-forms B_n^i , and has the following Cartan-Maurer equations:

$$d\sigma_m^A + \frac{1}{2} C^A_{BC} \sigma_n^B \sigma_{m-n}^C - (m-n) A_n \sigma_{m-n}^A = 0 \quad (3.3a)$$

$$d\sigma + k \pi \sigma_n^A \sigma_{-n}^B \eta_{AB} = 0 \quad (3.3b)$$

$$dB_m^i + C^i_{Aj} \sigma_n^A B_{m-n}^j + \frac{1}{6} C^i_{ABC} \sigma_n^A \sigma_s^B \sigma_{m-n-s}^C - (m-n) A_n B_{m-n}^i = 0 \quad (3.3c)$$

$$dA_m + \frac{2k-m}{2} A_n A_{m-n} = 0 \quad (3.4a)$$

$$dA + \frac{c}{24} (\pi^2 - \pi) A_n A_{-n} = 0 \quad (3.4b)$$

The closure of this algebra^{*} entails the same conditions on the structure constants C^i_{ABC} and C^i_{Aj} as in (2.21) (see Appendix A).

To construct the BRST operator, we follow the procedure of the previous Section, and find the BRST transformation rules:

$$\begin{aligned} s\sigma_m^A &= -dg_m^A - C^A_{BC} g_n^B \sigma_{m-n}^C + (m-n) A_n g_{m-n}^A + (m-n) a_n \sigma_{m-n}^A \equiv -(Dg)_m^A + (m-n) a_n \sigma_{m-n}^A \\ s g_m^A &= -\frac{1}{2} C^A_{BC} g_n^B g_{m-n}^C + (m-n) a_n g_{m-n}^A \\ s\sigma &= -2k\pi \sigma_n^A g_{-n}^B \eta_{AB} - d\Sigma \\ s\Sigma &= -k\pi g_n^A g_{-n}^B \eta_{AB} \\ sB_m^i &= -db_m^i - C^i_{Aj} g_n^A B_{m-n}^j - C^i_{Aj} \sigma_n^A b_{m-n}^j - \frac{1}{2} C^i_{ABC} \sigma_n^A \sigma_s^B g_{m-n-s}^C + (m-n) A_n b_{m-n}^i + (m-n) a_n B_{m-n}^i \\ s b_m^i &= -dg_m^i - C^i_{Aj} g_n^A b_{m-n}^j - C^i_{Aj} \sigma_n^A g_{m-n}^j - \frac{1}{2} C^i_{ABC} \sigma_n^A \sigma_s^B g_{m-n-s}^C + (m-n) a_n b_{m-n}^i + (m-n) A_n g_{m-n}^i \\ s g_m^i &= -C^i_{Aj} g_n^A g_{m-n}^j - \frac{1}{6} C^i_{ABC} g_n^A g_s^B g_{m-n-s}^C + (m-n) a_n g_{m-n}^i \end{aligned}$$

^{*} We could also consider a central charge b and introduce the dual two-form B such that $dB + b T_{ABC} \sigma_n^A \sigma_s^B \sigma_{m-n-s}^C = 0$, with $T_{ABC} = G$ -invariant tensor, e.g. $C^D_{BC} \eta_{AD}$. Again this will not affect our results for the anomalies of $\widehat{FDA1} \in \text{Diff}(S^1)$.

$$\begin{aligned}
 sA_m &= -da_m - \frac{2n-m}{2} A_n a_{m-n} - \frac{2n-m}{2} a_n A_{m-n} \equiv -(Da)_m \\
 sam &= -\frac{2n-m}{2} a_n a_{m-n} \\
 sA &= -da - \frac{c}{12} (n^3-n) A_n a_n \\
 sa &= \frac{c}{24} (n^3-n) a_n a_n
 \end{aligned} \tag{3.5}$$

a_n , a and Σ are the ghosts of A_n , A and σ respectively.
The corresponding BRST operator is (normal ordering is implicit):

$$\begin{aligned}
 \widehat{\Phi}_{FDA1} \in \text{Diff}(S^1) = & \\
 & g_m^A \pi_A^m + b_m^i \pi_i^m + g_m^i E_i^m + a_m L^m + \Sigma k + a c \\
 & + \frac{1}{2} C^A_{BC} g_r^B g_s^C h_c^{r+s} + C^i_{Aj} g_r^A b_s^j k_i^{r+s} + C^i_{Aj} g_r^A g_s^j h_i^{r+s} \\
 & + \frac{1}{2} (m-m) a_m a_m \omega^{m+m} - (m-n) a_n g_{m-n}^A h_A^m - (m-n) a_n b_{m-n}^i k_i^m \\
 & - (m-n) a_n g_{m-n}^i h_i^m \\
 & - \frac{1}{2} C^i_{ABC} \sigma_n^A \sigma_s^B g_t^C K_i^{n+s+t} - \frac{1}{2} C^i_{ABC} \sigma_n^A g_s^B g_t^C k_i^{n+s+t} \\
 & - \frac{1}{2} C^i_{ABC} g_n^A g_s^B g_t^C h_i^{n+s+t}
 \end{aligned} \tag{3.6}$$

with

$$\begin{aligned}
 g_m^A \pi_A^m = & g_m^A T_A^m - C^B_{AC} g_n^A g_s^C h_B^{n+s} - C^i_{Aj} g_n^A b_s^j k_i^{n+s} \\
 & - C^i_{Aj} g_n^A g_s^j h_i^{n+s} + (m-n) a_n g_{m-n}^A h_A^m \\
 & - 2kn \sigma_n^A g_{-n}^B \eta_{AB} \pi - kn g_n^A g_{-n}^B \eta_{AB} \pi
 \end{aligned} \tag{3.7a}$$

$$b_m^i \pi_i^m = b_m^i T_i^m - C^j_{Ai} b_n^i g_s^A k_j^{n+s} + (m-n) a_n b_{m-n}^i k_i^m \tag{3.7b}$$

$$g_m^i E_i^m = g_m^i E_i^m - C^j_{Ai} g_n^i g_s^A h_j^{n+s} + (m-n) a_n g_{m-n}^i h_i^m \tag{3.7c}$$

$$\begin{aligned} a_m \mathbb{L}^m &= a_m L^m - (m-m) a_m a_m \omega^{m+m} \\ &+ (m-m) a_m g_{m-m}^A h_A^m + (m-m) a_m b_{m-m}^i k_i^m + (m-m) a_m g_{m-m}^i h_i^m \\ &- \frac{c}{12} (m^3-m) A_m a_{-m} \Omega + \frac{c}{24} (m^3-m) a_m a_{-m} \omega \end{aligned} \quad (3.7d)$$

$$\sum k = -(d\Sigma) \pi \quad (3.7e)$$

$$a \mathbb{C} = -(da) \Omega \quad (3.7f)$$

and

$$\begin{aligned} g_m^A T_A^m &= (-dg_m^A - C^A_{BC} g_m^B \sigma_{m-m}^C + (m-m) A_m g_{m-m}^A) \pi_A^m \\ &- C^i_{Aj} g_m^A B_j^i K_i^{nts} \end{aligned} \quad (3.7g)$$

$$b_m^i T_i^m = (-db_m^i - C^i_{Aj} \sigma_{m-m}^A b_{m-m}^j + (m-m) A_m b_{m-m}^i) K_i^m \quad (3.7h)$$

$$g_m^i E_i^m = (-dg_m^i - C^i_{Aj} \sigma_{m-m}^A g_{m-m}^j + (m-m) A_m g_{m-m}^i) k_i^m \quad (3.7i)$$

In Table 1 we give a list of all fields and their conjugate momenta, and of all ghosts and antighosts. The operators in (3.7a-f) have been defined according to the procedure of Appendix B. They close on the algebra:

$$\begin{aligned} [\pi_A^m, \pi_B^m] &= C^C_{AB} \pi_C^{m+m} + k k \eta_{AB} \delta_{m,-m} \\ [\mathbb{L}^m, \pi_A^m] &= -m \pi_A^{m+m} \\ [\mathbb{L}^m, \mathbb{L}^m] &= (m-m) \mathbb{L}^{m+m} + \frac{c}{12} (m^3-m) \delta_{m,-m} \\ [\pi_A^m, \pi_i^m] &= C^j_{Ai} \pi_j^{m+m} \\ [\pi_A^m, E_i^m] &= C^j_{Ai} E_j^{m+m} \end{aligned} \quad (3.8)$$

all other commutators vanishing. Applying the techniques outlined in Appendix B for the computation of Q^2 , and defining

$$C^C_{AD} C^D_{BC} \equiv -C_{ADj} \eta_{AB}, \quad C^i_{Aj} C^j_{Bi} = -C_{R} \eta_{AB} \\ (i, j = 1, \dots, \dim R)$$

we find the conditions

$$k = -2\mathbb{C}_A + 3\mathbb{C}_R \quad (3.9a)$$

$$c = 2(26 + 2\dim G - 3\dim R) \quad (3.9b)$$

for $Q^2_{\widehat{\text{FDA1}}} \in \text{Diff}(S^1) = 0$. Notice that classically (i.e. without normal ordering), we have

$$[T_A^m, T_B^m] = C^c_{AB} T_C^{m+m} + k \eta_{AB} \delta_{m,-m} \quad (3.10)$$

with T_A^m defined as in (3.7g). However normal ordering in the $\sigma_m^A, \pi_B^m, B_i^j, K_i^r$ field modes of the explicit realization (3.7g) yields the modified (quantum) commutations:

$$[T_A^m, T_B^m] = C^c_{AB} T_C^{m+m} + k' \eta_{AB} \delta_{m,-m}, \quad k' = k + \mathbb{C}_{\text{Adj}} - \mathbb{C}_R \quad (3.11)$$

Similarly, the quantum Virasoro commutators are

$$[L^m, L^n] = (m-n) L^{m+n} + \frac{c'}{12} (m^3 - m) \delta_{m,-n}, \quad c' = c - 26 - 2\dim G + 2\dim R \quad (3.12)$$

In terms of k' and c' , the $Q^2 = 0$ conditions become

$$k' = -\mathbb{C}_{\text{Adj}} + 2\mathbb{C}_R \quad (3.13a)$$

$$c' = 26 + 2(\dim G - 2\dim R) \quad (3.13b)$$

and reduce to the known conditions of refs. 11) for the Kac-Moody $\in \text{Diff}(S^1)$ algebra:

$$k' = -\mathbb{C}_{\text{Adj}} \quad (3.14a)$$

$$c' = 26 + 2\dim G \quad (3.14b)$$

when we truncate the FDA1 to its Lie algebra G .

4. Conclusions

All geometrical approaches to string field theory involve the gauging of infinite dimensional Lie algebras, e.g. the Virasoro algebra. In the loop field theory of ref. 2), the relevant algebra is the semidirect sum of a Kac-Moody type extension of a free differential algebra with the (super)Diff(S^1) algebra. The requirement of quantum consistency (i.e. anomaly-free quantum algebra) does restrict the choice of FDA.

In this paper we have found the conditions for quantum nilpotency of the BRST

operator in the case of the simplest extension of a Kac-Moody $\mathfrak{g}$ $\text{Diff}(S^1)$ algebra to a $\widehat{\text{FDA}} \in \text{Diff}(S^1)$ algebra containing an infinite set of 2-form potentials B_n^i . The generalization to the supersymmetric case, and the applications for string field theory will be reported elsewhere.

Note that our discussion has focused on the "rigid" algebras (with vanishing curvatures), and could be repeated for the more general case of "soft" algebras. In that case $Q^2 = 0$ yields also the Bianchi identities for the curvatures, and in certain supersymmetric algebras even the field equations¹⁷⁾. The rigid case, however, is important: indeed vanishing curvatures correspond, in the framework of ref.2), to free strings, and one should recover in that limit the results of the first quantized theory.

Acknowledgements

It is a pleasure to thank L. Alvarez-Gaumé, S. Cecotti, A. D'Adda and R. Stora for interesting discussions.

TABLE - The fields of $\widehat{\text{FDA1}} \in \text{Diff}(S^1)$

field	generators	bidegree	canonical commutations
σ_m^A π_A^m	T_A^m	$(1, 0)$ $(-1, 0)$	$\{\sigma_m^A, \pi_B^m\} = \delta_B^A \delta_m^m$
g_m^A h_A^m		$(0, 1)$ $(0, -1)$	$\{g_m^A, h_B^m\} = \delta_B^A \delta_m^m$
σ π	k	$(1, 0)$ $(-1, 0)$	$\{\sigma, \pi\} = 1$
Σ Π		$(0, 1)$ $(0, -1)$	$\{\Sigma, \Pi\} = 1$
B_m^i K_i^m	T_i^m	$(2, 0)$ $(-2, 0)$	$[B_m^i, K_j^m] = \delta_j^i \delta_m^m$
b_m^i k^m	E_i^m	$(1, 1)$ $(-1, -1)$	$[b_m^i, k_j^m] = \delta_j^i \delta_m^m$
g_m^i h_i^m		$(0, 2)$ $(0, -2)$	$[g_m^i, h_j^m] = \delta_j^i \delta_m^m$
A_m Ω^m	L^m	$(1, 0)$ $(-1, 0)$	$\{A_m, \Omega^m\} = \delta_m^m$
a_m ω^m		$(0, 1)$ $(0, -1)$	$\{a_m, \omega^m\} = \delta_m^m$
A Ω	ϵ	$(1, 0)$ $(-1, 0)$	$\{A, \Omega\} = 1$
a ω		$(0, 1)$ $(0, -1)$	$\{a, \omega\} = 1$

Appendix A - Integrability of $\widehat{FDA1} \in \text{Diff}(S^1)$

We only have to prove the closure of d in eq. (3.3c), since $d^2 = 0$ for the other Cartan-Maurer equations is ensured by the Jacobi identities of the ordinary Kac-Moody $\in \text{Diff}(S^1)$ system. From (3.3c) we have

$$d^2 B_m^i = - \left[C^i_{A_j} d\sigma_n^A B_{m-n}^j - C^i_{A_j} \sigma_n^A dB_{m-n}^j + \frac{1}{3} C^i_{ABC} d\sigma_n^A \sigma_s^B \sigma_{m-n-s}^C + \frac{1}{6} C^i_{ABC} \sigma_n^A \sigma_s^B d\sigma_{m-n-s}^C - (m-n) dA_n B_{m-n}^i + (m-n) A_n dB_{m-n}^i \right] \quad (A.1)$$

Now we substitute the derivatives on the fields by minus the nonderivative part of the Cartan-Maurer equations (3.3), and consider for example the $A\sigma\sigma\sigma$ terms:

$$\frac{1}{3} C^i_{ABC} (n-t) A_t \sigma_{n-t}^A \sigma_s^B \sigma_{m-n-s}^C + \frac{1}{6} C^i_{ABC} \sigma_n^A \sigma_s^B (m-n-s-t) A_t \sigma_{m-n-s-t}^C + (m-n) A_n \frac{1}{6} C^i_{ABC} \sigma_s^A \sigma_t^B \sigma_{m-n-s-t}^C \quad (A.2)$$

The first term, after the redefinition $u=n-r-s$, becomes

$$\frac{1}{3} C^i_{ABC} (m-u-s-t) A_t \sigma_{m-u-s}^A \sigma_s^B \sigma_u^C \quad (A.3)$$

and is clearly equal to zero because of $s \leftrightarrow u$ antisymmetrization due to $\sigma_s^B \sigma_u^C$. For analogous reasons the two other terms in (A.2) vanish separately.

The $\sigma\sigma\sigma\sigma$ terms vanish because of (2.21b). Indeed we have:

$$\begin{aligned} & -\frac{1}{6} C^i_{A_j} \sigma_n^A C^j_{BCD} \sigma_s^B \sigma_t^C \sigma_{m-n-s-t}^D + \frac{1}{6} C^i_{EAB} C^E_{CD} \sigma_t^C \sigma_{n-t}^D \sigma_s^A \sigma_{m-n-s}^B \\ & + \frac{1}{12} C^i_{ABE} \sigma_n^A \sigma_s^B C^E_{CD} \sigma_t^C \sigma_{m-n-s-t}^D = \\ & -\frac{1}{6} C^i_{A_j} C^j_{BCD} (\sigma_n^A \sigma_s^B \sigma_t^C \sigma_{m-n-s-t}^D) + \frac{1}{6} C^i_{EAB} C^E_{CD} (\sigma_t^C \sigma_{n-t}^D \sigma_s^A \sigma_{m-n-s}^B) \\ & + \frac{1}{12} C^i_{EAB} C^E_{CD} (\sigma_n^A \sigma_s^B \sigma_t^C \sigma_{m-n-s-t}^D) \end{aligned} \quad (A.4)$$

It is easy to prove, via index redefinitions, that the terms in the parentheses are equal and completely antisymmetric in A, B, C, D .

All other projections of (A.1) are similarly shown to vanish. This proves the closure of the $\widehat{FDA1} \in \text{Diff}(S^1)$ algebra.

These arguments easily generalize to prove the integrability of any $\widehat{FDA} \in \text{Diff}(S^1)$ system.

Appendix B - The computation of $Q^2 \widehat{FDA1} \in \text{Diff}(S^1)$

The key observation is that terms with C_{ABC}^i cannot contribute to the anomaly. Indeed the only terms in Q responsible for an anomaly have the form

$$: C^\alpha{}_\beta q^\beta p_\alpha : \quad (B.1)$$

where $C^\alpha{}_\beta$ are field dependent coefficients, and q^β, p_α are canonically conjugated fields. For example, the terms ^{*)}

$$: C^A{}_{BC} g_n^B g_s^C h_A^{n+s} : \quad (B.2)$$

give rise, when multiplied by themselves in Q^2 , to an anomalous contribution

$$g_n^A g_s^B C^C{}_{AD} C^D{}_{BC} \delta_{n,-s} = -g_n^A g_{-n}^B \eta_{AB} C_{Adj} \quad (B.3)$$

which explains the $-C_{Adj}$ in eq (3.14a). Now the terms with C_{ABC}^i do not contain canonically conjugated pairs, and their normal ordering is not essential.

We have therefore the quantum equality

$$Q^2 \widehat{FDA1} \in \text{Diff}(S^1) = \tilde{Q}^2 \quad (B.4)$$

where

$$\tilde{Q} = Q \widehat{FDA1} \in \text{Diff}(S^1) - \text{terms containing } C_{ABC}^i \quad (B.5)$$

and the $\widehat{FDA1} \in \text{Diff}(S^1)$ algebra will be anomaly free when $\tilde{Q}^2 = 0$. ($\tilde{Q}^2 = 0$ classically because of (2.21a)).

We can define the "improved" generators

$$\begin{aligned} \pi_A^m &\equiv \{\tilde{Q}, h_A^m\} & L^m &\equiv \{\tilde{Q}, \omega^m\} \\ \mathbb{T} &\equiv [\tilde{Q}, k_i^m] & k &\equiv \{\tilde{Q}, \pi\} \\ E_i^m &\equiv [\tilde{Q}, h_i^m] & c &\equiv \{\tilde{Q}, \omega\} \end{aligned} \quad (B.6)$$

Provided $\tilde{Q}^2 = 0$, these generators are $\tilde{Q}$ invariant i.e.

$$[\pi_A^m, \tilde{Q}] = 0 \quad \text{etc.} \quad (B.7)$$

^{*)} negative moded ghosts and positive moded antighosts are annihilation operators, normal ordered to the right.

and close on the algebra (3.8). Indeed

$$\begin{aligned} [\mathbb{T}_A^m, \mathbb{T}_B^m] &= [\mathbb{T}_A^m, \{\tilde{\phi}, h_B^m\}] = \{\tilde{\phi}, [\mathbb{T}_A^m, h_B^m]\} = \\ &= \{\tilde{\phi}, C_{AB}^C h_C^{m+m} + k \eta_{AB} \delta_{m,-m} \pi\} = C_{AB}^C \mathbb{T}_C^{m+m} + k \eta_{AB} \delta_{m,-m} \pi \end{aligned} \quad (B.8)$$

Also, we have

$$\tilde{\phi}^2 = g^\alpha g^\beta ([\mathbb{T}_\alpha, \mathbb{T}_\beta] - C_{\alpha\beta}^\gamma \mathbb{T}_\gamma) \quad (B.9)$$

where $\mathbb{T}_\alpha$ are any of the generators (B.6), and $C_{\beta\gamma}^\alpha$ are the structure constants of the algebra (3.8).

Therefore, to verify that the generators $\mathbb{T}_\alpha$ indeed commute as in (3.8) is equivalent to verify $\tilde{Q}^2 = 0$.

The standard trick to compute the anomalies of an algebra such as (3.8) is to evaluate the commutators on the ghost oscillator vacuum $|0\rangle$:

$$\langle 0 | [\mathbb{T}_\alpha, \mathbb{T}_\beta] | 0 \rangle \quad (B.10)$$

For example, the term $-C_{AC}^B : g_s^C h_B^{n+s} :$ in $\mathbb{T}_A^m$ ($n > 0$) can be split as

$$-C_{AC}^B \sum_{s=1}^{m-1} g_{-s}^C h_B^{m-s} - C_{AC}^B g_{-m}^C h_B^0 - C_{AC}^B g_0^C h_B^m + \dots \quad (B.11)$$

where the dots indicate terms that give vanishing contributions in the commutators (B.10). Using (B.11), it is straightforward to find

$$\begin{aligned} \langle 0 | \left[- \sum_s C_{AC}^B : g_s^C h_B^{m+s} : , - \sum_{s'} C_{A'C'}^{B'} : g_{s'}^{C'} h_{B'}^{m+s'} : \right] | 0 \rangle = \\ = - C_{Adj} \quad_{AA'} \delta_{m,-m} \end{aligned} \quad (B.12)$$

where

$$C_{AD}^C C_{BC}^D \equiv - C_{Adj} \eta_{AB} \quad (B.13)$$

Similarly, we find that $-C_{AC}^B : g_s^C \pi_B^{s+m} :$ gives the same anomalous contribution (B.13) (since it has the identical structure), whereas the three terms

$$-C^i_{A_j} B_s^j K_i^{m+s} : - C^i_{A_j} b_s^j k_i^{m+s} : - C^i_{A_j} : g_s^j h_i^{m+s} : \quad (B.14)$$

give the contribution

$$+ 3 \epsilon_R \eta_{AA'} \delta_{m, -m} \quad (B.15)$$

where

$$C^i_{A_j} C^i_{B_i} = - \epsilon_R \eta_{AB} \quad (B.16)$$

The plus sign in (B.15) is due to the opposite statistics of the conjugated pairs in (B.14). Summing all contributions we obtain the result in (3.9a). By similar arguments one easily deduces (3.9b).

REFERENCES

- 1) E. Witten, Nucl. Phys. B268(1986)253;
M. Bowick and B. Rajeev, Phys. Rev. Lett. 58(1987)535 and MIT preprints;
I. Bars and S. Yankielowicz, Phys. Rev. D35(1987)3878, and USC preprints;
M. Kaku, HEP-CCNY-14/86 and CCNY preprints;
P. Ramond, UFTP-86-20 and University of Florida preprints;
J. L. Gervais, Nucl. Phys. B276(1986)339 and LPTENS preprints;
K. Bardakci, Nucl. Phys. B284(1987)334;
I.Ya Aref'eva and I.V. Volovich, Trieste preprint ICTP-87/61;
P. Oh, UFTP-87-12;
S.R. Das and M.A. Rubin, Fermilab 85/142-T;
T. Hori, Phys. Lett. B194(1987)487;
J. Greensite and F.R. Klinkhamer, NBI-HE-88-39 /UCRL-98882;
N. Bralic, Progr. of Theor. Phys. Suppl., 86(1986).
- 2) L. Castellani, Phys. Lett. B, 206(1988)47.
- 3) Y. Ne'eman and T. Regge, Phys. Lett. 74B(1978)31 and Riv. Nuovo Cim. 1(1978)5;
A. D'Adda, R. D'Auria, P. Fré and T. Regge, Riv. Nuovo Cim. 3(1980)6;
R. D' Auria, P. Fré and T. Regge, Riv. Nuovo Cim. 3(1980)12.
- 4) L. Castellani, R. D' Auria and P. Fré, Seven Lectures on the Group Manifold approach to Supergravity and the spontaneous compactification of extra dimensions, Proceedings of the XIX-th Winter School, Karpacz 1983, ed. B. Milewski, World Scientific;
L. Castellani, R. D' Auria and P. Fré, "Supergravity Theory: a Geometrical Perspective", World Scientific 1988, to appear.
- 5) V.G. Kac, Infinite Dimensional Lie Algebras, Birkhäuser 1983;
P. Goddard and D. Olive, Kac-Moody and Virasoro algebras in relation to Quantum Physics, DAMTP 86-5.
- 6) R. D' Auria and P. Fré, Nucl. Phys. B201(1982)101;
- 7) L. Castellani, P. Fré, F. Giani, K. Pilch and P. van Nieuwenhuizen, Ann. of Physics 146(1983)35;
P. van Nieuwenhuizen, "Free graded differential superalgebras", in Group theoretical methods in physics, Lect. notes in Phys. n. 180, Springer Verlag 1983.
- 8) C. Becchi, A. Rouet and R. Stora, Phys. Lett. 52B(1974)344 and Ann. of Phys. 98(1976)287.

- 9) E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. 55B(1975)224;
L.A. Batalin and G.A. Vilkovisky, Phys. Lett. 69B(1977)309;
E.S. Fradkin and T.E. Fradkina, Phys. Lett. 72B(1978)343;
T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. 66(1979)1.
- 10) L. Faddeev and V.N. Popov, Phys. Lett. 25B(1967)30.
- 11) Z. Hlousek and K. Yamagishi, Phys. Lett. B173(1986)65;
A. Bilal and J.L. Gervais, Phys. Lett. B177(1986)313.
- 12) D. Sullivan, Infinitesimal Computations in Topology, Bull. de L'Institut
des Hautes Etudes Scientifiques, Publ. Math. n.47 (1977).
- 13) C. Chevalley and S. Eilenberg, Trans. Amer. Math. Soc. 63(1948)85;
- 14) R. Stora, Algebraic structure and topological origin of anomalies, in
Recent Progress in Gauge Theories, G. Lehmann et al eds., Plenum 1984.
- 15) L. Baulieu and M. Bellon, Nucl. Phys. B266(1986)75.
- 16) S. Boukraa, Nucl. Phys. B303(1988)237.
- 17) L. Castellani, in preparation.