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Moving Mirrors in Schrödinger Picture

M. Reuter

and

C.T. Hill⁽¹⁾

CERN Theory Division, CH-1211

Geneva, 23, Switzerland

Abstract

We formulate the problem of accelerating mirrors upon which quantum fields satisfy Dirichlet boundary conditions, with the associated Hawking effect, in a functional Schrödinger picture. Systems of this kind model some aspects of gravitational collapse leading to the formation of a black-hole. The criterion for the non-separability of the mode expansion in space- and time-dependences is the non-vanishing of a certain second-rank antisymmetric functional, the analogue of an electromagnetic field strength. Arbitrarily moving mirrors involve the functional analogue of the quantum mechanics of a charged particle in an external magnetic field. It is solved explicitly for a special case corresponding to the analogue of gravitational collapse.

⁽¹⁾On leave from Fermi National Accelerator Laboratory

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I. Introduction

The Hawking effect [1] is an intriguing feature of quantum field theory in curved spacetimes [2]. Hawking’s original argument applied to the situation of gravitational collapse leading to a black-hole, but the ideas underlying the mechanism are more general. The key problem of quantum field theory in arbitrary Riemannian spaces is that, in general, there is no preferred choice of the time-like vector field with respect to which positive and negative frequencies are defined. Different choices lead generally to different Fock spaces and, in particular, to different vacuum states. In the usual discussions one treats the solution of the wave-equation as an operator; but there is an implicit dependence in the definition of the local operator algebra upon the timelike vector field. This is reflected in the non-trivial Bogoliubov transformation relating the field operators arising from different choices of the “vacuum” [2].

In the Schrödinger (or Heisenberg) picture one must construct a Hamiltonian which propagates the wave-functional (or local operators) on the time slices defined by a time-like vector field. One selects a family of space-like surfaces, a foliation, $\{\Sigma_s\}$, each foil labelled by the “time” parameter, s , and locally orthogonal to the time-like vector field. The Schrödinger state is a wave-functional, $\Psi[\phi, s]$, of “snap-shot” (s -independent) field configurations on Σ_s . ϕ is the fundamental variable, analogous to $\vec{x}$ in ordinary single particle quantum mechanics, and the probability of finding configuration ϕ_1 on the slice labelled by s_1 is $|\Psi[\phi_1, s_1]|^2$.

The Hamiltonian, H_Σ , governs the propagation of the state from surface Σ_s to $\Sigma_{s'}$, and is a functional of fields, ϕ and their conjugate momenta, π_ϕ . In the field-space representation, ϕ is a multiplicative c-number while π_ϕ is represented by the functional derivative, $\sim -i\delta/\delta\phi$. Schrödinger picture is covariant in the sense that different observers using the same foliation agree on the evolution of the wave-functional, irrespective of the coordinate description of the foliation. However, two constructions on different foliations need not yield the same physics, e.g., the ground states may be different. This typically occurs when one of the foliations is singular, such as the case for (i) Rindler coordinates where the foils are labelled by the Rindler time, τ [3, 4], where the Schrödinger representation has been stud-

ied by [5, 6, 7] (ii) the “eternal black-hole” in Schwarzschild coordinates, [7] and (iii) deSitter space in static coordinates [8], where it is possible to give a Schrödinger representation of the deSitter group [9] and 2-dimensional conformal group [10].

In most applications of the functional Schrödinger picture one chooses a set $\{\phi_n(\xi^i|s)\}$ of basis functions on each surface Σ_s . (ξ^i are the coordinates on Σ_s .) Since arbitrary field configurations on Σ_s can be expanded in terms of the $\{\phi_n\}$, the Schrödinger equation decomposes into an infinite set of coupled ordinary differential equations for the expansion coefficients. In all applications of the Schrödinger picture discussed so far it was possible to choose the same set $\{\phi_n(\xi^i)\}$ on all surfaces Σ_s . A typical example is the evolution of a scalar field in a spatially flat Robertson–Walker spacetime [8] with metric:

$$ds^2 = dt^2 - a^2(t)\eta_{ij}d\xi^i d\xi^j . \quad (1.1)$$

Identifying s (not to be confused with the proper time element $\sqrt{ds^2}$) with the cosmic time, t , and $\Sigma_s \equiv \Sigma_t$ with the sections $t = \text{const}$, it is obvious that for every t the surface Σ_t is the euclidean space R^3 . Therefore, despite the scale factor in (1.1) we can use the same basis on each hypersurface (plane waves $\{\exp[ik_i \xi^i]\}$, say). Foliations of a given space for which this is possible we shall call separable. In general an arbitrary foliation is non-separable. This includes cosmological models with time-dependent spatial sections, for instance. Another example is (flat) spacetime bounded by a moving mirror, *i.e.* a surface on which the fields are forced to fulfil some appropriate boundary conditions [11-15]. For suitable mirror trajectories two-dimensional systems of this kind can model the radial world of a spherically collapsing star. Here the generation of Hawking radiation can be studied without the complications of a generally curved background [11,12]. Consider a real massless scalar field $\phi(t, x)$ on two-dimensional Minkowski space with coordinates (t, x) bounded by a mirror (actually a path) at $x = z(t)$. The field is required to fulfil Dirichlet boundary conditions at the location of the mirror,

$$\phi(t, z(t)) = 0 . \quad (1.2)$$

The ϕ equation of motion is simply:

$$(\partial_t^2 - \partial_x^2) \phi = 0 . \quad (1.3)$$

If we quantize the field in the Schrödinger picture with Σ_s chosen as the $s \equiv t = \text{const}$ lines, for the basis on Σ_t , we can use:

$$\phi_\omega(x|t) = \sqrt{\frac{2}{\pi}} \sin[\omega(x - z(t))], \quad \omega > 0 , \quad (1.4)$$

since the set $\{\sin(\omega x), \omega > 0\}$ is complete for $x > 0$.

Because of the time-dependent boundary conditions the ϕ_ω 's are necessarily time-dependent too. Introducing a new spatial coordinate¹ $x' \equiv x - z(t)$ we can get a time independent boundary condition, but this changes the Minkowski metric to:

$$ds^2 = (1 - \dot{z}^2)dt^2 - 2\dot{z}dx'dt - dx'^2 \quad (1.5)$$

and the field equation to:

$$[\partial_t^2 - 2\dot{z}\partial_t\partial_{x'} - \ddot{z}\partial_{x'} - (1 - \dot{z}^2)\partial_{x'}^2] \phi = 0. \quad (1.6)$$

Except for the trivial case $\dot{z} = \text{const}$, the solutions of eq.(1.6) are not separable as $\phi = \phi_\omega(x') \exp[-i\omega t]$. Hence the standard Fock-space construction, introducing the concept of "particles," does not apply. Quite generally, the impossibility of having time-independent basis functions and the fact that the wave equation cannot be separated are two aspects of the same difficulty. If, on the contrary, a wave equation like $\square\phi = 0$ can be separated, the solutions $\{\Phi_\omega\}$ of the space part $(\nabla^2 + \omega^2)\Phi_\omega = 0$ are complete on the surfaces $t = \text{const}$ (assuming $\nabla^2 + \omega^2$ to be Hermitian). Assuming this implies that the foliation $t = \text{const}$ is separable and in this case the Schrödinger Hamiltonian decouples into a set of harmonic oscillator Hamiltonians, possibly with time-dependent frequencies. In this case it is fairly easy to study, for instance, cosmological particle creation. Making a Gaussian product ansatz for the ground state wave-functional, the Schrödinger equation becomes an ordinary differential equation for its covariance. Once

¹By transforming both x and t it is possible for most trajectories to have time-independent boundary conditions and satisfy an ordinary Klein-Gordon equation. [11] This amounts to a change of the foliation, however.

this is solved, it is straightforward to obtain particle number expectation values [5].

The purpose of this paper is two-fold. First, we set up a general formalism for field quantization on non-separable foliations. Second, we apply this formalism to the moving mirror model for two different foliations, one of which will be relevant to the discussion of Hawking radiation. In section (II) we show that a free field theory on a non-separable foliation does not lead to a set of harmonic oscillators, but rather to a set of oscillators interacting with a “functional gauge connection” [16]. In section (III) we discuss the moving mirror model with a $t = \text{const}$ foliation as an example. In section (IV) Hawking radiation is studied in the framework of a moving mirror with a Schrödinger picture quantization on null hypersurfaces. Using a special mirror trajectory we find a ground state with a thermal flux of particles which, in the language of the black-hole analogy, is radiated by the hole. For the eternal black-hole it was shown by Lee [7] that the thermal character of the Hawking radiation persists even for massive and interacting field theories. In section (V) we investigate whether this is the case for our dynamical picture of the collapse. It turns out that an analogue of Lee’s theorem can be obtained, but only for a free massless field can it be used to demonstrate the thermal nature of the density matrix for the outgoing quanta.

II. Covariant Schrödinger Picture for Non-Separable Foliation

Before we begin the discussion of the novel features coming from the non-separability of the foliation, let us briefly repeat the main ingredients of the functional Schrödinger picture [5]. Given a spacetime manifold, we fix a coordinate system², x^μ ($\mu = 0, 1, \dots, d$) and choose a family $\{\Sigma_s\}$ of spacelike hypersurfaces labelled by a parameter s . On Σ_s we choose coordinate ξ^i ($i = 1, \dots, d$). The embedding of the hypersurfaces into spacetime is defined by a set of equations $x^\mu = x^\mu(s, \xi^i)$. Then the timelike vector-field

²For notational simplicity we ignore possible complications due to the necessity of having to introduce several coordinate charts to cover the entire manifold.

$\partial x^\mu / \partial s$ is normal to Σ_s . The volume element of the surface reads:

$$d^d \Sigma_\mu = \sqrt{-g} \epsilon_{\mu\nu_1 \dots \nu_d} \frac{\partial x^{\nu_1}}{\partial \xi^1} \dots \frac{\partial x^{\nu_d}}{\partial \xi^d} d^d \xi \equiv \frac{D \Sigma_\mu}{D \xi} d^d \xi. \quad (2.1)$$

It is invariant under changes of the coordinates ξ^i on Σ_s . Now consider a scalar field $\phi(s, \xi) \equiv \phi(x^\mu(s, \xi))$ with action S and energy momentum tensor $T_{\mu\nu}$. The theory is quantized by defining the canonical momentum as:

$$\pi = |g|^{-1/2} \frac{\partial x^\mu}{\partial s} \frac{\delta S}{\delta(\partial^\mu \phi)} \quad (2.2)$$

and requiring the following commutation relations on each hypersurface:

$$[\phi(s, \xi), \pi(s, \xi')] = i \delta_\Sigma(\xi, \xi'). \quad (2.3)$$

Here the δ -function:

$$\delta_\Sigma(\xi, \xi') = \left| \frac{\partial x^\mu}{\partial s} \frac{\partial x_\mu}{\partial s} \right|^{1/2} \left| \frac{D \sigma^\nu}{D \xi} \frac{D \Sigma_\nu}{D \xi} \right|^{-1/2} \prod_{i=1}^d \delta(\xi^i - \xi'^i) \quad (2.4)$$

it is invariant under both $s \rightarrow s'(s)$ transformations and ξ -reparametrization on Σ_s . The commutation relation eq.(2.3) is fulfilled if on each Σ_s the operator ϕ is represented by the multiplication with an s -independent function $\phi(\xi)$ and the momentum by:

$$\pi(\xi) = -i \left| \frac{\partial x^\mu}{\partial s} \frac{\partial x_\mu}{\partial s} \right|^{1/2} \left| \frac{D \sigma^\nu}{D \xi} \frac{D \Sigma_\nu}{D \xi} \right|^{-1/2} \frac{\delta}{\delta \phi(\xi)}. \quad (2.5)$$

The functional derivative is to be understood in a d -dimensional sense.

The evolution of the wave-functional $\Psi[\phi(\xi); s]$ is governed by the Schrödinger equation:

$$\begin{aligned} H \Psi[\phi(\xi); s] &= \int d^d \Sigma^\mu \frac{\partial x^\nu}{\partial s} T_{\mu\nu}[\phi(\xi), \pi(\xi)] \Psi[\phi(\xi); s] \\ &= i \frac{d}{ds} \Psi[\phi(\xi); s]. \end{aligned} \quad (2.6)$$

The Hamiltonian H is s -independent only if the spacetime under consideration allows $\partial x^\mu / \partial s$ to be chosen as a timelike Killing vector field. The Schrödinger equation, eq.(2.6), is form invariant under changes of the space-time coordinates x^μ , changes of the surface coordinates ξ^i and under reparametrizations $s \rightarrow s'(s)$ of the evolution parameter. It is in this

sense that the formalism is covariant. The quantization procedure does depend on the choice of the foliation, however. Different choices of $\{\Sigma_s\}$ lead to different quantum theories. Eq. (2.6) was solved explicitly for Rindler space [5] and for spatially flat Robertson-Walker spacetimes [8]. For $\phi(\xi)$ an ansatz:

$$\phi(\xi) = \sum_n a_n \phi_n(\xi) \quad (2.7)$$

was made with an s -independent basis $\{\phi_n\}$ on all Σ_s . Then the wave functional $\Psi[\phi; s] \equiv \Psi[\{a_n\}; s]$ is a (product-Gaussian) function of the a_n 's, which may be viewed as generalizing the position vector, $\vec{x}$, of ordinary quantum mechanics. [The index n can, of course, be continuous, *e.g.* a momentum label.] It turns out that H is diagonalized as the ϕ_n 's are chosen as solutions of the classical field equations with the time-dependence separated off as $\exp[-i\omega_n s]$.

We consider now the case of non-separable foliations. By definition this means that it is impossible to use the same system $\{\phi_n(\xi)\}$ on all hypersurfaces Σ_s . One has to introduce a family $\{\phi_n(\xi|s)\}$ of complete systems, one for each surface. The time evolution of $\Psi[\{a_n\}; s]$ consists of two pieces then: there is the usual explicit s -dependence of the wave functional, but there is also an implicit time dependence due to the fact that the basis functions are s -dependent. Therefore the Hamiltonian no longer has the simple form of eq.(2.6). We now study the time evolution of Ψ including the effect of the s -dependence of ϕ_n . For definiteness we consider a minimally coupled, real scalar field with the action:

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2\} . \quad (2.8)$$

Let $\phi = \phi(x^\mu)$ be an arbitrary field configuration defined on the entire spacetime manifold. Using the relations $x^\mu = x^\mu(s, \xi^i)$ defining the embedding, we write ϕ as a function of s and ξ^i ; $\phi(s, \xi^i) \equiv \phi(x^\mu(s, \xi^i))$. This function is expanded in terms of the complete sets $\{\phi_n(\xi|s)\}$ as:

$$\phi(s, \xi^i) = \sum_n \{a_n(s) \phi_n(\xi|s) + a_n^*(s) \phi_n^*(\xi|s)\}. \quad (2.9)$$

[For later application it is useful to allow for complex ϕ_n 's.] Note that the basis vectors $\{\phi_n(\xi|s)\}$ are complete on each Σ_s , but this does not imply that they are complete when considered as one function $f_n(x^\mu) =$

$\phi_n(\xi(x^\mu)|s(x^\mu))$ on the spacetime manifold. This is obvious from the extreme case of a separable foliation where ϕ_n has no s -dependence at all. Thus, in the expansion eq.(2.9) *the coefficients are generally required to be s -dependent.*

We now express the action in terms of the a_n 's by inserting eq.(2.9) into eq.(2.8). One obtains:

$$\begin{aligned}
 S = \frac{1}{2} \int ds \sum_{n,m} & \left\{ (\dot{a}_n, \dot{a}_n^*) \begin{pmatrix} T_{nm}^{(1)} & T_{nm}^{(2)} \\ T_{nm}^{(2)} & T_{nm}^{(1)*} \end{pmatrix} (\dot{a}_m, \dot{a}_m^*)^T \right. \\
 & - (a_n, a_n^*) \begin{pmatrix} V_{nm}^{(1)} & V_{nm}^{(2)} \\ V_{nm}^{(2)} & V_{nm}^{(1)*} \end{pmatrix} (a_m, a_m^*)^T \\
 & \left. + 2 (\dot{a}_n, \dot{a}_n^*) \begin{pmatrix} f_{nm}^{(1)} & f_{nm}^{(2)} \\ f_{nm}^{(2)} & f_{nm}^{(1)*} \end{pmatrix} (a_m, a_m^*)^T \right\} \quad (2.10)
 \end{aligned}$$

where:

$$\begin{aligned}
 T_{nm}^{(1)}(s) &= \int d^d \tau \tilde{g}^{ss} \phi_n \phi_m \\
 T_{nm}^{(2)}(s) &= \int d^d \tau \tilde{g}^{ss} \phi_n \phi_m^* \\
 V_{nm}^{(1)}(s) &= \int d^d \tau \{ m^2 \phi_n \phi_m - \tilde{g}^{ss} \dot{\phi}_n \dot{\phi}_m - 2 \tilde{g}^{sj} \dot{\phi}_n \partial_j \phi_m - \tilde{g}^{ij} \partial_i \phi_n \partial_j \phi_m \} \\
 V_{nm}^{(2)}(s) &= \int d^d \tau \{ m^2 \phi_n \phi_m^* - \tilde{g}^{ss} \dot{\phi}_n \dot{\phi}_m^* - 2 \tilde{g}^{sj} \dot{\phi}_n \partial_j \phi_m^* - \tilde{g}^{ij} \partial_i \phi_n \partial_j \phi_m^* \} \\
 f_{nm}^{(1)}(s) &= \int d^d \tau \{ \tilde{g}^{ss} \phi_n \dot{\phi}_m + \tilde{g}^{sj} \phi_n \partial_j \phi_m \} \\
 f_{nm}^{(1)}(s) &= \int d^d \tau \{ \tilde{g}^{ss} \phi_n \dot{\phi}_m^* + \tilde{g}^{sj} \phi_n \partial_j \phi_m^* \} \quad (2.11)
 \end{aligned}$$

and:

$$\begin{aligned}
 d^d \tau &= \frac{\partial x^\mu}{\partial s} \frac{D \Sigma_\mu}{D \xi} d^d \xi \\
 \tilde{g}^{ss} &= g^{\mu\nu} \frac{\partial s}{\partial x^\mu} \frac{\partial s}{\partial x^\nu} \\
 \tilde{g}^{sj} &= g^{\mu\nu} \frac{\partial s}{\partial x^\mu} \frac{\partial \xi^j}{\partial x^\nu} \\
 \tilde{g}^{ij} &= g^{\mu\nu} \frac{\partial \xi^i}{\partial x^\mu} \frac{\partial \xi^j}{\partial x^\nu}. \quad (2.12)
 \end{aligned}$$

Here $\dot{\phi}$ and $\partial_i \phi$ denote the derivative with respect to s and ξ^i , respectively. In an obvious matrix notation the action can be written as:

$$S = \frac{1}{2} \int ds \sum_{N,M} \{ \dot{A}_N T_{NM} \dot{A}_M - A_N V_{NM} A_M + 2 \dot{A}_N F_{NM} A_M \} \quad (2.13)$$

where $A_N \equiv (a_n, a_n^*)$ and the matrices T , V and F can be read off from eq.(2.10) and eq.(2.11); in general they depend on s .

The canonical variables are the A_N 's and their conjugate momenta:

$$P_N = T_{NM} \dot{A}_M + F_{NM} A_M. \quad (2.14)$$

They are required to satisfy the canonical commutation relations

$$[A_N, P_M] = i\delta_{N,M}; \quad [A_N, A_M] = [P_N, P_M] = 0. \quad (2.15)$$

Acting on a Schrödinger wave functional $\Psi[A_N; s]$ we represent A_N multiplicatively and P_N as $-i\delta/\delta A_N$. The Hamiltonian, H , describing the time evolution of Ψ is obtained from eq.(2.13) via the usual Legendre transformation. In the next section we write it down for a special example. In general it is rather complicated, since it not only contains s -dependent kinetic and potential energy matrices T and V , but also a term coupling A_N to $\dot{A}_N$ or P_N , respectively. As is seen from eq.(2.11), the origin of this term is the s -dependence of the basis functions. It couples A_N to a “functional gauge potential”³ $\mathcal{A}_N = \mathcal{F}_{NM} A_M$ with a constant, *i.e.* A -independent field strength tensor $\mathcal{F}_{[N,M]}$. This is reminiscent of a term $L_{int} = e x_i F^{ij} \dot{x}_j$ in particle mechanics, where $A^i = \frac{1}{2} F^{ij} x_j$ is the vector potential of a constant magnetic field F^{ij} . A consequence of this “functional magnetic field” is that the velocity operators do not commute:

$$[\dot{A}_M, \dot{A}'_M] = 2iT_{MN}^{-1} T_{M'N'}^{-1} \mathcal{F}_{[N',N]} \quad (2.16)$$

Hence there are correlations between the modes A_N similar to the correlation between the x - and y - motion of a charged particle in a magnetic field directed along the z -axis. By the standard argument a “pure gauge,” $\mathcal{A}_N = \delta\Lambda[A]/\delta A_N$, does not affect the dynamics. By inspection of $f_{nm}^{(1)}$ and $f_{nm}^{(2)}$ in equation (2.11) one can readily be convinced that a pure gauge

³For a discussion of functional gauge connection in a different context see ref. [16].

would correspond to a situation where we are using an s -dependent basis system without being forced to do so, *i.e.* when on each Σ_s there exists an s -dependent unitary transformation changing $\{\phi_n(\xi|s)\}$ to a new s -independent basis $\{\phi'_n(\xi)\}$. By definition this means that $\{\Sigma_s\}$ is separable. Hence the statement $\mathcal{F}_{MN} \neq 0$ is a coordinate system independent characterization of non-separable foliations.

In principle, at least, one can solve the Schrödinger equation resulting from the above construction and one then could compute arbitrary expectation values. However, the association of “particles” with the excitation A_N is even more dubious than it is for separable foliations (in curved space) already. There we can choose the basis functions on Σ_s to be the spatial part of a complete set of solutions of the classical field equations. Because these modes diagonalize the Hamiltonian, they do not mix during the time evolution. In this sense they preserve their identity and could be considered as “particles” in a restricted sense. In the non-separable cases, however, it is not possible to find modes with this property. The closest analogue one could imagine is that it is possible (in general it is not) to diagonalize T and V , and to simultaneously skew-diagonalize F . Then the Hamiltonian would be diagonalized by pairs of A_N ’s corresponding to the x - and y -coordinates of our quantum mechanical analogue model.

III. Moving Mirrors: Spacelike Surfaces

To illustrate the method of Section (II) we consider two dimensional Minkowski space with coordinates (t, x) bounded by a moving mirror at $x = z(t)$. An example is shown in Fig. 1. For $x > z(t)$ we define a scalar field $\phi(t, x)$ with the action:

$$S = \frac{1}{2} \int dt \int_{z(t)}^{\infty} dx \{(\partial_t \phi)^2 - (\partial_x \phi)^2 - m^2 \phi^2\}. \quad (3.1)$$

ϕ is assumed to obey Dirichlet boundary conditions of the location of the mirror:

$$\phi(t, z(t)) = 0. \quad (3.2)$$

The hypersurfaces $\Sigma_s \equiv \Sigma_t$ are taken to be the lines $t = \text{const}$. As a basis on Σ_t we use the functions of eq.(1.3). A general configuration $\phi = \phi(t, x)$, $x >$

$z(t)$, is expanded as:

$$\phi(t, x) = \sqrt{\frac{2}{\pi}} \int_0^\infty d\omega a_\omega(t) \sin[\omega(x - z(t))]. \quad (3.3)$$

The action reads now:

$$\begin{aligned} S = & \frac{1}{2} \int dt \int_0^\infty d\omega \{ (\dot{a}_\omega)^2 - \omega^2 [1 - \dot{z}^2] a_\omega^2 - m^2 a_\omega^2 \} \\ & - \int dt \dot{z} \int_0^\infty d\omega d\omega' F(\omega, \omega') \dot{a}_\omega a_{\omega'} \end{aligned} \quad (3.4)$$

with the antisymmetric “field strength tensor:”

$$F(\omega, \omega') = \frac{2}{\pi} \int_0^\infty dx \sin(\omega x) \omega' \cos(\omega' x). \quad (3.5)$$

The momenta conjugate to a_ω are:

$$p_\omega = \dot{a}_\omega - \dot{z} \int_0^\infty d\omega' F(\omega, \omega') a_{\omega'} \quad (3.6)$$

and the Hamiltonian becomes:

$$\begin{aligned} H = & \frac{1}{2} \int_0^\infty d\omega \left\{ (p_\omega + \dot{z} \int_0^\infty d\omega' F(\omega, \omega') a_{\omega'})^2 \right. \\ & \left. + [\omega^2 (1 - \dot{z}^2) + m^2] a_\omega^2 \right\}. \end{aligned} \quad (3.7)$$

We again note the similarity with the Hamiltonian of a particle subject to the Lorentz force and an additional harmonic restoring force. In the Schrödinger picture the system is quantized by maintaining a_ω as a c-number and replacing p_ω by the derivative operator $-i\delta/\delta a_\omega$. Then the velocity commutators are:

$$[\dot{a}_\omega, \dot{a}_{\omega'}] = 2i\dot{z}F(\omega, \omega'). \quad (3.8)$$

The Schrödinger equation becomes:

$$H(t)\Psi[\{a_\omega\}; t] = i \frac{d}{dt} \Psi[\{a_\omega\}; t] \quad (3.9)$$

with the Hamiltonian:

$$H = \frac{1}{2} \int_0^\infty d\omega \left\{ -\frac{\delta^2}{\delta a_\omega^2} - 2i\dot{z} \int_0^\infty d\omega' F(\omega, \omega') a_{\omega'} \frac{\delta}{\delta a_\omega} + [\omega^2 + m^2] a_\omega^2 \right\}. \quad (3.10)$$

In going from eq.(3.7) to eq.(3.10) we have omitted a divergent normal ordering constant arising from $\delta a_\omega / \delta a_\omega = \delta(0)$. The second term on the RHS of eq.(3.10) has a simple interpretation. In terms of instantaneous field configurations $\phi(x)$ it is proportional to $\int_{z(t)}^\infty dx \dot{\phi}(x) \delta / \delta \phi(x)$. Therefore, acting on $\Psi[\phi(x); t]$, it generates that part of the time dependence of Ψ which compensates for the implicit time dependence in the basis functions.

Since H does explicitly depend on time, there exists no distinguished ground state assuming the lowest energy eigenvalue. If we nevertheless choose Ψ to be a product of Gaussians on some initial surface, time evolution will take this state into a more complicated one containing cross terms between the a_ω 's. In a typical separable situation this is forbidden by spatial momentum conservation, which, in our case, is spoiled by the presence of the mirror. Except for the trivial case $\dot{z} = \text{const}$, it is in general not possible to obtain closed form solutions to the Schrödinger equation eq.(3.9) with eq.(3.10). There are two regimes where approximate solutions can be obtained. If $\dot{z}(t) \ll 1$ for all t between the initial and final surface, perturbation theory with respect to the second term on the RHS of equation (3.10) is possible. On the other hand, if $z(t)$ is constant except for a very small time interval during which it changes from z_1 to z_2 , the final state is simply determined by the overlap integrals between $\sin[\omega(x - z_1)]$ and $\sin[\omega'(x - z_2)]$.

IV. Moving Mirrors: Null Surfaces

In this section we quantize the moving mirror system in the Schrödinger picture employing a family of null surfaces. These are the foliations for which the analogy between the receding mirror and a spherically symmetric gravitational collapse occurs [11,12]. Introducing light-cone coordinates $u = t - x$ and $v = t + x$ the hypersurfaces $\Sigma_s \equiv \Sigma_u$ are taken to be the lines $u = \text{const}$. The dynamical variables are "instantaneous" field configurations $\phi = \phi(v)$. As sketched in Figs. (1) and (2) we are using a mirror trajectory with a future horizon at $v = 0$ and with no past horizon. We shall impose initial conditions for Ψ on null past infinity $\mathcal{J}^-$. The Schrödinger equation determines the wave functional on future null infinity $\mathcal{J}^+$. All information about the Hawking radiation produced by the mirror

is contained in $\Psi(\mathcal{J}^+) \equiv \Psi(u = +\infty)$. Future null infinity consists of two components $\mathcal{J}_L^+$ and $\mathcal{J}_R^+$. In the black-hole analogy left moving massless quanta asymptotically approaching $\mathcal{J}_L^+$ correspond to particles trapped by the black-hole. They leave $\mathcal{J}^-$ at positive v -values. Quanta leaving $\mathcal{J}^-$ at negative v values are reflected by the mirror and asymptotically approach $\mathcal{J}_R^+$. They correspond to particles which can escape from the black-hole. The last trapped ray, $v = 0$, acts as a horizon. In general null plane quantization is not equivalent to quantization on spacelike hypersurfaces. The lines $u=\text{const}$ and $v=\text{const}$ are characteristics of the Klein-Gordon equation. Initial conditions should be imposed on both u - and v -lines. Otherwise a wave packet travelling along some line $u = \text{const}$ at the speed of light can never be predicted from the initial data specified on $u = -\infty$. In the present case, however, such a wave packet would be reflected by the mirrors and hence would have to originate on $\mathcal{J}^-$. In Fig. (3) we have indicated how the evolution on the $u = \text{const}$ surface can be obtained as the limit of a quantization on spacelike hypersurfaces.

We now consider the theory, eq.(3.1), for a special trajectory $x = z(t)$ implicitly defined by the equation:

$$t + z(t) = -\kappa^{-1}e^{-\kappa(t-z(t))}, \quad \kappa > 0. \quad , \quad (4.1)$$

Its ray tracing function [11] is given by:

$$p(u) = -\kappa^{-1}e^{-\kappa u}. \quad (4.2)$$

In terms of $p(u)$ the boundary condition reads:

$$\phi(u, v = p(u)) = 0. \quad (4.3)$$

This trajectory has been introduced by Walker [15] and by Carlitz and Willey [14]. It is plotted in figs. (1) and (2). It has a null asymptote at $v = 0$ but no u -asymptote. Applying the strategy of Section (II) we start from the following expansion for a general field configuration $\phi = \phi(u, v)$:

$$\phi(u, v) = \int d\omega \{a_\omega(u)\phi_\omega(v|u) + a_\omega^*\phi_\omega^*(v|u)\}. \quad (4.4)$$

Here, $\{\phi_\omega(v|u)\}$ is a complete set of functions of v for each Σ_u . It proves advantageous to make a particular choice for the basis, namely to set

$\phi_\omega(v|u) \equiv \phi_\omega(u, v)$ where $\{\phi_\omega(u, v)\}$ is a basis in the space of solutions of the classical field equation:

$$\left[\partial_u \partial_v + \frac{1}{4} m^2 \right] \phi_\omega(u, v) = 0 \quad (4.5)$$

subject to the boundary condition, eq.(4.3). The ϕ_ω 's are assumed to be orthonormalized with respect to the inner product:

$$(\phi_{\omega'}, \phi_\omega) \equiv i \int_{p(\omega)}^\infty \phi_{\omega'}^* \overleftrightarrow{\partial}_v \phi_\omega(u, v) = \delta(\omega' - \omega). \quad (4.6)$$

This choice fixes the parametrical u -dependence of $\phi_\omega(v|u)$ in a way which simplifies the dynamics of the a_ω 's. Inserting eq.(4.4) into eq.(3.1) and using eq.(4.6) yields:

$$S = \frac{i}{2} \int du d\omega [a_\omega^* \partial_u a_\omega - a_\omega \partial_u a_\omega^*] - V(a_\omega, a_\omega^*) \quad (4.7)$$

with:

$$\begin{aligned} V(a_\omega, a_\omega^*) = & \frac{m^2}{4} \int du d\omega d\omega' [a_{\omega'} a_\omega \int_{p(u)}^\infty dv \phi_\omega \phi_{\omega'} \\ & + a_{\omega'} a_\omega^* \int_{p(u)}^\infty dv \phi_{\omega'} \phi_\omega^* + h.c.]. \end{aligned} \quad (4.8)$$

Our aim is to derive a Schrödinger equation for $\Psi(u)$ from eq.(4.7). Since S is first order in u , Dirac's formalism for constrained systems [17] has to be applied. As was shown in [18], this leads to the following commutator relations:

$$[a_{\omega'}, a_\omega^\dagger] = \delta(\omega - \omega') \quad [a_{\omega'}, a_\omega] = [a_{\omega'}^\dagger, a_\omega^\dagger] = 0. \quad (4.9)$$

The Hamiltonian is given by

$$H = V(a, a^*) . \quad (4.10)$$

Together with eq.(4.9) this form guarantees that the Heisenberg equations reproduce the classical equations of motion [19]. In the Schrödinger picture the algebra (4.9) is realized by the differential operators:

$$a_\omega = \frac{1}{\sqrt{2}} \left[\frac{\delta}{\delta q_\omega} + q_\omega \right], \quad a_\omega^\dagger = \frac{1}{\sqrt{2}} \left[-\frac{\delta}{\delta q_\omega} + q_\omega \right], \quad (4.11)$$

acting on wave-functionals $\Psi[\{q_\omega\}; u]$ with q_ω real. They are adjoints of each other with respect to the scalar product:

$$\langle \Psi_1 | \Psi_2 \rangle \equiv \int Dq \Psi_1^*[q] \Psi_2[q] \quad (4.12)$$

where $Dq \equiv \prod_\omega dq_\omega$ and $q \equiv \{q_\omega\}$. Obviously the main difference between the spacelike quantization and the null plane quantization is the following. In the first case the field amplitudes and momenta are realized as c-numbers, a_ω , and derivative operators, $-i\delta/\delta a_\omega$, respectively. Raising and lowering operators can be introduced as linear combinations of these. In the latter case, $a_\omega^\dagger$ and a_ω , themselves, act as creation and annihilation operators (as in Dirac theory).

We first discuss massless fields. Then $V \equiv 0$ and the Hamiltonian vanishes identically. Comparing eq.(4.7) with eq.(2.10), we observe that in the limit when Σ_s becomes lightlike the kinetic part of the action eq.(4.7) arises from the term in eq.(2.10) containing the functional gauge potential. The Schrödinger equation simply reads

$$\frac{\partial}{\partial u} \Psi[\{q_\omega\}; u] = V \Psi[\{q_\omega\}; u] \equiv 0. \quad (4.13)$$

This means that the whole time evolution is already contained in the parametric time-dependence of the basis functions.⁴ Only for non-vanishing mass does Ψ acquire an explicit u -dependence. This is related to the fact that for $m \neq 0$ the excitations also can propagate within the light cone.

It is easy to find solutions of the massless equation eq.(4.5) vanishing at $(u, p(u))$ [11]:

$$\phi_\omega(v|u) \equiv \phi_\omega(u, v) = (4\pi\omega)^{-1/2} [e^{-i\omega v} - e^{-i\omega p(u)}], \quad \omega > 0. \quad (4.14)$$

These modes effectively reduce to exponentials, $\exp[-i\omega v]$ on $\mathcal{J}^-$. Hence the degrees of freedom labeled by q_ω correspond to left-moving waves leaving $\mathcal{J}^-$ in the distant past. The “vacuum” Ψ_0 of the theory we define by a initial configuration $\Psi_0(u = -\infty)$ with no ϕ_ω -excitations present on $\mathcal{J}^-$, *i.e.* it is annihilated by the destruction operators a_ω . In the Schrödinger

⁴The time independence of Ψ is reminiscent of a Heisenberg state vector. In a general massive and interacting theory, however, Ψ (unlike a Heisenberg state) does depend on u .

picture this means that:

$$\left(\frac{\delta}{\delta q_\omega} + q_\omega\right) \Psi_0[\{q_\omega\}, u = -\infty] = 0. \quad (4.15)$$

The ground state Ψ_0 is defined independently of the time evolution operator. Here an analogous rôle is played by the integrated momentum flux through $\mathcal{J}^-$. It is measured by the stress tensor component $T_{vv} = (\partial_v \phi)^2$. Using eq.(4.4), eq.(4.11) and eq.(4.14) one finds:

$$\int_{-\infty}^{\infty} dv T_{vv} = \frac{1}{2} \int_0^{\infty} d\omega \omega \left[-\frac{\delta^2}{\delta q_\omega^2} + q_\omega^2 \right]. \quad (4.16)$$

Obviously Ψ_0 is the lowest eigenstate of $\int dv T_{vv}$. It is explicitly given by the Gaussian⁵:

$$\Psi_0[q] = \exp \left[-\frac{1}{2} \int_0^{\infty} d\omega q_\omega^2 \right]. \quad (4.17)$$

Since we know that there is no explicit time-dependence for $m = 0$, the wave functional has the same form on each surface Σ_u . An instantaneous field configuration $\phi_\omega(v|u)$ can be addressed as a “particle” in the sense that the ϕ_ω ’s do not mix during the u -evolution. To achieve this, even in the present non-separable case, we had to give a special parametric time-dependence to ϕ_ω (In the separable case ϕ_ω is time independent; it solves the field equations with the time-dependence separated off). On $\mathcal{J}^+$, Ψ_0 is still given by eq.(4.17). However, there the modes of eq.(4.14) do not possess a simple particle (plane wave) interpretation. Therefore we introduce a new complete set of basis functions [14]:

$$\begin{aligned} \phi_\omega^R(v|u) &= (4\pi\omega)^{-1/2} \left[e^{i\omega \ln(-\kappa v)/\kappa} \theta(-v) - e^{-i\omega u} \right], & \omega > 0; \\ \phi_\omega^L(v|u) &= (4\pi\omega)^{-1/2} e^{-i\omega \ln(\kappa v)/\kappa} \theta(v) & \omega > 0; \end{aligned} \quad (4.18)$$

The modes ϕ_ω^R reduce to plane waves on $\mathcal{J}_R^+$. The precise form of ϕ_ω^L is chosen for mathematical convenience. Since the particles trapped by the black-hole are not detected, it is not necessary that they reduce to plane waves on $\mathcal{J}_L^+$. We now quantize the theory using the modes eq.(4.18). Equation (4.4) becomes:

$$\phi(u, v) = \sum_{I=L,R} \int_0^{\infty} d\omega \left\{ a_\omega^I(u) \phi_\omega^I(v|u) + a_\omega^{I\dagger}(u) \phi_\omega^{I*}(v|u) \right\} \quad (4.19)$$

⁵For simplicity, the normalization factors of wave functionals and density matrices are omitted throughout the paper.

and eq.(4.9) is replaced by:

$$[a_{\omega'}^I, a_{\omega}^{J\dagger}] = \delta^{IJ} \delta(\omega - \omega') \quad [a_{\omega'}^I, a_{\omega}^J] = [a_{\omega'}^{I\dagger}, a_{\omega}^{J\dagger}] = 0 \quad (4.20)$$

The Schrödinger representation of this algebra reads

$$a_{\omega}^I = \frac{1}{\sqrt{2}} \left[\frac{\delta}{\delta q_{\omega}^I} + q_{\omega}^I \right], \quad a_{\omega}^{I\dagger} = \frac{1}{\sqrt{2}} \left[-\frac{\delta}{\delta q_{\omega}^I} + q_{\omega}^I \right]. \quad (4.21)$$

Our aim is to express Ψ_0 in terms of the new variables $\{q_{\omega}^I, I = L, R\}$. Using the orthonormality properties eq.(4.6) for $u \rightarrow -\infty$ we can express $\{\phi_{\omega}\}$ in terms of $\{\phi_{\omega}^L\}$ and $\{\phi_{\omega}^R\}$. For the trajectory eq.(4.1) the necessary scalar products can be evaluated in closed form. This basis change implies the following transformation of q_{ω} and $\delta/\delta q_{\omega}$:

$$\begin{aligned} q_{\omega'} &= \frac{1}{2} \sum_{I=L,R} \int_0^{\infty} d\omega \left\{ A_{\omega,\omega'}^I q_{\omega}^I + B_{\omega,\omega'}^I \frac{\delta}{\delta q_{\omega}^I} \right\}, \\ \frac{\delta}{\delta q_{\omega'}} &= \frac{1}{2} \sum_{I=L,R} \int_0^{\infty} d\omega \left\{ \tilde{B}_{\omega,\omega'}^I q_{\omega}^I + \tilde{A}_{\omega,\omega'}^I \frac{\delta}{\delta q_{\omega}^I} \right\}. \end{aligned} \quad (4.22)$$

The coefficients are given by (see ref.[5] for a related calculation):

$$\begin{aligned} A_{\omega,\omega'}^R &= A_{\omega,\omega'}^L = \tanh\left(\frac{\pi\omega}{2\kappa}\right) \tilde{A}_{\omega,\omega'}^R = \tanh\left(\frac{\pi\omega}{2\kappa}\right) \tilde{A}_{\omega,\omega'}^L \\ &= \frac{1}{\pi\kappa} \left(\frac{\omega}{\omega'}\right) \sinh\left(\frac{\pi\omega}{2\kappa}\right) \left\{ \left(\frac{\omega'}{\kappa}\right)^{-i\omega/\kappa} \Gamma(i\omega/\kappa) + \left(\frac{\omega'}{\kappa}\right)^{i\omega/\kappa} \Gamma(-i\omega/\kappa) \right\} \\ B_{\omega,\omega'}^R &= -B_{\omega,\omega'}^L = \tanh\left(\frac{\pi\omega}{2\kappa}\right) \tilde{B}_{\omega,\omega'}^R = -\tanh\left(\frac{\pi\omega}{2\kappa}\right) \tilde{B}_{\omega,\omega'}^L \\ &= \frac{1}{\pi\kappa} \left(\frac{\omega}{\omega'}\right) \sinh\left(\frac{\pi\omega}{2\kappa}\right) \left\{ \left(\frac{\omega'}{\kappa}\right)^{-i\omega/\kappa} \Gamma(i\omega/\kappa) - \left(\frac{\omega'}{\kappa}\right)^{i\omega/\kappa} \Gamma(-i\omega/\kappa) \right\} \end{aligned} \quad (4.23)$$

The transformation eq.(4.22) is the Schrödinger picture version of the Bogoliubov transformation of the (Heisenberg picture) creation and annihilation operators. If we consider q_{ω} a canonical position variable and $-i\delta/\delta q_{\omega}$ its conjugate momentum, we find that from the point of view of the new coordinates, $\{q_{\omega}^I, -i\frac{\delta}{\delta q_{\omega}^I}\}$, that the “position space representation” with q_{ω} diagonal corresponds to a representation in which a complicated linear combination of q_{ω}^I and $-\delta/\delta q_{\omega}^I$ is diagonal. Only for transformations with

$B_{\omega,\omega'}^I = \tilde{B}_{\omega,\omega'}^I = 0$ is there no mixing between coordinates and momenta. Then it is possible to express a state, such as $\Psi_0[\{q_\omega\}]$, in terms of $\{q_\omega^I\}$ by simply inserting eq.(4.22a) into eq.(4.17). To achieve this we introduce still another set of basis functions, $\{\Phi_\omega^I; I = L, R; \omega > 0\}$, given by [14]:

$$\begin{aligned}\Phi_\omega^R(v|u) &= [4\pi\omega(1 - e^{-\omega/T})]^{-1/2} [(-\kappa v + i\epsilon)^{i\omega/\kappa} - e^{-i\omega u}], \\ \Phi_\omega^L(v|u) &= [4\pi\omega(e^{\omega/T} - 1)]^{-1/2} [(-\kappa v + i\epsilon)^{-i\omega/\kappa} - e^{-i\omega u}],\end{aligned}\quad (4.24)$$

Here we used:

$$T = \frac{\kappa}{2\pi}. \quad (4.25)$$

The new modes are normalized according to:

$$(\Phi_{\omega'}^I, \Phi_\omega^J) \equiv i \int_{p(u)}^\infty \Phi_{\omega'}^{I*} \overleftrightarrow{\partial}_v \Phi_\omega^J = \delta^{IJ} \delta(\omega' - \omega) \quad (4.26)$$

Quantization again starts from the expansion:

$$\phi(u, v) = \sum_{I=L,R} \int_0^\infty d\omega \{A_\omega^I(u) \Phi_\omega^I(v|u) + A_\omega^{I\dagger}(u) \Phi_\omega^{I*}(v|u)\} \quad (4.27)$$

Introducing new variables $\{Q_\omega^I\}$ the Schrödinger picture operators are:

$$A_\omega^I = \frac{1}{\sqrt{2}} \left[\frac{\delta}{\delta Q_\omega^I} + Q_\omega^I \right], \quad A_\omega^{I\dagger} = \frac{1}{\sqrt{2}} \left[-\frac{\delta}{\delta Q_\omega^I} + Q_\omega^I \right]. \quad (4.28)$$

The basis, eq.(4.24), has the property that all inner products between Φ_ω^I and the negative energy solutions, ϕ_ω^* , vanish. This implies that the Bogoliubov transformation between Φ_ω^I and ϕ_ω is trivial, *i.e.*, it does not mix creation and annihilation operators [14]. Consequently Ψ_0 is also annihilated by A_ω^I :

$$\frac{1}{\sqrt{2}} \left[\frac{\delta}{\delta Q_\omega^I} + Q_\omega^I \right] \Psi_0 = 0, \quad I = L, R. \quad (4.29)$$

The only difference between a_ω^+ and A_ω^{I+} is that A_ω^{I+} does not create pure plane waves and $\mathcal{J}^-$, but more complicated multi-body excitations. Eq. (4.29) has the solution:

$$\Psi_0 = \exp \left[-\frac{1}{2} \int_0^\infty d\omega \{Q_\omega^{L2} + Q_\omega^{R2}\} \right] \quad (4.30)$$

which guarantees zero flux on $\mathcal{J}^-$. The advantage of the basis eq.(4.24) is that the transformation relating $\{q_\omega^I\}$ and $\{Q_\omega^I\}$ does not mix “position”

and “momentum” operators. Furthermore, the transformation is diagonal in the frequency ω . The analogue of eq.(4.22a) is found to be:

$$\begin{aligned} Q_\omega^R &= (1 - e^{-\omega/T})^{-1/2} [q_\omega^R - e^{-\omega/2T} q_\omega^L], \\ Q_\omega^L &= (1 - e^{-\omega/T})^{-1/2} [q_\omega^L - e^{-\omega/2T} q_\omega^R]. \end{aligned} \quad (4.31)$$

Since now the representation with $\{Q_\omega^I\}$ diagonal coincides with the representation in which $\{q_\omega^I\}$ is diagonal, we can insert eq.(4.31) into eq.(4.30) to obtain the ground state in terms of the $\mathcal{J}^+$ -variables $\{q_\omega^I\}$:

$$\Psi_0 = \exp \left[-\frac{1}{2} \int_0^\infty d\omega \left[\coth(\omega/2T) \{q_\omega^{L2} + q_\omega^{R2}\} - \frac{2}{\sinh(\omega/2T)} q_\omega^L q_\omega^R \right] \right]. \quad (4.32)$$

With Ψ_0 given in this form, vacuum expectation values are calculated according to:

$$\langle F \rangle = N^{-1} \int Dq \Psi_0 F(q^I, \delta/\delta q^I) \Psi_0 \quad (4.33)$$

where:

$$N = \int Dq |\Psi_0|^2 \quad (4.34)$$

and:

$$Dq = \prod_{\omega=0}^\infty dq_\omega^L dq_\omega^R. \quad (4.35)$$

Evaluating the density of particles asymptotically reaching $\mathcal{J}^+$, for instance, one finds for the particle number per unit volume:

$$\langle a_\omega^{R\dagger} a_\omega^R \rangle = \frac{1}{2\pi} \left\{ \frac{1}{\exp(\omega/T) - 1} \right\}. \quad (4.36)$$

To arrive at eq.(4.32) we heuristically interpreted a $\delta(0)$ as 2π times the “volume” of $\mathcal{J}_R^+$. Eq. (4.36) suggests that there is an outgoing flux of particles with a frequency spectrum given by the Bose–Einstein distribution for the temperature $T \equiv \kappa/2\pi$. We note that in the black–hole analogy κ is the surface gravity $1/4M$ of the hole. This leads to Hawking’s result $T = 1/8\pi M$. The interpretation of eq.(4.36) as a thermal radiation is confirmed by calculating the expectation value of the stress tensor component $T_{uu} = (\partial_u \phi)^2$ measuring the flux towards $\mathcal{J}_R^+$. Using eq.(4.18) and eq.(4.19) one finds after omitting a divergent normal ordering constant:

$$\langle T_{uu} \rangle = \int_0^\infty \frac{d\omega}{2\pi} \frac{\omega}{\exp(\omega/T) - 1} = \frac{\pi}{12} T^2. \quad (4.37)$$

In the derivation of eq.(4.36) and eq.(4.37) we used the identities:

$$\begin{aligned} N^{-1} \int Dq \, q_{\omega}^I q_{\omega'}^I \Psi_0^2 &= \frac{1}{2} \coth(\omega/2T) \delta(\omega - \omega') \\ N^{-1} \int Dq \, q_{\omega}^L q_{\omega'}^R \Psi_0^2 &= \frac{1}{2} [\sinh(\omega/2T)]^{-1} \delta(\omega - \omega'). \end{aligned} \quad (4.38)$$

The excitations q_{ω}^L do not contribute to $\langle T_{uu} \rangle$. In particular, the integrated flux operator can be expressed as:

$$\int_{-\infty}^{+\infty} du \, T_{uu} = \frac{1}{2} \int_0^{\infty} d\omega \, \omega \left[-\frac{\delta^2}{\delta q_{\omega}^{R2}} + q_{\omega}^{R2} \right]. \quad (4.39)$$

Formally it looks like a harmonic oscillator Hamiltonian with position coordinates q_{ω}^R . Eq. (4.37) shows that there is a constant flux of particles leaving the mirror. Its magnitude coincides with the result of Fulling and Davies [11] when specialized to the trajectory eq.(4.1). It is a special property of this trajectory that $\langle T_{uu} \rangle$ is independent of u . (For some other trajectories used in the literature [2] equation eq.(4.37) holds only asymptotically.)

V. Density Matrix and Lee's Theorem

In this section we take the point of view that the field coordinates $\{q_{\omega}^L\}$ corresponding to particles approaching $\mathcal{J}_L^+$, *i.e.* particles trapped by the black-hole, may be integrated out in the wave functional since these particles are not observed. This leads to an effective density matrix description of the observations on $\mathcal{J}_R^+$. The pure state, Ψ_0 , seems to be a mixed state due to the impossibility of detecting particles on the other side of the horizon. For separable cases, such as Rindler space, the eternal black-hole, or deSitter space, it is known that this density matrix has a thermal structure if the (pure) quantum state of the field is the ground state of the Hamiltonian. This is true even for massive and interacting field theories [7]. Here we investigate the question of the thermal nature of the Hawking radiation within the moving mirror model. This approach is complementary to the eternal black-hole (field theory on the Kruskal manifold) which can be treated in terms of separable foliations.

We again consider the wave function Ψ_0 , yielding no flux on $\mathcal{J}^-$. For the

moment we continue to maintain $m = 0$, so that $\Psi_0[u] \equiv \Psi_0$ is independent of u . We introduce the density matrix:

$$\varrho_0(q^R, q^{R'}) = \int Dq^L \Psi_0^*[q^L, q^R] \Psi_0[q^L, q^{R'}] \quad (5.1)$$

with $Dq^L \equiv \prod_{\omega=0}^{\infty} dq_{\omega}^L$. Expectation values of observables, $F_R\{q_{\omega}^R, \delta/\delta q_{\omega}^R\}$, referring only to R-modes can be written in terms of ϱ_0 as:

$$\langle F_R \rangle = \text{Tr}[F_R \varrho_0] \quad (5.2)$$

Here “ Tr ” means an integration over $\{q_{\omega}^R\}$ and F_R and ϱ_0 are understood to be matrices in q_{ω}^R -space. Inserting eq.(4.32) into eq.(5.1) one finds:

$$\varrho_0(q^R, q^{R'}) = \exp \left[-\frac{1}{2} \int_0^{\infty} d\omega \left[\coth(\omega/T) \{q_{\omega}^{R2} + q_{\omega}^{R'2}\} - \frac{2}{\sinh(\omega/T)} q_{\omega}^R q_{\omega}^{R'} \right] \right]. \quad (5.3)$$

Eq.(5.3) has the well-known form [20] of a thermal density matrix for the canonical ensemble. It obeys the differential equation:

$$\frac{1}{2} \int_0^{\infty} \omega d\omega \left[-\frac{\delta^2}{\delta q_{\omega}^{R2}} + q_{\omega}^{R2} \right] \varrho_0(q^R, q^{R'}) = \frac{\partial}{\partial \beta} \varrho_0(q^R, q^{R'}) \quad (5.4)$$

where $\beta \equiv T^{-1} = 2\pi/\kappa$. Comparison with eq.(4.37) shows that

$$\varrho_0(q^R, q^{R'}) = \langle q^R | \exp[-\beta \int du T_{uu}] | q^{R'} \rangle. \quad (5.5)$$

Eq. (5.5) makes it obvious that for an observer measuring only the $\{q_{\omega}^R\}$ degrees of freedom the system with the wave function Ψ_0 appears to be in a thermal state with the temperature given by $\kappa/2\pi$. Note that the operator $\int du T_{uu}$ in the Boltzmann factor is not the Hamiltonian, *i.e.* not the time evolution operator. In our model H is time-dependent and has no stationary eigenstates. In the present formulation this rôle is taken over by the flux eigenstate. It is intriguing that ϱ_0 has a very similar functional form as Ψ_0 :

$$\varrho_0(q^L, q^R, \beta = (2T)^{-1}) = \Psi_0(q^L, q^R). \quad (5.6)$$

This equality shows that the density matrix of the $\{q_{\omega}^R\}$ -observations determines the complete state Ψ_0 which also contains information about the $\{q_{\omega}^L\}$ -modes, describing the physics “behind the horizon”. Freese *et.al.* note this relationship in the case of Rindler space [5], while Lee [7] has

shown that relations analogous to eq.(5.6) hold for the Kruskal, and de-Sitter manifolds as well. These manifolds all contain horizons. Hence the Schrödinger picture variables can be divided into two sets, namely $\{q_\omega^R\}$ for the particles which can be detected and $\{q_\omega^L\}$ for excitations localized behind the horizon. Then the ground state wave-functional for the entire manifold is given by:

$$\Psi_0(q^L, q^R) = \langle q^L | \exp[-\beta H/2] | q^R \rangle. \quad (5.7)$$

Without explicitly calculating Bogoliubov coefficients, Lee gave an abstract proof of eq.(5.7) for a general massive and interacting field theory. Using eq.(5.7) as an input, it is easy to see that the density matrix of the R -sector is thermal:

$$\left\langle F\left\{q^R, \frac{\delta}{\delta q^R}\right\}\right\rangle = \int Dq^R \langle q^R | F e^{-\beta H} | q^R \rangle. \quad (5.8)$$

The thermal structure of the density matrix only holds if the system is in the ground state Ψ_0 . Excited states lead to different density matrices.

We now return to the moving mirror and allow for a non-vanishing mass $m \neq 0$. We also could imagine adding an interaction term to the Lagrangian. We again define $\Psi_0(u = -\infty)$ by the requirement of zero flux on $\mathcal{J}^-$. Therefore $\Psi_0(u = -\infty)$ still has the form eq.(4.30) or eq.(4.32). Hence the equality eq.(5.6) with ϱ_0 given by eq.(5.3) continues to hold for $\Psi_0 = \Psi_0(u = -\infty)$. However, the flux on $\mathcal{J}_R^+$ is determined by $\Psi_0(u = +\infty)$. To calculate Ψ_0 for $u \rightarrow +\infty$ one has to solve the Schrödinger equation $V\Psi_0 = i\partial_u\Psi_0$ where V is given by eq.(4.8) with eq.(4.11). This is not possible in closed form, but it is obvious that in general $\Psi_0(u = +\infty)$ will be different from $\Psi_0(u = -\infty)$. In view of eq.(5.8) this implies that the density matrix on $\mathcal{J}_R^+$ no longer will be given by eq.(5.3), *i.e.* mass and interaction terms destroy the thermal character of the density matrix. There are various reasons why in the moving mirror model Lee's theorem does not generalize to massive and interacting theories. First of all we must keep in mind that for moving mirrors there is *a priori* no reason for a thermal density matrix analogous to the imaginary time periodicity of the manifolds mentioned above. As is well known [2,15], even for free massless fields only a very limited class of trajectories leads to a thermal particle spectrum. Furthermore, Lee's proof is based on an analytic continuation to Euclidean field theory. This step can not be repeated here since in general

the function $x = z(t)$ does not remain real for imaginary time arguments. Also in the cases where Lee's theorem applies the operator H appearing in ϱ_0 is the true Hamiltonian (time evolution operator). In the present case it is a flux operator different from the time evolution operator.

Next we investigate the dependence of the $\mathcal{J}_R^+$ -density matrix on the state functional $\Psi[q^L, q^R]$. We return to the free massless theory. As an example of an excited state we construct a wave packet "behind the horizon," *i.e.* we excite some of the $\{q_\omega^L\}$ -modes. The (u-independent) state is chosen as:

$$\Psi_f = \left[f_0 + \int_0^\infty d\omega f_1(\omega) a_\omega^{L+} \right] \Psi_0 \quad (5.9)$$

with a_ω^{L+} given by eq.(4.21) and Ψ_0 by eq.(4.32). Here $f_0(f_i)$ is an arbitrary complex number (function). The density matrix associated with Ψ_f is:

$$\varrho_f(q^R, q^{R'}) = \int Dq^L \Psi_f^*[q^L, q^R] \Psi_f[q^L, q^{R'}]. \quad (5.10)$$

Inserting eq.(4.32), the integral yields:

$$\begin{aligned} \varrho_f(q^R, q^{R'}) &= \varrho_0(q^R, q^{R'}) \left\{ F^*(q^R) F(q^{R'}) \right. \\ &+ \frac{1}{\sqrt{2}} \int_0^\infty d\omega \{ F^*(q^R) f_1(\omega) + F(q^{R'}) f_1^*(\omega) \} \left[\frac{e^{\omega/2T}}{\sinh(\omega/T)} \right] (q_\omega^R + q_\omega^{R'}) \\ &+ \frac{1}{4} \left| \int_0^\infty d\omega f_1(\omega) \left[\frac{e^{\omega/2T}}{\sinh(\omega/T)} \right] (q_\omega^R + q_\omega^{R'}) \right|^2 \\ &\left. + \int_0^\infty d\omega |f_1(\omega)|^2 \left[\frac{e^{\omega/T}}{\sinh(\omega/T)} \right] \right\}, \quad (5.11) \end{aligned}$$

where:

$$F(q^R) = f_0 - \frac{1}{\sqrt{2}} \int_0^\infty d\omega f_1(\omega) q_\omega^R / \sinh(\omega/2T) \quad (5.12)$$

Obviously equation eq.(5.11) is very different from a thermal density matrix. The presence of quanta asymptotically reaching $\mathcal{J}_L^+$ changes the expectation values for observers on $\mathcal{J}_R^+$, despite the fact that they never can detect them directly. In particular, by measuring appropriate Hermitian operators, the observer on $\mathcal{J}_R^+$ can determine whether or not he is in a truly thermal state. For example, the field operator:

$$\phi^R(u, v) = \int_0^\infty d\omega \left[a_\omega^R \phi_\omega^R(v|u) + a_\omega^{R\dagger} \phi_\omega^{R*}(v|u) \right] \quad (5.13)$$

has a canonical ensemble average of zero. On the other hand, the average formed with ϱ_f reads

$$\langle \Psi_f | \phi^R(u, v) | \Psi_f \rangle = \frac{1}{2} \int_0^\infty \frac{d\omega}{\sinh(\omega/2T)} \left[f_0 f_1^*(\omega) \phi_\omega^R(v|u) + f_0^* f_1(\omega) \phi_\omega^{R*}(v|u) \right]. \quad (5.14)$$

Excitations above Ψ_0 lead to a non-thermal density matrix from which an observer at $v > 0$ (“outside the horizon”) can extract information about the region $v < 0$ (“inside the horizon”) which is inaccessible for direct observation. The reason for this possibility is the global nature of the wave functional. H is expressed in terms of the basis functions $\{\phi_\omega^L, \phi_\omega^R\}$ which are well defined both inside and outside the horizon. The cross term $q_\omega^L q_\omega^R$ in the ground state eq.(4.32) implies a correlation between L- and R-modes of the same frequency [14]. The additional terms in eq.(5.11) modifying the thermal density matrix in presence of left moving wave packet are a direct consequence of this correlation. This is the same kind of *BCS*-type pair correlation which is also present for separable foliations on manifolds with horizons (Rindler, Kruskal, etc.). The density matrix, ϱ_f , also leads to the phenomenon of stimulated emission [21]. Dropping particles into a black-hole can lead to an enhanced emission of quanta. This is modelled by the state Ψ_f containing a wave packet of L-quanta. Taking for simplicity $f_0 = 0$ and $f_1(w') = \delta(\omega' - \omega)$, one finds for the particle number expectation value:

$$\langle \Psi_f | a_\omega^{R+} a_\omega^R | \Psi_f \rangle = 2 \langle \Psi_0 | a_\omega^{R+} a_\omega^R | \Psi_0 \rangle \quad (5.15)$$

with the RHS of eq.(5.15) given by eq.(4.34). A left-moving quantum leads to a doubled density of outgoing quanta of the same frequency. The emission probability for the other frequencies remains unchanged.

VI. Conclusions

In this paper we have extended the covariant functional Schrödinger formalism to the treatment of non-separable spacetime foliations. As a special example of this class of systems we discussed in some detail Minkowski space field theory with time dependent boundary conditions, *i.e.*, those of moving mirrors.

In these problems the fundamental degrees of freedom of the Schrödinger

picture necessarily acquire an implicit time dependence. Hence, the RHS of the Schrödinger equation, $H\Psi = id\Psi/dt$, seemingly becomes subject to interpretation, *i.e.* should the derivative be interpreted as a partial derivative *wrt* the explicit time dependences, or should it be interpreted as a total time derivative, *e.g.*, as $id/dt \rightarrow i\partial/\partial t + i \int dx \dot{\phi}(x) \partial/\partial \phi(x)$? We give a rigorous canonical analysis which confirms the latter.

This in turn leads to an interesting functional criterion for the separability of the manifold. If a certain antisymmetric functional vanishes, then the manifold is separable and time independent basis functions can always be used. This is formally akin to a magnetic field strength, and correspondingly, the problems on nonseparable manifolds, such as the general moving mirror, are the functional equivalents of charged particles moving in external magnetic fields.

We consider a mirror trajectory which, for a massless field (quantized on a set of null hypersurfaces), leads to a constant Hawking flux with a *thermal* energy distribution. A relation between the density matrix of the out-going quanta and the complete wave-function can be derived. It is reminiscent of a result due to Lee [7], but, contrary to the case of an eternal black-hole, we find that the thermal character of the radiation is destroyed by mass and interaction terms.

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Figures

Fig.(1): The trajectory of eq.(4.1) and a family of spacelike hypersurfaces $\{\Sigma_t\}$.

Fig.(2): Penrose diagram of a spacetime bounded by a mirror. The function, $p(u)$ is defined by requiring the point $(u, v = p(u))$ to lie on the trajectory.

Fig.(3): Penrose diagram showing the family $\{\Sigma_{t_0}\}$ of spacelike hypersurfaces defined by $t = (1 - \epsilon)x + t_0$ for a small positive number ϵ . For $\epsilon \rightarrow 0$ the surfaces approach the u -lines. The time evolution maps $\mathcal{J}^-$ onto $\mathcal{J}_L^+ \cup \mathcal{J}_R^+$.

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Fig. 1

Fig. 2

Fig. 3