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GEOMETRICAL CONSTRUCTION OF A GENERALIZED SKYRME LAGRANGIAN

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ABSTRACT

A geometrical scheme is proposed that leads to a straight-forward generalization of a Skyrme Lagrangian, comprising higher-order terms up to the eighth power in pion fields.

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1. The model first proposed by Skyrme [1] has been the subject of many investigations due to a revival of interest in the past few years [2-4]. Geometrizations of the model have also been proposed; one of the most natural schemes is that proposed by Manton [5]. It exploits the analogy with the stress tensor of a deformation of a three-dimensional manifold into another three-dimensional manifold, and introduces a natural measure of the elastic strain. The target manifold need not be a group.

In the scheme we propose below, the target space has to be a Lie group. We shall propose a Lagrangian associated to a section of a fibre bundle whose structure group is $SU(2)$. Our Lagrangian generalizes the usual Skyrme Lagrangian in the sense that it reproduces it with some extra terms which contain higher powers of the derivatives of the pion field.

2. Consider a principal fibre bundle $P(V,G)$ with V a Riemannian manifold with metric g_V , and G a Lie group with a non-degenerate Cartan-Killing form K . Subsequently we shall use the following convention concerning the indices:

$$\begin{aligned} \mu, \nu &= 0, 1, 2, 3, & \text{space-time manifold } V \\ i, j &= 1, 2, 3, & \text{three-dimensional space} \\ A, B, \dots &= 1, 2, \dots, N & N = \dim G \end{aligned} \quad (2.1)$$

If not otherwise stated, we shall assume that the bundle $P(V,G)$ is globally trivial and admits a global section. Also, the trivial connection exists, equal to the Maurer-Cartan 1-form on the fibres, i.e.,

$$\omega = \omega_A dx^A \quad (2.2)$$

with

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \quad (2.3)$$

Any vector tangent to $P(V,G)$ can be projected onto V and G , yielding a horizontal and a vertical component:

$$X \longrightarrow d\pi(X) \in TV, \quad X \longrightarrow \overset{\text{ver}}{X} \in TG \sim \mathcal{A}_G \text{ in } TP \quad (2.4)$$

Let us define the metric on $P(V,G)$:

$$g_p(X, Y) = g_v(d\pi(X), d\pi(Y)) + \kappa(\omega(X)^{\text{ver}}, \omega(Y)^{\text{ver}}). \quad (2.5)$$

Using the explicit expression at $g \in G$ for ω ,

$$\omega(g) = dL_{g^{-1}} \quad (2.6)$$

(the differential of the left translation), we can also write

$$g_p(X, Y) = g_v(d\pi(X), d\pi(Y)) + \kappa(dL_{g^{-1}} X^{\text{ver}}, dL_{g^{-1}} Y^{\text{ver}}) \quad (2.7)$$

Now we are able to compute the linear Christoffel connection of g_p and the corresponding Riemann tensor on $P(V, G)$. If the connection is trivial, then not only is the metric a direct sum of two symmetric tensors, one vanishing on the vertical and the other on the horizontal vector field, but the Riemann tensor is also a sum of two tensors:

$$R_p = R_p^{\text{hor}} + R_p^{\text{ver}}, \quad (2.8)$$

$$R_p^{\text{hor}}(X, Y, Z, W) = 0 \quad \text{if one (or more) of four vectors is vertical} \quad (2.9)$$

$$R_p^{\text{ver}}(X, Y, Z, W) = 0 \quad \text{if one of four vectors is horizontal;}$$

$$R_p^{\text{hor}}(X, Y, Z, T) = R_{v_x}(d\pi(X), d\pi(Y), d\pi(Z), d\pi(T)),$$

$$R_p^{\text{ver}}(X, Y, Z, T) = \kappa([\omega(X), \omega(Y)], [\omega(Z), \omega(T)]). \quad (2.10)$$

We recognize R_{ver}^p as the Riemann tensor of the group G .

3. Suppose now that we have a local section of our bundle:

$$U: x \in V \longrightarrow \xi \in P(V, G); \quad \xi \sim (x, u(x));$$

$$u(x) \in G \quad (3.1)$$

under some trivialization of $P(V, G)$. Any tensor defined on $P(V, G)$ can be pulled down with the tangent mapping to $U(x)$, U^* (which can be conceived as a matrix of partial derivatives of u). In particular, the metric tensor g_P and the Riemann tensor R_P can be pulled down by means of the section U : let us denote them by

$$\overset{u}{g} \quad \text{and} \quad \overset{u}{R}; \quad (3.2)$$

In local co-ordinates,

$$\overset{u}{g}_{\mu\nu} = (U^* g_P)_{\mu\nu}, \quad \overset{u}{R}_{\mu\nu\lambda\varrho} = (U^* R_P)_{\mu\nu\lambda\varrho}. \quad (3.3)$$

Under local trivialization of $P(V, G)$, one may always choose the co-ordinates so that

$$U = (\text{Id}_V, u(x)) \quad (3.4)$$

On the other hand, as both tensors g_P and R_P are direct sums of horizontal and vertical parts, we get the expression in co-ordinates:

$$U^*(g_P^{\text{hor}}) = \text{Id}_V^*(g_V),$$

$$U^*(g_P^{\text{ver}}) = u^*(\kappa) \quad (3.5)$$

which yields

$$\overset{u}{g}_{\mu\nu} = g_{\mu\nu} + (u^*(\kappa))_{\mu\nu} \quad (3.6)$$

and

$${}^u R_{\mu\nu\lambda g} = (Id_V^* R_V)_{\mu\nu\lambda g} + u^*(\kappa([\omega, \omega], [\omega, \omega]))_{\mu\nu\lambda g}$$

More precisely, if TU is the tangent mapping of U , then

$$(U^*(g_P))_{\mu\nu} = g_P(TU(e_\mu), TU(e_\nu)) \quad (3.7)$$

with e_μ a basis in V , so that

$$g_P(TU(e_\mu), TU(e_\nu)) = g_V(e_\mu, e_\nu) + \kappa_{u(x)}(Tu(e_\mu), Tu(e_\nu))$$

in a holonomous basis

$${}^u g_{\mu\nu} = g_{\mu\nu}^V + \kappa\left(dL_{u^{-1}}Tu\left(\frac{\partial}{\partial x^\mu}\right), dL_{u^{-1}}Tu\left(\frac{\partial}{\partial x^\nu}\right)\right) \quad (3.8)$$

Let us define ${}^u g^{\mu\nu}$ as

$${}^u g^{\mu\nu} = g_{\lambda g} {}^u g^{\lambda\mu} g^{\nu} \quad (3.9)$$

Denoting by $T_\mu \in \mathcal{X}_G$ the combination

$$T_\mu = dL_{u^{-1}}Tu(e_\mu) \quad (3.10)$$

we obtain for the two tensors, g_P and R_P , when pulled down to V with the section u :

$${}^u g_{\mu\nu} = \dot{g}_{\mu\nu} + \kappa(T_\mu, T_\nu),$$

$$\overset{u}{R}_{\mu\nu\lambda\varrho} = \overset{o}{R}_{\mu\nu\lambda\varrho} + \kappa([T_\mu, T_\nu], [T_\lambda, T_\varrho]) \quad (3.11)$$

$\overset{o}{g}_{\mu\nu}, \overset{o}{R}_{\mu\nu\lambda\varrho}$ being the metric and curvature of the basis space V.

4. Let us now form a general Lagrangian out of three invariants computed on V:

$$\mathcal{L} = \sqrt{|g|} \left(\lambda + \mu \overset{u}{g}^{\mu\nu} \overset{u}{g}_{\mu\nu} + e \overset{u}{g}^{\mu\nu} \overset{u}{g}^{\lambda\varrho} \overset{u}{R}_{\mu\lambda\nu\varrho} \right) \quad (4.1)$$

λ, μ, e being some constants. We obtain then

$$\begin{aligned} \overset{u}{g}^{\mu\nu} \overset{u}{g}_{\mu\nu} &= \overset{u}{g}_{\mu\nu} \overset{u}{g}^{\lambda\varrho} \overset{o}{g}^{\mu\lambda} \overset{o}{g}^{\nu\varrho} = \\ &= \overset{o}{g}_{\mu\nu} \overset{o}{g}^{\mu\nu} + 2 \overset{o}{g}^{\mu\nu} \kappa(T_\mu, T_\nu) + \overset{o}{g}^{\mu\nu} \overset{o}{g}^{\lambda\varrho} \kappa(T_\mu, T_\lambda) \kappa(T_\nu, T_\varrho) \end{aligned} \quad (4.2)$$

and the second term becomes

$$\begin{aligned} \overset{u}{g}^{\mu\nu} \overset{u}{g}^{\lambda\varrho} \overset{u}{R}_{\mu\lambda\nu\varrho} &= \\ &= \overset{o}{R} + 2 \overset{o}{R}^{\mu\nu} \kappa(T_\mu, T_\nu) + \overset{o}{R}^{\mu\nu\lambda\varrho} \kappa(T_\mu, T_\lambda) \kappa(T_\nu, T_\varrho) - \\ &- \frac{1}{4} \overset{o}{g}^{\mu\nu} \overset{o}{g}^{\lambda\varrho} \kappa([T_\mu, T_\lambda], [T_\nu, T_\varrho]) - \\ &- \frac{1}{2} \overset{o}{g}^{\mu\nu} \overset{o}{g}^{\lambda\varrho} \overset{o}{g}^{\alpha\beta} \kappa(T_\varrho, T_\beta) \kappa([T_\mu, T_\lambda], [T_\nu, T_\varrho]) - \\ &- \frac{1}{4} \overset{o}{g}^{\mu\nu} \overset{o}{g}^{\lambda\varrho} \overset{o}{g}^{\alpha\beta} \overset{o}{g}^{\gamma\delta} \kappa(T_\nu, T_\varrho) \kappa(T_\alpha, T_\gamma) \kappa([T_\mu, T_\beta], [T_\lambda, T_\delta]) \end{aligned} \quad (4.3)$$

In three dimensions, for the stationary solutions, $T_0 \equiv 0$; then, if the group is $SU(2)$, we can put $C_{AB}^A = \epsilon_{ABC}$, and use the following identities:

$$\begin{aligned} \kappa(X, X)\kappa(Y, Y) - (\kappa(X, Y))^2 &= 2\kappa([X, Y], [X, Y]), \\ \kappa(X, Y)\kappa(Z, T) - \kappa(X, T)\kappa(Y, Z) &= 2\kappa([X, Z], [Y, T]) \end{aligned} \quad (4.4)$$

Then the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \lambda + 4\mu + 2\mu\kappa(T_i, T^i) + \mu\kappa(T_i, T^i)\kappa(T_j, T^j) - \\ &- \left(2\mu + \frac{e}{4}\right)\kappa(T_i, T_j)\kappa(T^i, T^j) - \\ &- \frac{e}{4}\kappa(T_i, T_\ell)\kappa([T^i, T^j], [T^\ell, T_j]) - \\ &- \frac{e}{4}\kappa([T_i, T_j], [T_\ell, T_m])\kappa(T^i, T^\ell)\kappa(T^j, T^m) \end{aligned} \quad (4.5)$$

We can normalize our constants in order to make the first two terms coincide with the Skyrme Lagrangian; by choosing $\mu = -1/4$, $e = 9/4$, $\lambda = 1$,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\kappa(T_i, T^i) - \frac{1}{16}\kappa([T_i, T_j], [T^i, T^j]) - \\ &- \frac{1}{4}\kappa(T_i, T^i)\kappa(T_j, T^j) - \\ &- \frac{9}{16}\kappa(T_i, T_\ell)\kappa([T^i, T^j], [T^\ell, T_j]) - \\ &- \frac{9}{16}\kappa([T_i, T_j], [T_\ell, T_m])\kappa(T^i, T^\ell)\kappa(T^j, T^m) \end{aligned} \quad (4.6)$$

In local co-ordinates, given a section $u: R^3 \rightarrow SU(2)$, T_i has to be replaced by $u^+(\partial u / \partial x^i)$.

5. The above expression has been calculated for the case when the space is R^3 , i.e., flat. If one assumes that our space is of constant curvature, i.e., S^3 in the three-dimensional case, then its Riemann tensor is equal to

$$\begin{aligned}\overset{\circ}{R}_{ij\ell k} &= k (g_{ik} g_{j\ell} - g_{i\ell} g_{jk}), \\ \overset{\circ}{R}_{ij} &= k (n-1) g_{ij} = 2k g_{ij}, \\ \overset{\circ}{R} &= k (n-1) n = 6k\end{aligned}\tag{5.1}$$

Then the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= (\lambda + 4\mu + 6k) + 2(\mu + 2k) \kappa(T_i, T^i) + \\ &+ \mu \kappa(T_i, T^i) \kappa(T_j, T^j) + (2k - 2\mu - \frac{e}{4}) \kappa([T_i, T_j], [T^i, T^j]) \\ &- \frac{e}{4} \kappa(T_i, T_\ell) \kappa([T_j, T^i], [T^j, T^\ell]) - \\ &- \frac{e}{4} \kappa([T_i, T_j], [T_\ell, T_m]) \kappa(T^i, T^\ell) \kappa(T^j, T^m)\end{aligned}\tag{5.2}$$

Again, in order to normalize the constants so that the first terms should reproduce the Skyrme model, we must put

$$e = \frac{9}{4} + 24k, \quad \mu = -\frac{1}{4} - 2k,\tag{5.3a}$$

and

$$\lambda = 1 + 2k \quad (5.3b)$$

It is also useful to express everything in terms of fundamental invariants of the 3×3 matrix

$$\{D\}_\ell^k = U^\dagger \partial_\ell U U^\dagger \partial^k U, \quad (5.4)$$

which are

$$\begin{aligned} \text{Tr } D &= I_1, \\ \frac{1}{2} \left[(\text{Tr } D)^2 - \text{Tr}(DD) \right] &= I_2, \\ \text{and } \det D &= I_3. \end{aligned} \quad (5.5)$$

In terms of these, we have

$$\begin{aligned} \mathcal{L} &= (2\mu + 4k) I_1 + \left(2\mu + \frac{e}{4} - 2k \right) I_2 - \mu I_1^2 + \\ &+ \frac{3e}{8} I_3 - \frac{e}{8} I_1 I_2 + \frac{e}{4} I_2^2 - \frac{e}{2} I_1 I_3 \end{aligned} \quad (5.6)$$

It is clear that k , the curvature of S^3 , is positive.

With the spherically-symmetric Skyrme ansatz

$$U = e^{i \vec{\sigma} \cdot \vec{n} F(r)} = I \cos F + i \vec{\sigma} \cdot \vec{n} \sin F, \quad (5.7)$$

the fundamental invariants are

$$\begin{aligned}
 I_1 &= \text{Tr } D = F'^2 + \frac{2 \sin^2 F}{r^2}, \\
 I_2 &= \frac{1}{2}[(\text{Tr } D)^2 - \text{Tr}(DD)] = 2F'^2 \frac{\sin^2 F}{r^2} + \frac{\sin^4 F}{r^4}, \\
 I_3 &= \det D = F'^2 \frac{\sin^4 F}{r^4}
 \end{aligned}
 \tag{5.8}$$

and the Lagrangian [with the constants expressed by means of k only, due to (5.3)] is:

$$\begin{aligned}
 \mathcal{L} &= - \left(F'^2 + \frac{2 \sin^2 F}{r^2} \right) - \left(2F'^2 \frac{\sin^2 F}{r^2} + \frac{\sin^4 F}{r^4} \right) - \\
 &- (1+8k) \left(F'^4 + 4F'^2 \frac{\sin^2 F}{r^2} + \frac{4 \sin^4 F}{r^4} \right) - \\
 &- 6(3+32k) \left(2F'^4 \frac{\sin^2 F}{r^2} + 5F'^2 \frac{\sin^4 F}{r^2} + 2 \frac{\sin^6 F}{r^6} \right) - \\
 &- 24(3+32k) \left(4F'^4 \frac{\sin^4 F}{r^4} + 4F'^2 \frac{\sin^6 F}{r^6} + \frac{\sin^8 F}{r^8} \right) + \\
 &+ 18(3+32k) F'^2 \frac{\sin^4 F}{r^4} + \\
 &+ 48(3+32k) \left(F'^4 \frac{\sin^4 F}{r^4} + 2F'^2 \frac{\sin^6 F}{r^6} \right).
 \end{aligned}
 \tag{5.9}$$

It is straightforward to verify that the identity mapping given by the substitution

$$r = \sin \eta, \quad F = \eta \quad (5.10)$$

solves the corresponding Euler-Lagrange equations. It has to be noted that not any function of the invariants I_1, I_2, I_3 has this property. It would be interesting to know if the Lagrangian density (5.6) is the unique combination (up to the power eight in the fields) that displays this property.

Although we do not yet have the numerical solutions of (5.9), some other examples of this type have been treated numerically in Ref. [4]. The stability of generalized skyrmions is assured if the signs of the higher-order terms are not arbitrary. In our case, this stability will be assured if $k > -(3/32)$. This means that even the negative-curvature target spaces can be considered as well, provided that their curvature is not too high.

As a final remark, let us note that we do not yet have a clear dynamical principle that would dictate the choice of these simplest invariants. Most probably, a more general field theory is needed in order to justify these terms in the Lagrangian as a part of some low-energy limit development.

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