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SYMMETRIES AND COUPLINGS IN HETEROTIC SUPERCONFORMAL FIELD THEORIES

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ABSTRACT

We discuss the symmetries of the superpotential in compactified heterotic superstring theories formed from the product of minimal $N = 2$ superconformal field theories. It is shown how these symmetries can ensure flatness of the potential involving the moduli, and we derive new results (going beyond those obtained by superconformal techniques alone) for the flatness of the potential involving other massless fields.

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Recently Gepner[1] introduced a method for the compactification of heterotic string theories based on the product of $N = 2$ minimal superconformal field theories. These models are the only known non-trivial realisations of the $N = 2$ superalgebra and the method provides a generalisation of previous constructions based on trivial free field realisations. Remarkably, some of these models are closely related to known Calabi-Yau compactifications[2], both in the massless multiplet and discrete symmetry structure. This correspondence has been extended by Distler and Greene[3] and Gepner[4] who showed that even the non-vanishing couplings at $O(27^3)$ in the superpotential agree (up to field normalisations which are unknown in the Calabi-Yau compactification). However, the conformal field theory construction has a much richer symmetry structure than is known in the Calabi-Yau case and it is the purpose of this letter to determine these symmetries and their implications for the structure of the superpotential*. Although our interest has mainly been stimulated by the correspondence with Calabi-Yau theories, the implications of these symmetries is of much broader interest because other examples of the tensor construction correspond to orbifold (or toroidal) compactifications, or new compactifications in their own right[7].

The $N = 2$ superconformal models used in the tensor construction are labelled by a positive integer k and carry central charge $c = 3k/(k+2)$. Products of such models are used to obtain the desired total central charge. For example, the choice of five copies of the level $k = 3$ model has $c = 9$ as required for a four dimensional string construction and is found to correspond to the Calabi-Yau model $Y_{4,5}$ at a particular point in moduli space given by the maximally symmetric choice of the quintic polynomial in $\mathbf{P}_4$.

The models have an $N = 1$ supersymmetry in four spacetime dimensions plus an $E_8 \times E_6 \times G$ gauge group, where G is a group of rank $(r-1)[1]$. Further symmetries of the construction are revealed using the relationship between the $N = 2$ minimal superconformal models and the Z_k parafermion models introduced by Fateev and Zamolodchikov[8]. The primary fields of the $N = 2$ superconformal theory $\Phi_{q,s;\bar{q},\bar{s}}^{l,\bar{l}}(z, \bar{z})$ are given in terms of the parafermionic primary fields $\chi_{m;\bar{m}}^{l,\bar{l}}(z, \bar{z})$, with total isospin $l, \bar{l}$ and third components $m, \bar{m}$ given by[9]

$$\Phi_{q,s;\bar{q},\bar{s}}^{l,\bar{l}}(z, \bar{z}) = \chi_{q-s;\bar{q}-\bar{s}}^{l,\bar{l}}(z, \bar{z}) : e^{i\alpha_{q,s}\phi(z)+i\alpha_{\bar{q},\bar{s}}\bar{\phi}(\bar{z})} : \quad (1)$$

* In the case of orbifolds the selection rules for the superpotential has been investigated in refs.[5,6].

where $\phi(z)$ and $\bar{\phi}(\bar{z})$ are free bosons, and $\alpha_{q,s}$ is given by

$$\alpha_{q,s} = [k(k+2)]^{-\frac{1}{2}} [-q + \frac{1}{2}(k+2)s]. \quad (2)$$

For the ‘standard’ range, defined as $|q| \leq l$ in the Neveu-Schwarz sector and $|q-s| \leq l$ in the Ramond sector, the conformal dimensions and $U(1)$ charges of the superconformal fields are

$$\begin{aligned} h &= \frac{l(l+2) - q^2}{4(k+2)} + \frac{s^2}{8} \\ Q &= -\frac{q}{k+2} + \frac{s}{2}. \end{aligned} \quad (3)$$

The parafermion theory has a $Z_k \times \bar{Z}_k$ symmetry associated with the left and right moving sectors (the simplest example of such a system is the Z_k Ising model). Due to this and the self-duality of the parafermions, we have the identifications

$$\chi_{m;\bar{m}}^{l;\bar{l}} \simeq \chi_{m+2k;\bar{m}}^{l;\bar{l}} \simeq \chi_{m;\bar{m}+2k}^{l;\bar{l}} \simeq \chi_{m+k;\bar{m}+k}^{k-l;k-\bar{l}} \quad (4)$$

which immediately implies that the superconformal fields satisfy

$$\Phi_{q,s;\bar{q},\bar{s}}^{l;\bar{l}} \simeq \Phi_{q+2(k+2),s+4;\bar{q},\bar{s}}^{l;\bar{l}} \simeq \Phi_{q,s;\bar{q}+2(k+2),\bar{s}+4}^{l;\bar{l}} \simeq \Phi_{q+k+2,s+2;\bar{q}+k+2,\bar{s}+2}^{k-l;k-\bar{l}} \quad (5)$$

Note that these transformations preserve the charges $\alpha_{q,s}$ of eq(2). The correlation functions of parafermions conserve separately the charges m and $\bar{m}$ and the total isospins l and $\bar{l}$ up to these field identifications.

These results for the parafermionic correlation functions may immediately be used to determine the symmetries of the correlation functions of the $N=2$ superconformal fields via eq(1), because the correlation functions differ only by a trivial factor involving free boson correlation functions. In order to do this we must be careful to look after the superconformal ghost anomaly[5,10,11]. We will compute correlation functions of the form

$$\langle V_{-1}^1 V_{-\frac{1}{2}}^2 V_{-\frac{1}{2}}^3 V_0^4 \dots V_0^n \rangle, \quad (6)$$

where $V_{-1(0)}$ is a vertex operator for a space-time scalar in the $(-1)((0))$ picture and $V_{-\frac{1}{2}}$ is a vertex operator for a space-time fermion in the $(-\frac{1}{2})$ picture. The correlation function

(6) has the correct ghost charge (-2) for a correlator computed on the sphere (string tree level). The $N = 1$ space-time supersymmetry means that a non-vanishing correlator of the form given in eq(6) may only arise from the F-term of a product of all left-handed (or all right-handed) chiral superfields. We adopt the convention that they are all left-handed; the others may be constructed via charge conjugation.

The vertex operator in the (-1) picture has the form

$$V_{-1}^\alpha(z, \bar{z}) = e^{-\rho(z)} O(z, \bar{z}) e^{ikx} \tau^\alpha. \quad (7)$$

Here $\rho(z)$ is the free boson field which comes from bosonising the ghost system, and $O(z, \bar{z})$ is an operator in the internal theory of the form

$$O(z, \bar{z}) = \bigotimes_{i=1}^r \Phi_{q_i, s_i; \bar{q}_i, \bar{s}_i}^{l_i, \bar{l}_i}(z, \bar{z}), \quad (8)$$

where the internal theory is constructed from the tensor product of r minimal superconformal theories at levels k_i ($i = 1, 2, \dots, r$). The factor τ^α in eq(7) describes the gauge degrees of freedom in the bosonic sector. In what follows we will be interested primarily in the internal sector structure so we will often suppress all but the $O(z, \bar{z})$ dependence of the vertex operators.

The massless fields which transform as the **10** in the **27** ($\bar{\mathbf{10}}$ in the $\bar{\mathbf{27}}$) representation of the $SO(10)$ in E_6 correspond to $O(z, \bar{z})$ with (l_i, q_i, s_i) and $(\bar{l}_i, \bar{q}_i, \bar{s}_i)$ chosen such that $h_{TOT} = \bar{h}_{TOT} = \frac{1}{2}$ and $Q_{TOT} = \pm \bar{Q}_{TOT} = -1$, subject to the heterotic condition relating the left and right sectors. The choice of sign of Q_{TOT} is in accord with our convention of working with only left-handed chiral superfields. Using eq(3) gives $h_{TOT} \geq \frac{1}{2}|Q_{TOT}|$ and $\bar{h}_{TOT} \geq \frac{1}{2}|\bar{Q}_{TOT}|$, with equality iff $l_i = |q_i|$, $s_i = 0$ and $\bar{l}_i = |\bar{q}_i|$, $\bar{s}_i = 0$, respectively, when all labels are in the standard range. Imposing the heterotic condition restricts the allowed values of q_i relative to $\bar{q}_i$. With the standard affine invariant $\delta_{l, \bar{l}}$ connecting the left and right sector the allowed states are $(l_i, l_i, 0) \otimes (l_i, \pm l_i, 0)$ and $(l_i, l_i, 0) \otimes (l_i, (k_i + 2) \pm (k_i - l_i), 2)$. In most cases the latter do not satisfy the heterotic condition, but we do not have an algebraic proof that this happens in every case.

Vertex operators in the $(-\frac{1}{2})$ picture may be constructed from those in the (-1) picture

by operating with the supersymmetry charge[3]

$$Q^\mu = \oint \frac{dz}{2\pi i} e^{-\rho(z)/2} s^\mu(z) S(z) \quad (9)$$

where s^μ is the spin operators for the four free (space-time) fermions ψ^μ , and $S(z)$ is an $N = 2$ superconformal field

$$S(z) = \bigotimes_{i=1}^r I_i e^{\frac{i}{2} \sqrt{\frac{k}{k+2}} \phi_i(z)}, \quad (10)$$

with I_i the identity operator on parafermionic fields. $S(z)$ just produces a shift of bosonic charges. Applied to the operators of eq(7) we have

$$S(z)O(z, \bar{z}) = \bigotimes_{i=1}^r \Phi_{q_i+1, s_i+1; \bar{q}_i, \bar{s}_i}^{l_i; \bar{l}_i} \quad (11)$$

Finally, vertex operators in the (0) picture may be constructed from those in the (-1) picture by operating with $e^{\rho(z)} T_F(z)$, where $T_F(z)$ is the superpartner of the stress tensor. Applied to the vertex operator of eq(6) this gives

$$V_0^\alpha = [G_{-\frac{1}{2}}^+(z) + G_{-\frac{1}{2}}^-(z) + ik\psi] O(z, \bar{z}) e^{ikx} \tau^\alpha, \quad (12)$$

where $G_{-\frac{1}{2}}^\pm(z)$ are the components of T_F acting on the internal sector. Acting on the scalar components of massless left-handed superfields $G_{-\frac{1}{2}}^-$ annihilates $O(z, \bar{z})$ while the action of $G_{-\frac{1}{2}}^+(z)$ on $O(z, \bar{z})$ is given by

$$G_{-\frac{1}{2}}^+(z)O(z, \bar{z}) = \sum_{j=1}^r \bigotimes_{i=1}^{j-1} \Phi_{q_i, s_i; \bar{q}_i, \bar{s}_i}^{l_i; \bar{l}_i} \otimes \Phi_{q_j, s_j+2; \bar{q}_j, \bar{s}_j}^{l_j; \bar{l}_j} \otimes \bigotimes_{i'=j+1}^r \Phi_{q_{i'}, s_{i'}; \bar{q}_{i'}, \bar{s}_{i'}}^{l_{i'}; \bar{l}_{i'}} \quad (13)$$

In the correlation functions considered here the (k, ψ) term in eq(12) never contributes due to the conservation of Q_{TOT} [3].

Using eqs(1),(7),(11),(12) and (13) we may now evaluate the correlation function of eq(6) in terms of a correlation function of parafermionic fields and a free bosonic correlation

function. The latter gives the following selection rules for non-vanishing correlators

$$\sum_{J=1}^n \alpha_{\bar{q}_i^J, \bar{s}_i^J} = 0 \quad (\forall i) \quad (14)$$

$$\sum_{J=1}^n \alpha_{q_i^J, s_i^J} + \sqrt{\frac{k_i+2}{k_i}} \left[\sum_{J=4}^n d_i^J + 2\left(-\frac{1}{k_i+2} + \frac{1}{2}\right) \right] = 0 \quad (\forall i). \quad (15)$$

The second term in eq(15) takes account of the action of the picture change to the (0) picture. From eqs(13) and (2) we have $d_i^J = 0, 1$. The third term in eq(15) arises when picture changing the two operators to the $(-\frac{1}{2})$ picture. Here we adopt the convention that $\alpha_{q,s}$ refers to the field in the (-1) picture (eq(7)), while $\alpha_{\bar{q},\bar{s}}$ refers to an appropriate choice of representation leading to a gauge invariant correlation function. Evaluating the parafermion correlator gives rise to the additional selection rules

$$\begin{aligned} \sum_{J=1}^n \vec{l}_i^J &= 0 \\ \sum_{J=1}^n \vec{\bar{l}}_i^J &= 0 \end{aligned} \quad (16)$$

$$\begin{aligned} \sum_{J=1}^n (q_i^J - s_i^J) - 2 \sum_{J=4}^n d_i^J &= 0 \pmod{2k_i} \\ \sum_{J=1}^n (\bar{q}_i^J - \bar{s}_i^J) &= 0 \pmod{2k_i} \end{aligned} \quad (17)$$

where eq(16) means that the vector addition of the contributing isospins should vanish.

The first of these selection rules, eq(14), is just the conservation law associated with the $U(1)^r$ gauge factors which appear in the tensor construction. The diagonal sum of these provides the $SO(10)$ singlet component of the E_6 adjoint of gauge bosons. The remaining $U(1)^{r-1}$ are additional gauge bosons not found in the Calabi-Yau models, which are attributed to the fact that the tensor construction corresponds to a particularly symmetric point in the moduli space of a Calabi-Yau space.

Due to the picture changing there is not an equivalent $U(1)^r$ conservation in the left fermionic sector but instead the result of eq(15) applies. We may use eqs(14) and (15)

together with the Z_k and $\tilde{Z}_k$ symmetries in eq(17) to derive a $Z_{k+2} \times \tilde{Z}_{k+2}$ symmetry given by

$$\sum_{J=1}^n \frac{1}{2} q_i^J + 1 = 0 \pmod{k_i + 2} \quad (\forall i) \quad (18)$$

$$\sum_{J=1}^n \frac{1}{2} \bar{q}_i^J = 0 \pmod{k_i + 2} \quad (\forall i) \quad (19)$$

The diagonal sum of eqs(18) and (19) defines a Z_{k_i+2} symmetry associated with each tensor factor and this is the symmetry that has been identified[1] with the known Calabi-Yau symmetries in the cases where the superconformal theory matches the complete intersection Calabi-Yau theory. However, the symmetries of eqs(15),(16),(18) and (19) represent a much larger set of symmetries than just these Z_{k_i+2} symmetries.

Indeed, we can immediately use these symmetries to derive interesting structure for the superpotential. Consider E_6 gauge singlet fields described by eq(8) with $q_i = \bar{q}_i$ ($\forall i$). As we will discuss shortly this applies to a large number of massless fields. For these fields eqs(18) and (19) clearly cannot be simultaneously satisfied and hence any correlator involving just these fields vanishes.

Examples of singlet fields satisfying $q_i = \bar{q}_i$ ($\forall i$) come from the h^{21} moduli, $\tilde{C}$, associated with the h^{21} fields transforming as the **27**. They may be obtained from the $O(z, \bar{z})$ of the **27**'s (in the **10** of $SO(10)$) by the action of $G_{-\frac{1}{2}}^+(\bar{z})$, which does not affect the $\bar{q}_i$. As we have noted above in most models with standard affine invariants the h^{21} **27** superfields, and hence the moduli $\tilde{C}$, are $q_i, \bar{q}_i$ symmetric.

Our result immediately establishes the vanishing of the potential $V(\tilde{C})$ to all orders, a result which is a consequence of the discrete symmetries of the tensor construction. This result is not new for it can be proved in general for superconformal theories but only by ‘stringy’ manipulations using the conformal geometry of the world sheet[11], not via symmetries of the effective lagrangian.

However we may immediately use our argument to establish a result that goes beyond the general superconformal structure. Namely, the absence of correlation functions between fields for which $q_i = \bar{q}_i$ implies the stronger result: that the potential involving any singlet fields, C , obtained from the $(q_i, \bar{q}_i$ symmetric) **27** by the action of $G_{-\frac{1}{2}}^+(\bar{z})$ on the i 'th tensor factor vanishes (and not just the symmetrised fields which are the moduli $\tilde{C}$).

To illustrate this point we give in Table 1 the massless multiplet structure of the $\mathbf{27}$, the $\bar{\mathbf{27}}$ and the singlet fields for the $Y_{4,5}$ model. We see that all but two groups of singlets have $q_i = \bar{q}_i$ ($\forall i$) and the potential involving these fields alone vanishes. A similar result immediately follows for the singlet fields with $q_i = -\bar{q}_i$ ($\forall i$) which includes the h^{11} moduli $\tilde{R}$, obtained from the $\bar{\mathbf{27}}$'s (in the $\mathbf{10}$ of $SO(10)$) by the action of $G_{-\frac{1}{2}}^-(\bar{z})$, and additional (non-symmetrised) fields, R . In the case that both types of fields are present a coupling may occur at some order. For example $(\mathbf{27}.\bar{\mathbf{27}})^n$ is non-vanishing if $n\bar{q}_i(\mathbf{27}) = 1 \pmod{k_i+2}$, for $\mathbf{27}, \bar{\mathbf{27}}$ fields with $q_i = \pm\bar{q}_i$, respectively. In the $Y_{4,5}$ example this means the first non-vanishing term occurs at order $(\mathbf{27}.\bar{\mathbf{27}})^4$.

Although general statements are difficult to establish, in practice the full content of the symmetries (15), (16), (18) and (19) may easily be worked out for any given model. In the $Y_{4,5}$ model we find that the discrete symmetries forbid all couplings of the form

$$\langle \mathbf{1}^n.(\mathbf{27}.\bar{\mathbf{27}})^m \rangle,$$

where $(m = 0; n = 0, 1, 2, 3)$ and $(m = 1, 2; n = 0, 1, 2)$, and $\mathbf{1}, \mathbf{27}$ and $\bar{\mathbf{27}}$ refer to any of the massless modes given in Table 1. Our analysis is limited by the time needed to search all possible combinations, so we have not been able to check to very high order, particularly in the singlet sector. However it is easy to establish that the discrete symmetries do not imply complete flatness of the potential involving gauge singlet fields - for example they do allow the term $(RS)^4$ where R and S are defined in Table 1.

There are however some results that clearly do not follow from the symmetries alone. For example the coupling $(\tilde{R}\tilde{S})^4$, where $\tilde{R}$ and $\tilde{S}$ are the fields R and S symmetrised in the position of the $s = 2$ factor, must vanish because $\tilde{R}$ and $\tilde{S}$ are moduli. This can be proved by superconformal techniques but is not ruled out by the symmetries. Furthermore Distler and Greene[3] have proved, using the superconformal techniques developed by Dixon[11], that the terms $\mathbf{27}^3\tilde{R}^p\tilde{C}^q$ and $\bar{\mathbf{27}}^3\tilde{R}^q\tilde{C}^p$ ($p > 0, q \geq 0$) vanish. The symmetries of eqs(15)-(19) play no role in this because using the symmetric $\tilde{R}$ and $\tilde{C}$ discussed above, we may always construct terms $\prod_{i=0}^p \tilde{R}_{i,0}$ and $\prod_{j=0}^q \tilde{C}_{j,0}$ which are simultaneously invariant under all the symmetries in both the left and right sector for suitable choice of $\tilde{R}_i$ and $\tilde{C}_i$. Thus the symmetries do not forbid a correlator of the form

$$\langle \mathbf{27}_{l,-1}\mathbf{27}_{m,-\frac{1}{2}}\mathbf{27}_{n,-\frac{1}{2}}\prod_{i=0}^p \tilde{R}_{i,0}\prod_{j=0}^q \tilde{C}_{j,0} \rangle \quad (23)$$

for combinations (l, m, n) which correspond to a nonvanishing 27^3 vertex operator.

To summarise, we have determined the discrete symmetries of the heterotic superconformal field theories. They go far beyond those found in the equivalent Calabi-Yau construction and ensure the flatness of the potential for fields with symmetric forms for the $q_i, \bar{q}_i$ assignments. For the moduli these results have been proved by more general arguments applicable to all superconformal theories but we were able to show that the discrete symmetries imply flatness of the potential for a much larger class of fields than have been derived from the superconformal structure alone.

We investigated the implications of the discrete symmetries in some detail for the specific example $Y_{4,5}$, where it was found that many couplings vanished that are allowed by the symmetries of the corresponding Calabi-Yau theory. This should have an important bearing on the phenomenological implications of such theories.

The structure which is emerging shows that many couplings may be reliably computed in the Calabi-Yau theories. The absence of the terms $27^3 \bar{R}^p \tilde{C}^q$ ($p > 0, q \geq 0$) means that the trilinear 27^3 coupling for arbitrary complex structure is not changed[3] in going from the heterotic superconformal field theory to the Calabi-Yau with arbitrary values of the moduli $\tilde{R}$. These couplings may therefore be reliably determined (up to string loop corrections) using the heterotic superconformal field theory. Similarly, the vanishing of R^n terms proved above may be seen to apply to the full Calabi-Yau theory. In the case of 27^3 couplings we do expect a difference as the moduli change, but in this case the coupling is determined by the topology of the Calabi-Yau manifold[12]. Provided world-sheet non-perturbative effects are small (as may be expected if the compactification scale is less than the Planck scale) then these couplings too may be reliably estimated in the Calabi-Yau theory.

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TABLE 1. Massless spectrum of the model 3^5 . The notation is that each mode (up to permutations of the factors) is denoted by: $\prod_i (l_i, q_i, s_i) \times \prod_j (\bar{l}_j, \bar{q}_j, \bar{s}_j)$.

Family modes (27 of E_6) ($\# = 101$):

$(0,0,0)(0,0,0)(0,0,0)(2,2,0)(3,3,0)X(0,0,0)(0,0,0)(0,0,0)(2,2,0)(3,3,0)$
 $(0,0,0)(0,0,0)(1,1,0)(1,1,0)(3,3,0)X(0,0,0)(0,0,0)(1,1,0)(1,1,0)(3,3,0)$
 $(0,0,0)(0,0,0)(1,1,0)(2,2,0)(2,2,0)X(0,0,0)(0,0,0)(1,1,0)(2,2,0)(2,2,0)$
 $(0,0,0)(1,1,0)(1,1,0)(1,1,0)(2,2,0)X(0,0,0)(1,1,0)(1,1,0)(1,1,0)(2,2,0)$
 $(1,1,0)(1,1,0)(1,1,0)(1,1,0)(1,1,0)X(1,1,0)(1,1,0)(1,1,0)(1,1,0)(1,1,0)$

Antifamily modes ($\bar{27}$ of E_6) ($\# = 1$):

$(1,1,0)(1,1,0)(1,1,0)(1,1,0)(1,1,0)X(1,-1,0)(1,-1,0)(1,-1,0)(1,-1,0)(1,-1,0)$

Singlet modes (1 of E_6) ($\# = 330$):

$(0,0,0)(0,0,0)(0,0,0)(2,2,0)(3,3,0)X(0,0,0)(0,0,0)(0,0,0)(2,2,0)(3,3,2)$
 $(0,0,0)(0,0,0)(0,0,0)(2,2,0)(3,3,0)X(0,0,0)(0,0,0)(0,0,0)(2,2,2)(3,3,0)$
 $(0,0,0)(0,0,0)(1,1,0)(1,1,0)(3,3,0)X(0,0,0)(0,0,0)(1,1,0)(1,1,0)(3,3,2)$
 $(0,0,0)(0,0,0)(1,1,0)(1,1,0)(3,3,0)X(0,0,0)(0,0,0)(1,1,0)(1,1,2)(3,3,0)$
 $(0,0,0)(0,0,0)(1,1,0)(2,2,0)(2,2,0)X(0,0,0)(0,0,0)(1,1,0)(2,2,0)(2,2,2)$
 $(0,0,0)(0,0,0)(1,1,0)(2,2,0)(2,2,0)X(0,0,0)(0,0,0)(1,1,2)(2,2,0)(2,2,0)$
 $(0,0,0)(1,1,0)(1,1,0)(1,1,0)(2,2,0)X(0,8,2)(1,-1,0)(1,-1,0)(1,-1,0)(2,0,0)$
 $(0,0,0)(1,1,0)(1,1,0)(1,1,0)(2,2,0)X(0,0,0)(1,1,0)(1,1,0)(1,1,0)(2,2,2)$
 $(0,0,0)(1,1,0)(1,1,0)(1,1,0)(2,2,0)X(0,0,0)(1,1,0)(1,1,0)(1,1,2)(2,2,0)$
 $S=(1,1,0)(1,1,0)(1,1,0)(1,1,0)(1,1,0)X(1,1,0)(1,1,0)(1,1,0)(1,1,0)(1,1,2)$
 $R=(1,1,0)(1,1,0)(1,1,0)(1,1,0)(1,1,0)X(1,-1,0)(1,-1,0)(1,-1,0)(1,-1,0)(1,-1,-2)$