CERN.TH-5137/88

UNIFICATION OF POINCARÉ, BRST AND PARISI-SOURLAS SUPERSYMMETRIES*

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Abstract

Poincaré, BRST and Parisi-Sourlas supersymmetries are usually viewed as unrelated. Here we argue that they are in fact closely related. We argue that fermions in Poincaré SUSY theories can be interpreted as negative dimensional phase space coordinates, in the Parisi-Sourlas sense. The Bose-Fermi balance then implies that the effective phase space dimensionality vanishes. We show that interacting Poincaré SUSY theories are related to the corresponding free theories by superrotations similar to BRST transformations. For superstrings we find that spacetime can be viewed as a zero dimensional manifold in the sense of Parisi-Sourlas.

*Dedicated to the memory of Bruce McClain

[†]Work supported in part by the U.S. Department of Energy Outstanding Junior Investigator Grant under contract No. DE-AC02-76ER01545

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Ref.CERN.TH-5137/88

July 1988

Poincaré SUSY theories [1] can be characterized by the existence of a Nicolai transformation [2]. Except for zero energy states [3] this can also be seen as an exact cancellation between bosonic and fermionic functional integrals [4].

A cancellation between bosonic and fermionic integrations also occurs in the Parisi-Sourlas formulation of stochastic quantization [5,6], where anticommuting variables are identified with negative dimensional coordinates. Recently this formalism has been applied to BRST quantization, by extending the BRST supersymmetry of a constrained system into an $\text{Osp}(1,1|2)$ (Parisi-Sourlas) supersymmetry [7-9]. The cancellation between unphysical gauge and ghost degrees of freedom in a BRST quantized theory can then be viewed as a manifestation of the Parisi-Sourlas mechanism: The unphysical phase space of a BRST quantized constrained system is a zero dimensional manifold.

Poincaré superalgebras can be obtained from orthosymplectic (Parisi-Sourlas) superalgebras by Inönü-Wigner [10,11] contractions. Hence it might also be possible to understand the Bose-Fermi cancellation [2-4] in Poincaré SUSY theories in terms of the Parisi-Sourlas mechanism. In this Letter we shall argue that this is indeed the case. We shall argue that fermions in a Poincaré SUSY theory can indeed be interpreted as negative dimensional phase space coordinates in sense of [5,6]. Consequently both Poincaré and BRST supersymmetries can be understood as manifestations of the Parisi-Sourlas supersymmetry.

Consider a superspace with m commuting coordinates x^μ and $2n$ anticommuting coordinates θ^i such that $\theta^i\theta^j + \theta^j\theta^i = 0$. A supercoordinate has components $y^\alpha = (x^\mu, \theta^i)$; $\alpha = 1, \dots, 2n + m$, and $\text{Osp}(m|2n)$ is the set of linear transformations that leave the inner product

$$(y_1, y_2) = G_{\alpha\beta} y_1^\alpha y_2^\beta = g_{\mu\nu} x_1^\mu x_2^\nu + c_{ij} \theta_1^i \theta_2^j \quad (1)$$

invariant, with $g_{\mu\nu}$ a symmetric real and c_{ij} an antisymmetric real matrix.¹ The Lie algebra of $\text{Osp}(m|2n)$ can be abstracted by considering infinitesimal invariances of (1), [12]

$$[R^{\alpha\beta}, R^{\gamma\delta}] = G^{\beta\gamma} R^{\alpha\delta} - (-)^{\gamma\delta} G^{\beta\delta} R^{\alpha\gamma} - (-)^{\alpha\beta} G^{\alpha\gamma} R^{\beta\delta} + (-)^{\alpha\beta} (-)^{\gamma\delta} G^{\alpha\delta} R^{\beta\gamma} \quad (2)$$

with the differential representation $R^{\alpha\beta} = (-)^{\alpha\beta} y^\alpha \partial^\beta - y^\beta \partial^\alpha$. From (2) we conclude that e.g. in a basis with $g_{\mu\nu} = \delta_{\mu\nu}$ generators such as $Q_{\mu\nu} = R^{\mu\theta} + iR^{\nu\theta}$, $Q_{\mu\nu}^\dagger = R^{\mu\theta} - iR^{\nu\theta}$ ($\mu \neq \nu$) are nilpotent. With $H = R^{\theta\theta}$ these generators then determine the following

¹With $g_{\mu\nu}$ antisymmetric and c_{ij} symmetric we obtain $\text{Osp}(2n|m)$ instead. With $g_{\mu\nu}$ indefinite we get $\text{Osp}(m-k, k|2n)$.

"supersymmetry" algebra

$$[Q_{\mu\nu}, Q_{\mu\nu}] = [Q_{\mu\nu}^\dagger, Q_{\mu\nu}^\dagger] = [H, H] = [Q_{\mu\nu}, H] = [Q_{\mu\nu}^\dagger, H] = 0 \quad (3.a)$$

$$[Q_{\mu\nu}, Q_{\mu\nu}^\dagger] \propto H \quad (3.b)$$

With $g_{\mu\nu}$ indefinite, light-cone generators such as $R^{-\theta}$ are nilpotent. Indeed, the BRST operator of a constrained system can be identified with the generator $R^{-\theta}$ in an $\text{Osp}(1,1|2)$ algebra [7-9,13].

The Parisi-Sourlas identification of θ^i as negative dimensional coordinates emerges when we set $y_1 = y_2 = y$ in (1) and consider integrals of $\text{Osp}(m|2n)$ invariant functions $F(y^2)$ in the superspace. From the results of [5,6],

$$\int d^m x d^{2n} \theta F(y^2) = \frac{1}{\sqrt{s \det |G_{\alpha\beta}|}} \int d^{m-2n} x F(x^2) \quad (4)$$

and if $m - 2n$ is nonpositive, we define (4) by analytic continuation. In this sense the variables θ^i are then negative dimensional coordinates. In particular, for $m = 2n$ i.e. Bose-Fermi balance, we get integrals over zero dimensional spaces

$$\int d^{2n} x d^{2n} \theta F(y^2) = \frac{1}{\sqrt{s \det |G_{\alpha\beta}|}} F(0) \sim \lim_{d \rightarrow 0} \int d^d x F(x^2) \quad (5)$$

Consider a position dependent superrotation in the l.h.s. of (4),

$$y^\alpha \rightarrow y^\alpha + \frac{1}{N} [y^\alpha, R^{\mu\theta}] \Psi \quad (6)$$

with Ψ an arbitrary function of y , and N an integer. The integrand in (4) is invariant under (6), but since Ψ depends on y the Jacobian is nontrivial²

$$\int d^m x d^{2n} \theta F(y^2) = \int d^m x d^{2n} \theta \left[1 - \frac{1}{N} [\Psi, R^{\mu\theta}] + \mathcal{O}\left(\frac{1}{N^2}\right) \right] F(y^2) \quad (7.a)$$

We select a basis (e.g. the basis (3), or light-cone basis) where $R^{\mu\theta}$ is nilpotent and repeat (6) N times. In the limit $N \rightarrow \infty$ the Jacobian exponentiates, and we have

$$\int d^m x d^{2n} \theta F(y^2) \stackrel{N \rightarrow \infty}{=} \int d^m x d^{2n} \theta e^{[\Psi, R^{\mu\theta}]} F(y^2) \quad (7.b)$$

²Here we ignore global issues and complications such as Gribov ambiguities that may arise with $\Psi(y)$ such that (6) is not invertible.

Since the *l.h.s.* is independent of Ψ the *r.h.s.* must also be independent of Ψ , and this constitutes a proof of the Fradkin-Vilkovisky theorem [8,9,14,15].

With $R^{\nu\bar{\theta}}$ another nilpotent generator such that $[R^{\nu\bar{\theta}}, [\Psi, R^{\mu\theta}]] = 0$ and Φ another arbitrary function of y , we can generalize (7.b) into

$$\int d^m x d^{2n} \theta F(y^2) = \int d^m x d^{2n} \theta \exp([\Psi, R^{\mu\theta}] + [\Phi, R^{\nu\bar{\theta}}]) F(y^2) \quad (7.c)$$

and so on.

The previous formalism has been applied to BRST quantization of constrained systems [8,9,14,15]. For this consider a $2N$ dimensional phase space $\Gamma \sim p_i, q^i$ with $M < N$ first class constraints $\phi_a \sim 0$ i.e. $\{\phi_a, \phi_b\} = U_{ab}^c \phi_c$. The BRST extended phase space is obtained by including additional (bosonic) canonical variables λ_a, π^a and (fermionic) canonical variables $\eta^a, \bar{P}_a$, and $\bar{P}^a, \bar{\eta}_a$. The Poisson brackets of these variables determine a closed (graded) symplectic two-form ω . Locally, in patches ω is then an exterior differential of a one-form, $\omega = d\vartheta$. Following [9] we include the hamiltonian as part of the constraints, and write the path integral of the quantum theory in the coordinate independent Hamilton-Jacobi form

$$Z \sim \int \prod_{\text{patches}} \omega_t \wedge \dots \wedge \omega_t e^{i \int \vartheta} \quad (8)$$

In a coordinate system (locally) related to canonical coordinates by a change of variables of the form (6) and BRST operator Ω identified with the (light-cone basis) nilpotent $R^{-\theta}$ as in [8,9], (8) becomes [9,14,15]

$$Z \sim \int [dp_i \dots d\bar{P}^a] \exp(i \int p_i \dot{q}^i + \dots + \bar{\eta}_a \dot{\bar{P}}^a - [\Omega, \Psi]) \quad (9)$$

and different Ψ correspond to different gauge fixings, i.e. different coordinate systems on the phase space related to each other by superrotations of the form (6). In particular, the pertinent reduced phase space path integral [16]

$$Z \sim \int [dp_i dq^i] \delta(\phi_a) \delta(\psi^a) \det|\{\phi_a, \psi^b\}| \exp\{i \int p_i \dot{q}^i - H\} \quad (10)$$

with $\psi^a \sim 0$ the gauge condition, emerges from (9) by a Inönü-Wigner contraction with $\Psi = \kappa \bar{P}_a \psi^a + \bar{P}_a \lambda^a$, in the limit $\kappa \rightarrow \infty$ and after truncation of variables that decouple. This equivalence between (9) and (10) can then be seen as a manifestation of (4) and (7.b,c), with unphysical phase space variables cancelling each other by the Parisi-Sourlas

mechanism [9]: In the sense of (5) the unphysical submanifold of the extended phase space is a *zero* dimensional manifold.

Particularly curious in the following is the observation [8,9] that generators of canonical Parisi-Sourlas superrotations in (9) can be related to the differential representation of $\text{Osp}(1,1|2)$ by a simple conjugation, locally in the phase space. For this we introduce a local change of variables from the original ϕ_a, ψ^a to G_a, F^a with $\{G_a, G_b\} = \{F^a, F^b\} = 0$, $\{G_a, F^b\} = -\delta_a^b$. Identifying light-cone variables $y^{a\alpha} = (F^a, -\pi^a, \eta^a, \bar{\rho}^a)$ and $p_{a\alpha} = (G_a, \lambda_a, \mathcal{P}_a \bar{\eta}_a)$ (for each a) such that $\{p_{a\alpha}, y^{b\beta}\} = -\delta_a^b \delta_\alpha^\beta$ the canonical differential representation of $\text{Osp}(1,1|2)$ becomes

$$R_A^{\alpha\beta} = q_a^\alpha p_a^\beta - (-)^{\alpha\beta} q_a^\beta p_a^\alpha \sim (-)^{\alpha\beta} y^\alpha \partial^\beta - y^\beta \partial^\alpha \quad (11)$$

By a (locally defined) conjugation in the phase space,

$$R^{\alpha\beta} = e^\Phi R_A^{\alpha\beta} e^{-\Phi} = R_A^{\alpha\beta} + \{R_A^{\alpha\beta}, \Phi\} + \frac{1}{2!} \{\{R_A^{\alpha\beta}, \Phi\}, \Phi\} + \dots \quad (12)$$

we then obtain the original canonical representation of the $\text{Osp}(1,1|2)$ generators, with the BRST and antiBRST operators corresponding to the original ϕ_a identified with the generators $R^{-\theta}, R^{-\bar{\theta}}$ respectively, and the ghost number operator identified with $R^{\theta\bar{\theta}}$ [8,9].

An explicit construction of the generators Φ in (12) for (super)string theories has been presented in [8,9].

We shall now argue that the previous formalism is directly applicable to the quantization of Poincaré SUSY theories. In particular, we shall argue that the phase space functional integral of a Poincaré SUSY theory can always be viewed as an integral of the form (9), (7.b,c). Due to the Bose-Fermi balance this functional integral is then an integral over a *zero* dimensional manifold in the sense of (5).

We shall also consider conjugations such as (12). We shall find that in Poincaré SUSY theories these conjugations relate an interacting SUSY theory to the pertinent free SUSY theory, with the free SUSY generators of the abelian form (11). The relation between free and interacting SUSY theories is then an *exact* analog of the relation between abelian and nonabelian realizations of the canonical $\text{Osp}(1,1|2)$ algebra in constrained systems [8,9].

For free SUSY theories a Parisi-Sourlas interpretation is straightforward: After eliminating auxiliary fields, free SUSY actions have the generic form

$$S = \int \Phi(-\square - m^2)\Phi + \bar{\Psi}(i\partial - m)\Psi \sim \phi_a M_{ab} \phi_b + \psi_a N_{ab} \psi_b \quad (13)$$

with Φ a set of bosons and Ψ a set of Majorana fermions. The matrix M_{ab} is symmetric and N_{ab} antisymmetric³ so that we can view (13) as a bilinear of the form (1) with $y_1 = y_2$, and consequently the free Lagrangian is $\text{Osp}(\infty|\infty)$ invariant: With M_{ab} and N_{ab} invertible these $\text{Osp}(\infty|\infty)$ invariances are the *only* invariances that (13) can have. In particular, since SUSY transformations are invariances of (13), SUSY transformations are necessarily $\text{Osp}(\infty|\infty)$ rotations in the configuration space of the free theory.

We observe that various non-SUSY free Lagrangians also have the functional form (13), even though in general there is no Bose-Fermi balance and the masses of the Bose and Fermi fields are different. Hence non-SUSY free Lagrangians can also be Osp invariant. But since different masses label unitarily inequivalent representations of Poincaré algebra this Osp invariance cannot be unitarily implemented in the free quantum theory unless the Bose and Fermi masses are equal, as in SUSY. Furthermore, we shall argue that *only* for SUSY can the interacting quantum theory be related to the free theory by a change of variables. Hence *only* for SUSY is the $\text{Osp}(\infty|\infty)$ invariance of the free theory also an invariance of the interacting theory: Since a generic interacting theory cannot be related to the pertinent free theory by a change of variables, the Osp invariances of non-SUSY free theories become irrelevant when interactions are turned on.

Consider first supersymmetric quantum mechanics with

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}W_q^2 + i\bar{\eta}\dot{\eta} - i\bar{\eta}W_{qq}\eta \quad (14)$$

We wish to relate its phase space quantization to the BRST quantization (9). For this we observe that by introducing real Grassmann variables θ_1 and θ_2 such that $\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)$ and $\bar{\eta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$, the kinetic term in (14) becomes $\frac{i}{2}\theta_1\dot{\theta}_1 + \frac{i}{2}\theta_2\dot{\theta}_2$. Thus in (14) conjugate momenta π_1, π_2 have been eliminated by implementing (second class) primary constraints $\pi_1 = -\frac{i}{2}\theta_1, \pi_2 = -\frac{i}{2}\theta_2$ while in (9) this elimination has not been implemented, and both position and momentum variables appear. This is the only technical difference between Poincaré SUSY and BRST functional integrals. It could be circumvented by introducing auxiliary fields, but for the present purpose it is sufficient to work in terms of the variables θ_1, θ_2 and their Dirac brackets $\{\theta_{1,2}, \theta_{1,2}\} = 1$ instead. (Obviously, similar remarks apply to other SUSY theories with Majorana fermions.)

³In a more rigorous formulation we start from a discretized approximation to (13), by approximating M_{ab}, N_{ab} by $2n \times 2n$ matrices. The lattice Lagrangian is then $\text{Osp}(2n|2n)$ invariant, and the $\text{Osp}(\infty|\infty)$ is defined as the $n \rightarrow \infty$ limit.

The canonical supersymmetry charges are

$$Q = \frac{1}{\sqrt{2}}\eta(p + iW_q) \quad Q^\dagger = \frac{1}{\sqrt{2}}\bar{\eta}(p - iW_q) \quad (15)$$

and the supersymmetry algebra is of the form (3),

$$[Q, Q] = [Q^\dagger, Q^\dagger] = [H, Q] = [H, Q^\dagger] = [H, H] = 0 \quad (16.a)$$

$$[Q, Q^\dagger] = H = \frac{1}{2}p^2 + \frac{1}{2}W_q^2 + i\bar{\eta}W_{qq}\eta \quad (16.b)$$

and the quantum theory is defined by

$$Z \sim \int [dpdq][d\eta d\bar{\eta}] \exp(i \int p\dot{q} + \bar{\eta}\dot{\eta} + [Q, Q^\dagger]) \quad (17)$$

This is *identical* to the BRST functional integral (9) provided we identify Q (or $Q^\dagger$) with the BRST operator Ω in (9) and $Q^\dagger$ (or Q) with the gauge fermion Ψ . The Fradkin-Vilkovisky theorem (7.b) then ensures that the functional integral (17) is *independent* of the functional form of $Q^\dagger \sim \Psi$. Hence we may vary $Q^\dagger$ while keeping Q intact and this will not change the value of (17). (In this letter we are not interested in global aspects such as boundary conditions in the functional integral.)

We observe that a conjugation of the form (12) with $\Phi = iW - \frac{i}{2}q^2$, transforms the "BRST operator" Q in (15) onto the "BRST operator" of a supersymmetric harmonic oscillator,⁴

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \theta_2\theta_1; \quad Q = \frac{1}{\sqrt{2}}\eta(p + iq) \quad (18)$$

The interacting theory then differs from the free theory only by a superrotation of the form (5) and a conjugation of the form (12). In particular, the relation between (15,16) and (18) is *entirely* analogous to the relation between abelian and nonabelian constraint algebras discussed in [8,9]. (Notice that conjugations similar to (12) in Poincaré SUSY theories have also been considered in [3].)

Notice that H in (18) is an $\text{Osp}(2|2)$ invariant bilinear of the form (1), with $y_1 = y_2$. The pertinent functional integral (17) is then a Parisi-Sourlas integral as in (4). Since the interacting quantum theory is obtained from the free quantum theory by a change of variables with interaction identified with the pertinent jacobian, we conclude that the

⁴Since we have identified $Q^\dagger$ with the gauge fermion Ψ , the conjugated $Q^\dagger$ is not very interesting: The Fradkin-Vilkovisky theorem (7.b) ensures that (17) does not depend on the functional form of the "gauge fermion" $Q^\dagger$.

interacting quantum theory is also $\text{Osp}(2|2)$ invariant. In particular, from the Fermi-Bose balance we conclude that the phase space is a zero dimensional manifold, in the sense of (5).

A more careful analysis reveals that the superrotation from (15,16) to (18) is not necessarily globally defined and in particular fails to be invertible for the zero energy eigenstates of H . As a consequence it fails to provide direct information on nonperturbative aspects such as *e.g.* dynamical supersymmetry breaking and the Witten index [3] of the original theory. Obviously, this generalizes to higher dimensional SUSY field theories. However, here space does not permit a discussion of global aspects.

Our conclusions can be immediately generalized to Poincaré SUSY theories in higher dimensions. This follows since the ($D > 2$ for the moment) supersymmetry algebras (we use $\bar{Q}_\alpha^i = Q_{\alpha i}^\dagger$),

$$[Q_{\alpha i}, Q_{\beta j}^\dagger] = 2\delta_i^j (\sigma^\mu)_{\alpha\beta} P_\mu \quad (19.a)$$

$$[Q_{\alpha i}, Q_{\beta j}] = 2\epsilon_{\alpha\beta} Z_{ij} \quad [Q_{\alpha i}^\dagger, Q_{\beta j}^\dagger] = -2\epsilon_{\alpha\beta} Z^{ij} \quad (19.b)$$

with $Z_{ij} = -Z_{ji}$ central charges, can be viewed as Inönü-Wigner contractions of suitable $\text{Osp}(m|2n)$ algebras [11], with the Lorentz algebra in the $\text{Sp}(2n)$ of $\text{Osp}(m|2n)$ and $Q_{\alpha i}$ etc. nilpotent $\text{Osp}(m|2n)$ generators as in (3). From (19.a) we get after normalization

$$\sum_\alpha [Q_{\alpha i}, Q_{\alpha i}^\dagger] = H \quad (20)$$

Hence the quantum theory is determined by

$$Z \sim \int [d(\text{Liouville})] \exp(i \int \vartheta - \sum_\alpha [Q_{\alpha i}, Q_{\alpha i}^\dagger]) \quad (21.a)$$

a Parisi-Sourlas integral of the form (7.b,c). Since $[Q_{\beta j}, [Q_{\alpha i}, Q_{\alpha i}^\dagger]] = 0$ we can use the Fradkin-Vilkovisky theorem (7.b,c) to perform superrotations of the form (6) generated by the $Q_{\beta j}$ that reduces the functional integral into

$$Z \sim \int_{\text{patches}} [d(\text{Liouville})] \exp(i \int \vartheta - [Q_{\alpha i}, Q_{\alpha i}^\dagger]) \quad (\text{no sum}) \quad (21.b)$$

where now only a single pair $Q_{\alpha i}, Q_{\alpha i}^\dagger$ appears. This is a functional integral of the form (9), (7.b). Superrotations generated by the "BRST operator" $Q_{\alpha i}$ and the corresponding free "BRST operators" $Q_{\alpha i}^{\text{free}}, Q_{\beta j}^{\text{free}}$ then rotate (21.b) into the corresponding free SUSY quantum theory.

We conclude that an interacting SUSY theory differs from the corresponding free theory only by (local) Parisi-Sourlas superrotations (6), and the SUSY interaction can be viewed as the superjacobian corresponding to these superrotations. Furthermore, as a consequence of the Bose-Fermi balance (21) is a zero dimensional integral in the sense of (5): The phase space of a SUSY theory is a zero dimensional manifold.

Notice that since our superrotations in general do not define unitary automorphisms in the Hilbert space of the interacting quantum theory, our observations do not imply that supersymmetric quantum theories are trivial.

As an example, consider the four dimensional N=1 Yang-Mills theory. In the Weyl basis, the nilpotent supersymmetry generators are

$$Q_\alpha = \int (E_i^a + iB_i^a) \sigma_{\alpha\beta}^i \chi_\beta^a \quad (22)$$

with

$$\{E_i^a(x), A_j^b(y)\} = -\delta_{ij}^{ab}(x-y) \quad \{\chi_\alpha^a, \chi_\beta^b\} = -i\delta_{\alpha\beta}^{ab}(x-y) \quad (23)$$

The nontrivial anticommutator is

$$[Q_\alpha, Q_\beta^\dagger] = -2i \left(H\delta_{\alpha\beta} + \{P^i + \int A_a^i \mathcal{G}^a\} (\sigma_i)_{\alpha\beta} \right) \quad (24)$$

with $\mathcal{G}^a$ the Gauss' law constraint functional. The quantum theory can then be viewed as a double rotated Parisi-Sourlas integral over the constraint surface $\mathcal{G}^a \sim 0$,⁵

$$Z \sim \int [dE_i^a dA_i^a][d\psi d\psi^\dagger] \exp(i \int E \cdot \dot{A} + \psi^\dagger \dot{\psi} - [Q_1, Q_1^\dagger] - [Q_2, Q_2^\dagger]) \quad (25)$$

Since $[Q_2, [Q_1, Q_1^\dagger]] = 0$ we can eliminate the last term in (25) using a superrotation of the form (6) generated by Q_2 . Fradkin-Vilkovisky theorem (7.b,c) ensures that the remaining functional integral is independent of the functional form of the gauge fermion $Q_1^\dagger$, and the interacting quantum theory can be related to the pertinent free theory by additional superrotations of the form (6) with the interaction identified with the corresponding superjacobians. Since the free super-YM theory is a quadratic form such as (1), (13), *i.e.* $\text{Osp}(\infty|\infty)$ invariant, the interacting theory is also $\text{Osp}(\infty|\infty)$ invariant and the functional integral (25) is a Parisi-Sourlas integral of the form (7.b,c). In particular the Bose-Fermi balance implies that the effective phase space dimensionality vanishes.

⁵Here we do not address problems associated with constraints. This can be best discussed using canonical BRST [8,9,14,15]. The integral will then have the form (9) also in the ghost sector.

We observe that again the "BRST operator" Q_1 can be related to the corresponding free "BRST operator" by a conjugation (12) with

$$\Phi = -\frac{ig}{3!}\epsilon_{ijk}f^{abc}\int A_i^a A_j^b A_k^c \quad (26)$$

Consequently the relation between the free and interacting theory is again an exact analog of the relation between abelian and nonabelian realizations of the canonical $\text{Osp}(1,1|2)$ algebra in constrained systems [8,9].

Examples where the present construction cannot be directly applied appear in $D = 1, 2$ where in certain cases all combinations of supersymmetry generators fail to be nilpotent. The pertinent extension of (19-21) could be developed, but here we find it instructive to proceed differently. As an example we consider the $N = 1$ supersymmetry in two dimensions. In order to understand how this model can be included in the present formalism we first consider the two dimensional $N=2$ Wess-Zumino model,

$$L = \partial_\mu \varphi^* \partial^\mu \varphi + \bar{\psi} i \not{\partial} \psi - V_\varphi^* V_\varphi + \bar{\psi} \left(\frac{1-\gamma^3}{2} \right) \psi V_{\varphi\varphi} + \bar{\psi} \left(\frac{1+\gamma^3}{2} \right) \psi V_{\varphi^*\varphi^*} \quad (27)$$

Since (27) is related to the $N=1$ Wess-Zumino model in four dimensions by dimensional reduction and since the latter is a straightforward example of the present ideas, it is clear that our results apply to (27). However, it is instructive to discuss the details: We choose a basis where the canonical supersymmetry charges are

$$Q_1 = \pi_\varphi \chi_1 + 2\partial\varphi^* \chi_1 - iV_\varphi^* \chi_2^\dagger \quad Q_1^\dagger = \pi_{\varphi^*} \chi_1^\dagger + 2\partial\varphi \chi_1^\dagger + iV_\varphi \chi_2 \quad (28.a)$$

$$Q_2 = \pi_\varphi \chi_2 - 2\partial\varphi^* \chi_2 + iV_\varphi^* \chi_1^\dagger \quad Q_2^\dagger = \pi_{\varphi^*} \chi_2^\dagger - 2\partial\varphi \chi_2^\dagger - iV_\varphi \chi_1 \quad (28.b)$$

with

$$\{\pi_\varphi(x), \varphi(y)\} = \{\pi_{\varphi^*}(x), \varphi(y)\} = -\delta(x-y) ; \quad \{\chi_a(x), \chi_b^\dagger(y)\} = -i\delta_{ab}(x-y) \quad (29)$$

The combination $Q_1 + Q_2^\dagger$ is nilpotent and since

$$\begin{aligned} & \frac{i}{4}\{Q_1 + Q_2^\dagger, Q_1^\dagger + Q_2\} = H \\ & = \frac{1}{2}\pi_\varphi \pi_{\varphi^*} + 2\partial\varphi \partial\varphi^* + i\partial\chi_1 \chi_1^\dagger - i\partial\chi_2 \chi_2^\dagger + \frac{1}{2}V_{\varphi\varphi} \chi_1 \chi_2 + \frac{1}{2}V_{\varphi^*\varphi^*} \chi_2^\dagger \chi_1^\dagger + V_\varphi V_{\varphi^*} \end{aligned} \quad (30)$$

we can view $Q_1 + Q_2^\dagger$ as the "BRST operator", and $Q_1^\dagger + Q_2$ as the "gauge fermion". The quantum theory is then a functional integral (9), and by (7.b) it is independent of the

functional form of $\Psi \sim Q_1^\dagger + Q_2$. The interacting BRST operator can then be related to the free BRST operator by a conjugation (12),

$$\Omega_{free} = \mathcal{X}_1(\pi_\varphi + 2\partial\varphi^*) + \mathcal{X}_2^\dagger(\pi_{\varphi^*} - 2\partial\varphi) = e^{-\Phi}(Q_1 + Q_2^\dagger)e^\Phi \quad (31)$$

with $\Phi = -2i \int Re(V)$. Notice that a further conjugation with $\Phi = 2 \int \partial\varphi \partial\varphi^*$ maps (31) into the differential representation $\Omega_{free} = \mathcal{X}_1\pi_\varphi + \mathcal{X}_2^\dagger\pi_{\varphi^*}$ in (11).

Since the interacting theory is related to the free theory by a change of variables and since the free Hamiltonian is an $\text{Osp}(\infty|\infty)$ invariant quadratic form (1), all our previous conclusions apply. In particular the functional integral can be viewed as a Parisi-Sourlas integral over a zero dimensional phase space as in (5), and the interaction can be identified with the superjacobian corresponding to a superrotation (6).

The hamiltonian of the $N = 1$ SUSY scalar theory in two dimensions is obtained from (30) by a change of variables similar to the Inönü-Wigner contraction and truncation that relates (9) and (10): Define

$$\phi = \frac{1}{2}A + \frac{i}{2}B \quad ; \quad \mathcal{X}_1 = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \quad ; \quad \mathcal{X}_2 = \frac{1}{\sqrt{2}}(\theta_3 + i\theta_4) \quad (32)$$

and introduce the change of variables $B, \pi_B, \theta_2, \theta_3 \rightarrow \epsilon B, \epsilon\pi_B, \epsilon\theta_2, \epsilon\theta_3$. The corresponding Jacobian in the Liouville measure is trivial, and in the $\epsilon \rightarrow 0$ limit (30) yields the $N = 1$ Hamiltonian

$$H = \frac{1}{2}\pi_A^2 + \frac{1}{2}(\partial A)^2 - \frac{i}{2}\theta_1\partial\theta_1 + \frac{i}{2}\theta_3\partial\theta_4 + i\theta_1\theta_4W_{AA} + \frac{1}{2}W_A^2 \quad (33)$$

in analogy with the contraction from (9) to (10).

The contracted supersymmetry generators are

$$Q_1 \sim (\pi_A + \partial A)\theta_1 - W_A\theta_4 \quad Q_2^\dagger \sim i(\pi_A - \partial A)\theta_4 + iW_A\theta_1 \quad (34)$$

and we can write (33) as

$$H = \frac{i}{4}[Q_1 + Q_2^\dagger, Q_1^\dagger + Q_2] \quad (35)$$

Notice that no linear combination of Q_1, Q_2 is nilpotent. However, since (34), (35) are contractions and truncations of (28) and (30), we conclude that this model is nevertheless contained in the formalism (19-21).

It is interesting to understand the previous construction from the point of view of BRST operators for second class constrained systems: The contracted limit of $Q_1 + Q_2^\dagger$

is

$$Q_1 + Q_2^\dagger \rightarrow \Omega = (\pi_A + \partial A + iW_A)\theta_1 + i(\pi_A - \partial A + iW_A)\theta_4 \quad (36)$$

which fails to be nilpotent. If we introduce a conjugation by $\Phi = \int iW$,

$$e^{-\Phi}\Omega e^{\Phi} = (\pi_A + \partial A)\theta_1 + i(\pi_A - \partial A)\theta_4 \quad (37)$$

we recognize that the *r.h.s.* are analogous to the BRST operators for a left mover resp. right mover on the world sheet, except that the Kac-Moody central extensions of the second class "constraints" $\pi_A \pm \partial A \sim 0$ have been ignored, explaining the non-nilpotency of (36,37).

In general, a nilpotent BRST operator and the ensuing $\text{Osp}(1,1|2)$ algebra corresponding to a second class constraint algebra can be constructed either by introducing additional degrees of freedom [17] or by implementing certain canonical transformations on the extended phase space [18]. An $\text{Osp}(1,1|2)$ algebra for chiral bosons has also been constructed directly, by declaring half of the second class constraints as gauge fixing [19]. Comparing (36,37) with (28) we conclude that the contraction from (28) to (36) is in line with the proposal of [17], to construct BRST operators for second class algebras by extending the phase space. The truncated variables $\pi_B, B, \theta_2, \theta_3$ can be viewed as the variables that convert the second class algebra into an effective first class algebra along the lines of [17].

In superstring theories the configuration space is identified with the physical space-time: Since our general arguments imply that the configuration space of a SUSY theory is zero dimensional, we conclude that the *effective spacetime in superstring theories is zero dimensional!* In the following we shall explain how this emerges.

In the conformal gauge the (NRS) superstring Lagrangian is

$$L = \frac{1}{2}(\partial_a X^\mu)^2 - \frac{i}{2}\bar{\psi}^\mu \not{\partial} \psi_\mu + ib^{ab}\partial_a c_b + i\bar{\beta}^a \partial_a \lambda \quad (38)$$

and from our general discussion we conclude that all other gauge fixed versions can be related to (38) by a superrotation of the form (6), generated by the BRST operator of the superstring [20]. This ensures the gauge independence of the following observations.

The matter (*i.e.* X^μ, ψ^μ) action is a sum of free $N = 1$ actions. Hence our conclusions apply, as explained in (32)-(37). The ghost has the form

$$b^{ab}\partial_a c_b + \bar{\beta}^a \partial_a \lambda \sim M_{ab}(x_1^a x_2^b + \theta_1^a \theta_2^b); \quad M_{ab} = -M_{ba} \quad (39)$$

Since both g_{ab} and c_{ab} in (1) are now antisymmetric, this is not a bilinear of the form (1), (13). But since $g_{ab} = c_{ab}$, (39) nevertheless is an $\text{Osp}(1,1|2)$ invariant quadratic form separately for each a : For this we introduce (for each a) a four dimensional Parisi-Sourlas superspace $y_\alpha^a = (y_-^a, y_+^a, y_\theta^a, y_\beta^a) = (\theta_1^a, \theta_2^a, x_2^a, x_1^a)$ with light-cone metric and the corresponding inner product $y_\alpha^a y^{a\alpha} = 2\theta_1^a \theta_2^a + x_1^a x_2^a$. In this notation the ghost action (39) becomes

$$M_{ab}(x_1^a x_2^b + \theta_1^a \theta_2^b) = M_{ab} y_\alpha^a y^{b\alpha} \quad (40)$$

and introducing the pertinent differential representation (11) of $\text{Osp}(1,1|2)$ (for each a) one can verify that (40) is indeed $\text{Osp}(\infty|\infty)$ invariant. Hence the superstring functional integral is a Parisi-Sourlas integral (4-7) over a zero dimensional phase space. Since the phase space of a superstring is identified with the physical spacetime, this implies that in the Parisi-Sourlas sense the superstring moves in a zero dimensional spacetime. Since the interacting superstring theory is presumed to describe conventional interacting SUSY theories to which our conclusions apply, we argue that our conclusions remain valid also when superstring interactions are turned on.

Notice that our conclusions do not imply that superstrings are trivial. Even though the effective spacetime is zero dimensional, as a consequence of the conformal anomaly the $\text{Osp}(\infty|\infty)$ can be unitarily implemented in the Hilbert space of the quantum superstring only if there are ten bosonic coordinates X^μ . This ensures that the spacetime on of a superstring cannot collapse. Rather, our observation could be useful *e.g.* in understanding finiteness of superstrings: Since all integrals in a superstring quantum theory are zero dimensional in the sense of (5), and since zero dimensional integrals are both UV and IR finite, finiteness of superstrings appears quite natural.

Notice that the bosonic string action,

$$L = \frac{1}{2}(\partial_a X^\mu)^2 + i b^{ab} \partial_a c_b \quad (41)$$

has the functional form (13) and is consequently $\text{Osp}(\infty|\infty)$ invariant. This invariance corresponds to an $\text{Osp}(27,1|2)$ invariance at each mass level, ensuring Parisi-Sourlas cancellation between the unphysical degrees of freedom [9,13]. The effective spacetime dimension is then the familiar $D=26$. Finally, since all other gauge fixed versions are related to (41) by superrotations of the form (6) generated by the BRST operator of the bosonic string, these observations are gauge independent.

The zero mode of the (Ramond) superstring corresponds to a QM SUSY algebra of

the form

$$[H, H] = [H, Q] = 0 \quad [Q, Q] = -H \quad (42)$$

As in the $N=1$ model in two dimensions, the supersymmetry generator fails to be nilpotent. Consequently the formalism (19-21) is not directly applicable, but as before we can view (42) as a contraction and truncation of (14-17).

In [21] it has been shown that (42) can be related to a Dirac fermion: We find it conceptually very curious that a Dirac fermion can be viewed either as a 0+1 dimensional supergravity theory [21] or alternatively as a negative dimensional boson or as a zero dimensional supermanifold, as follows from the present considerations.

In conclusion, we have established that path integrals in Poincaré supersymmetric theories can be viewed as Parisi-Sourlas integrals of the Fradkin-Vilkovisky form (9), with fermions identified as negative dimensional phase space coordinates and interactions related to superjacobians of superrotations such as (6). We have also argued that as a consequence of the Fermi-Bose balance SUSY path integrals are formally zero dimensional in the Parisi-Sourlas sense. When applied to superstrings, this implies that the dimensionality of the physical spacetime vanishes. Our results are consistent and complementary to those in [2] and establish that Poincaré, BRST and Parisi-Sourlas supersymmetries can all be understood from a common point of view. Finally, even though the arguments presented in this letter are formal, a rigorous formulation is possible provided the SUSY theory in question can be regulated in a SUSY invariant manner. The regulated theory is then both finite and supersymmetric, and as a consequence all our conclusions apply.

We wish to thank G.'t Hooft for a discussion that prompted the present investigation. We also wish to thank L.Alvarez-Gaume, A.Chamseddine, J.Fröhlich, M.Günaydin, C.Kounnas, G.Parisi, A.Schwimmer, N.Sourlas and P.West for discussions. After this work was completed, B.Grossman informed us that relations between Poincaré and BRST supersymmetries are also considered in [22].

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