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ELLIPTIC GENERA AND STRING LOOP AMPLITUDES^{*}

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ABSTRACT

We review the computation of a certain class of superstring effective actions via high temperature expansions of supersymmetric σ -models. As the calculation is essentially the same as the one leading to the index of the Dirac-Ramond operator, these effective actions are closely related to the gauge and gravitational anomaly structure of the particular theories.

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We present a method for computing a certain type of heterotic string one-loop amplitudes, or rather, effective actions [1]. The special class of amplitudes we are considering here may be called "holomorphic". In fact, most *explicit* computations (where all modular integrals are performed) done so far deal with such amplitudes characterized by "almost" holomorphic modular integrands. Only for these cases it is known how to evaluate the τ -integrals explicitly.

Such a holomorphicity typically arises due to saturation of zero modes in the right-moving NSR-sector of the heterotic string. For example, in the right-moving periodic-periodic (PP) sector of the NSR-fermions one has

$$\left\langle \psi_0^{\mu_1} \psi_0^{\mu_2} \dots \psi_0^{\mu_n} \right\rangle_{PP} = e^{\mu_1 \mu_2 \dots \mu_n} = \text{const.} \quad (1)$$

If the number of external lines is such that the zero modes are just saturated, there are no Wick-contractions left, and there is no non-trivial contribution from the right-moving sector in the low energy limit (where the Koba-Nielsen factor $\chi^{\alpha' k_i k_j}(z_i - z_j)$ can be put equal to one). The string spectrum that contributes in the loop thus consists of the right-moving zero modes plus an infinite tower of left-moving states. Note that even though the left-moving states with $m_L \neq 0$ do not satisfy the physical state condition $m_L = m_R$, they all do contribute to the loop amplitude, in general. This is because the lower boundary of the fundamental domain $\mathcal{F}$ of the torus is not a straight line, but curved.

This type of spectrum occurs also in the context of loop space index theory. Our objective is to show that in fact *all* holomorphic amplitudes, i.e., the corresponding effective actions, can directly be related to and expressed in terms of the elliptic genus. The elliptic genus [2-4] is a loop space generalization of the Dirac genus that is the index density of the Dirac operator; that is, it is the index density associated with the Dirac-Ramond operator $\hat{G}_0$:

$$\text{ind} G_0(q) = \text{Tr}'_R \left[(-1)^F q^{L_0^L} \bar{q}^{L_0^R} \right]. \quad (2)$$

Here, F denotes the number operator of the NSR-fermions with space-time indices, and the trace runs over the Ramond-sector of these fields and over all sectors of all the other $2d$ fields in the theory. Thus, $\text{ind} G_0(q)$ is nothing but the partition function associated with the PP-sector of the space-time fermions (the prime above indicates omission of the zero modes). The spectrum that contributes to (2) is precisely of the above type: due to the world-sheet supersymmetry in the PP-sector, all states with $m_R \neq 0$ come in pairs with opposite $(-1)^F$ but same L_0^R eigenvalues, so that they cancel out in the trace. Thus, $\text{ind} G_0(q)$ can be expanded in a meromorphic power

series $\sum_{n=0}^{\infty} \text{ind}_n G_0 q^{n-1} \bar{q}^0$, where $\text{ind}_n G_0$ is the index of the n th left-moving level. It can be computed from a path integral [3][4]

$$\text{ind} G_0(q) = \int_{\text{Torus}} (d\psi)_{PP} [d\phi] e^{-S_E(F, R)}, \quad (3)$$

where $[d\phi]$ denotes integration over all other fields of the theory, and F, R are the background gauge and gravitational curvature two-forms. The result is

$$\text{ind} G_0(q) = \int \mathcal{A}(q, F, R) \Big|_{\text{top form}}, \quad (4)$$

where "top form" refers to the terms surviving the ψ_0^μ integration (1), and the elliptic genus is given by

$$\mathcal{A}(q, F, R) = \hat{A}(R) \sum_{k=0}^{\infty} q^{k-1} \text{Ch}(k, F) \text{Ch}(k, R). \quad (5)$$

Here, $\hat{A}(R)$ is the Dirac genus related to the right-moving spinor ground state, and $\text{Ch}(k, X) = \text{Tr}_{[k]} \exp(iX/2\pi)$ represents the Chern character of string level k . From (5) it is obvious that $\mathcal{A}(q, F, R)$ is intimately related to the gauge and gravitational anomaly structure of the theory.

The prototype of an holomorphic amplitude where the elliptic genus plays a role is the Green-Schwarz anomaly cancelling term [5] in $d = 2n + 2$ dimensions. It is associated with the PP-sector of the theory, as it involves the ϵ -tensor (1). It can be shown [6] that the Green-Schwarz effective action can be computed as

$$S_{1\text{-loop}}^{\text{eff}}(B, F, R) = \int d^{2n+2}x \left\{ \frac{1}{16\pi^2} B \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im}\tau)^2} \bar{\mathcal{A}}(q, F, R) \right\} \Big|_{2n+2 \text{ form}}, \quad (6)$$

where B is the antisymmetric tensor two-form field, and $\bar{\mathcal{A}}(q, F, R)$ is a minor modification of $\mathcal{A}(q, F, R)$ not to be explained here. The modular integral can be computed explicitly:

$$\int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im}\tau)^2} \bar{\mathcal{A}}(q, F, R) \Big|_{2n\text{-form}} = -64\pi^2 X_{2n}(F, R). \quad (7)$$

The $2n$ -form X_{2n} is a certain polynomial in terms of traces of powers of F and R , and is defined by the chiral anomaly polynomial I_{2n+4} of the particular theory:

$$I_{2n+4}(F, R) \equiv \mathcal{A}(q, F, R) \Big|_{\substack{2n+4 \text{ form} \\ \text{coeff. of } q^0}} = \frac{1}{2\pi} (\text{Tr } F^2 - \text{Tr } R^2) \cdot X_{2n}(F, R). \quad (8)$$

Because of modular invariance [2], the anomaly (8) has always this factorized form. It follows

$$S_{1\text{-loop}}^{\text{eff}}(B, F, R) = -4 \int B X_{2n}(F, R). \quad (9)$$

This is precisely the Green-Schwarz anomaly cancelling term [5].

There exist also other variants of holomorphic amplitudes: amplitudes can also be holomorphic due to saturation of zero modes of light-cone Green-Schwarz fields: $\langle S_0^{a_1} S_0^{a_2} \dots S_0^{a_n} \rangle = \epsilon^{a_1 a_2 \dots a_n}$. It is clear that the kind of amplitudes that will be holomorphic depends on the number of such zero modes, that is, on the degree of supersymmetry. For example, in $N=1$ $d=10$ theories there are eight zero modes, and since bosonic vertex operators contain two S^a 's, it follows that four-point amplitudes are holomorphic. Such graviton and gauge boson scattering amplitudes have been computed previously in [7].

We like to show that the corresponding effective actions are also related to the elliptic genus, by computing them directly from the path integral

$$S_{1\text{-loop}}^{\text{eff}}(F, R) = \int_{\text{Torus}} dS^a [d\phi] e^{-S_E(F, R)}, \quad (10)$$

using light-cone Green-Schwarz formalism. In general, to obtain the effective action by integrating out two dimensional fields [8] appears to be quite non-trivial, as one would have to solve the $2d$ theory explicitly. However, the point here is that as we are considering holomorphic amplitudes, we can evaluate the path integral in a certain limit, namely for $\text{Im}\tau \rightarrow 0$ (the high temperature limit). It follows then by analytic continuation that we obtain the exact result in this way! Indeed, in this limit, the euclidean $2d$ action for the non-zero modes becomes gaussian:

$$S_E^{(2)} = \frac{1}{2\pi} \int dt d\sigma \left[\partial X^i (\bar{\partial} \delta_{ij} + \frac{i}{2\pi} \hat{\mathcal{R}}_{ij}) X'^j - S'^a \partial S'_a - \lambda_A (\bar{\partial} \delta^{AB} + \frac{i}{2\pi} \hat{\mathcal{F}}^{AB}) \lambda_B \right]. \quad (11)$$

Here, λ_A are the left-moving fermions ($A = 1 \dots 32$, $a, i = 1 \dots 8$), and

$$\hat{\mathcal{R}}_{ij} = \frac{1}{2} \hat{R}_{ijab} S_0^a \wedge S_0^b, \quad \hat{\mathcal{F}}^{AB} = \frac{1}{2} \hat{F}_{ab}^{AB} S_0^a \wedge S_0^b \quad (12)$$

with

$$\hat{R}_{ijab} = \frac{1}{4} (\gamma^{[k} \gamma^{l]})_{ab} R_{ijkl}(x_0), \quad \hat{F}_{ab}^{AB} = -\frac{i}{4} (\gamma^{[i} \gamma^{j]})_{ab} F_{ij}^\alpha(x_0) T_\alpha^{AB}. \quad (13)$$

As $\hat{\mathcal{R}}_{ij}$ and $\hat{\mathcal{F}}^{AB}$ are valued in the exterior algebra of the spinor bundle generated by S_0^a , we may regard them as "spinor" two-forms. The leading part (11) of S_E is quadratic, and one can easily evaluate the non-zero mode integrals:

$$\int \mathcal{D}X \mathcal{D}S \mathcal{D}\lambda e^{-S_E^{(2)}} = \int dx_0^i dS_0^a \left[\frac{\det'(\partial)}{\det'(\bar{\partial})} \right]^{\frac{1}{2}} (\bar{q}) \left[\frac{\det(\bar{\partial} \delta^{AB} + \frac{i}{2\pi} \hat{\mathcal{F}}^{AB})}{\det'(\bar{\partial} \delta_{ij} + \frac{i}{2\pi} \hat{\mathcal{R}}_{ij})} \right]^{\frac{1}{2}} (q). \quad (14)$$

Summation over the spin structures of the λ_A is implicitly understood. The first bracket above cancels due to a world-sheet supersymmetry (which appears in light-cone gauge) between X^i and S'^a , hence (14) is holomorphic in q . The crucial point is that the calculation we are performing is very similar to the one leading to the index of the Dirac-Ramond operator [3][4]. In fact, the index is given by (14) if $\int dS_0^a$ is replaced by $\int d\psi_0^i$ and the spinor forms $\hat{\mathcal{R}}_{ij}, \hat{\mathcal{F}}^{AB}$ by

$$\mathcal{R}_{ij} = \frac{1}{2} R_{ijkl} \psi_0^k \wedge \psi_0^l, \quad \mathcal{F}^{AB} = -\frac{i}{2} F_{ij}^\alpha (T_\alpha)^{AB} \psi_0^i \wedge \psi_0^j, \quad (15)$$

which are two-forms in the exterior algebra generated by ψ_0^i . This is essentially a triality rotation in the $SO(8)$ transverse Lorentz group. Thus, we can borrow from the index calculation to express the effective action in terms of the elliptic genus, and to perform the modular integral:

$$\begin{aligned}
S_{1\text{-loop}}^{\text{eff}}(F, R) &\propto \int_{\mathcal{F}} \frac{d^2\tau}{(Im\tau)^2} \int d^8x \, \overline{\mathcal{A}}(q, \widehat{F}, \widehat{R}) \Big|_{\text{spinor 8-form}} \\
&= -64\pi^2 \int d^8x \, \epsilon_{a_1 a_2 \dots a_8} \widehat{X}_8^{a_1 a_2 \dots a_8}(\widehat{F}, \widehat{R}) .
\end{aligned} \tag{16}$$

The spinor form $\widehat{X}_8^{a_1 a_2 \dots a_8}(\widehat{F}, \widehat{R})$ is obtained from the eight-form $X_8(F, R)$ appearing in the anomaly polynomial (8) by simply replacing vector by spinor indices and the two-forms F, R by the spinor two-forms $\widehat{F}$ and $\widehat{R}$. Note that even though (16) contains an ϵ -tensor, the effective action does not violate parity. Rather, the contraction of $(\gamma^{[i}\gamma^{j]})_{ab}$ in $\widehat{F}$ and $\widehat{R}$ with $\epsilon^{a_1 \dots a_8}$ produces the well-known kinematical tensor $t^{ijklmnpq}$, an explicit expression for which can be found e.g. in [9]. We thus obtain the results derived by explicit string loop computations [7]. For example, from the formula for $X_8(F, R)$ corresponding to the $SO(32)$ heterotic string [5][6] one immediately deduces

$$\begin{aligned}
S_{1\text{-loop}}^{\text{eff}[SO(32)]}(F, R) &\propto \int d^8x \, t^{ijklmnpq} \left\{ 4 \text{Tr} R_{ij} R_{kl} R_{mn} R_{pq} + \text{Tr}(R_{ij} R_{kl}) \text{Tr}(R_{mn} R_{pq}) \right. \\
&\quad \left. + 32 \text{Tr} F_{ij} F_{kl} F_{mn} F_{pq} - 4 \text{Tr}(F_{ij} F_{kl}) \text{Tr}(R_{mn} R_{pq}) \right\} ,
\end{aligned} \tag{17}$$

where the traces are over the vector representations.

This method can easily be generalized, and one finds that for any $d = 2n + 2$ ($n = 1, 2, 4$) dimensional chiral $N = 1$ supersymmetric heterotic string theory, the one-loop n -point light-cone bosonic effective action can be mapped to the chiral index:

$$S_{1\text{-loop}}^{\text{eff}} = \int \widehat{X}_{2n} . \tag{18}$$

The complete scheme is given in the Table. Due to their close relation to the anomaly sector of the theory, it is conceivable the these effective actions do not get renormalized beyond one loop.

Table: *Classification of holomorphic one-loop amplitudes in supersymmetric heterotic string theories. They constitute the boundary between vanishing and generic non-vanishing amplitudes. The upper row indicates the even space-time dimensions above two in which chiral $N = 1$ theories exist, and the brackets below characterize the extended supersymmetries that arise upon torus compactification to $d = 4$.*

$N = 1$ Susy in: # of legs $\downarrow$	$d = 4$ ($N = 1$)	$d = 6$ ($N = 2$)	$d = 10$ ($N = 4$)	Type of eff. Action
1	holom.	0	0	Fayet-Iliopoulos D-term [10]
2	*	holom.	0	e.g., gauge coupling renormalization
4	*	*	holom.	e.g., equation (17)
≥ 5	*	*	*	*

Our results are also relevant for lower dimensional, torus compactified theories. For example, it can be shown [1] that upon torus compactification, the logarithmic infrared divergent gauge coupling constant renormalization of a $d=4$, $N=2$ theory can be related to the chiral index in $d=6$. More precisely, if T denotes a generator of the gauge group, then the β -function in $d=4$ is proportional to $g^3 X_4(T, 0)$, where X_4 characterizes the anomaly in $d=6$. This agrees with the particle theory result [11].

The calculation above was performed in light-cone Green-Schwarz formalism, and based on saturation of S_0^a zero modes. One might ask how the results are obtained in the NSR-formalism, where there are no such zero modes. In the NSR-formalism, the anomalies reside in the PP-sector of fermionic boundary conditions, while the parity conserving amplitudes that correspond to our effective actions come from the even spin structure sectors. Our result thus implies that in supersymmetric theories there has to be a non-trivial relation between the even and the odd spin-structure sectors.

This Ward-type relation [1] can be expressed in terms of identities between certain character valued partition functions, that are associated with the even and odd spin structure sectors. To characterize these identities, it is important to realize that supersymmetry in string theory is intimately related to exceptional groups [12-14]. In particular, the three generic cases in the table above correspond precisely to E_6 ($N=1$ supersymmetry in $d=1$), E_7 ($d=6$) and E_8 ($d=10$). It follows that the identities in question can be written in terms of identities between certain Kac-Moody characters that are related to exceptional groups.

To obtain these identities, decompose any conjugacy class (y) of E_n ($n = 6, 7, 8$)

$$(y) \rightarrow (\Delta_0^{(y)}, 0) \oplus (\Delta_v^{(y)}, v) \oplus (\Delta_s^{(y)}, s) \oplus (\Delta_c^{(y)}, c), \quad (19)$$

according to $E_n \rightarrow (K_{n-4}, D_4)$, where $K_2 = U(1)D_1$, $K_3 = A_1 D_2$ and $K_4 = D_4$. Then the one-loop partition function can be written as a sum of terms that have factors

$$Ch_{[\Delta_v^{(y)}]}^{K_{n-4}}(0|\tau) - Ch_{[\Delta_s^{(y)}]}^{K_{n-4}}(0|\tau) = 0. \quad (20)$$

Here, $Ch_{[\Delta_x^{(y)}]}^{K_{n-4}}(0|\tau)$ denotes the level one Kac-Moody character of K_{n-4} associated with the conjugacy class $\Delta_x^{(y)}$. The vanishing of (20) is a manifestation of space-time supersymmetry, and can be proven by the fact that the triality symmetry of the D_4 factor above ("ghost triality") becomes an inner automorphism when embedded into E_n [14]. The supersymmetric Ward-identities that relate even with odd spin structure sectors can easily be derived from a character valued generalization of (20),

$$Ch_{[\Delta_s^{(y)}]}^{K_{n-4}}(\vec{v}|\tau) - Ch_{[\Delta_c^{(y)}]}^{K_{n-4}}(\vec{v}|\tau) = Ch_{[\Delta_v^{(y)}]}^{K_{n-4}}(T\vec{v}|\tau) - Ch_{[\Delta_s^{(y)}]}^{K_{n-4}}(T\vec{v}|\tau), \quad (21)$$

that also can be proven via ghost triality. Above, T denotes the representation matrix of the triality transformation $s \rightarrow v$, $c \rightarrow s$, $v \rightarrow c$ in the Cartan sub-algebra of K_{n-4} . The LHS of (21) describes the odd spin structure sector, and the RHS represents a sum over all sectors. For the case with maximal supersymmetry (E_8), (21) is the same as the well-known Riemann identity that is an identity between characters of $K_4 = D_4$. For the other cases, (21) is a generalization of the Riemann identity. In fact, (21) extends also to the multiloop case [14], and thus is expected to be relevant for various non-renormalization theorems in supersymmetric string theories.

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