SIMPLE DERIVATION OF THE ENERGY-DEPENDENT EFFECTIVE STRENGTH
OF GRAVITATIONAL INTERACTION IN SMALL-ANGLE SCATTERING

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ABSTRACT

A simple derivation is given of the progression ($1 \rightarrow 2 \rightarrow 4$) of the effective strengths of the gravitational interaction in small angle scattering, from the Newtonian product of "charges", $Gm_1 m_2$, to the Einsteinian strength, $2GE_1 m_2/c^2$, for light scattered by a fixed mass, to the strength, $4GE_1 E_2/c^4$, for two highly relativistic particles. It is noted that the "complicated" amplitude of 't Hooft [1] is just the Glauber amplitude for a $1/r$ potential, equal (more or less by accident) to the exact "Coulomb" amplitude of the Schroedinger equation, with $\eta_C = z_1 z_2 e^2/\hbar v$ replaced by an appropriate gravitational parameter. The differential scattering cross section for small-angle gravitational scattering at any energy is $d\sigma/dt = 4\pi\eta^2/t^2$, where η is the energy-dependent strength, given in invariant form by Eq.(5).

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The small-angle gravitational scattering of spinless point masses or strings at ultrahigh energies has been discussed recently [1,2]. At small angles, corresponding to large impact parameters (where the interaction is weak and the size of the strings can be neglected), the scattering cross section becomes the familiar Rutherford form, but with an equivalent interaction 4 times as large as the relativistic Newtonian result (meaning Newton's law of gravitation, but with $m_1 m_2$ replaced by $E_1 E_2 / c^4$). The quantum-mechanical amplitude for the small-angle scattering of mass points [1] is just the Schrodinger Coulomb amplitude [3] with an appropriate modification of the parameter $\eta_c = z_1 z_2 e^2 / \hbar v$.

It does not seem to be generally known that these results follow straightforwardly from elementary considerations of the small-angle scattering of a test particle by a force center, together with simple Lorentz invariance arguments. The purpose of this note is to demonstrate in a transparent manner the smooth progression from the Newtonian scattering of a nonrelativistic particle (effective coupling, $Gm_1 m_2$), to the scattering of light or a relativistic particle (effective interaction, $2GE_1 m_2 / c^2$), to the gravitational scattering of two ultrarelativistic masses (effective coupling, $4GE_1 E_2 / c^4$). The use of Glauber's approximation [4] for the quantum -mechanical scattering of the test particle is shown to lead (because of a peculiarity of the inverse square law of force) to the exact "Coulomb" scattering amplitude, obtained in a different way by 't Hooft [1]^{#1}. One thus finds a generally applicable expression, Eq.(7), for the small-angle gravitational scattering cross section, valid at all energies, from Newtonian to Planckian.

The starting point of the discussion is the general relativistic motion of a test particle of mass m_1 in the presence of a fixed mass. It is convenient to use "isotropic" coordinates [5] or the conformal-Euclidean metric [6], rather than the Schwarzschild metric. Then the Lagrangian for the test particle, correct to second order in the gravitational potential $\phi = Gm_2/c^2 r$ and to all orders in $\beta = v/c$, is

$$L = mc^2 \left[1 - \frac{1}{\gamma} + \gamma\phi(1+\beta^2) - \gamma\phi^2 \left(1 - \frac{3}{4}\beta^2 \right) + \frac{1}{2}\gamma^3\phi^2(1+\beta^2)^2 \right] \quad (1)$$

where $\gamma = (1-\beta^2)^{-1/2}$. For our purposes, only the term linear in ϕ is needed; to discuss the

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^{#1} In [1] and [2] the authors remark a posteriori that their results are equivalent to eikonal approximations, but their derivations are rather different.

advance of the perihelion the quadratic terms are necessary. To first order in the potential, the equation of motion is^{#2}

$$\frac{d\mathbf{p}}{dt} = Gm_1m_2 \gamma(1+\beta^2) \nabla \left(\frac{1}{r} \right) \quad (2)$$

where the exact form of $\mathbf{p}$ from Eq.(1) need not concern us.

For a particle with an initial speed $v_0 = \beta_0 c$, projected at a large impact parameter b , the scattering (through an angle assumed small) can be computed approximately by treating its trajectory as a straight line at constant speed and evaluating the transverse impulse of the force. The elementary result is

$$\Delta p = \frac{2Gm_1m_2}{b v_0} \cdot \gamma_0(1+\beta_0^2) \quad (3)$$

Eq.(3) contains everything needed for our demonstration. Visible first of all, quite transparently, are the limiting Newtonian and Einsteinian results for the deflection of a nonrelativistic particle and of light (or relativistic particle), with a smooth transition in between. To obtain the result for the small angle scattering of ultrarelativistic particles, we consider a special-relativistic boost to a frame moving with an arbitrary speed along the direction of the incident particle m_1 . Since both Δp and b are components of four-vectors perpendicular to this direction, they are Lorentz-invariant. To make Eq.(3) manifestly invariant, we define the dimensionless strength parameter,

$$\eta = \frac{Gm_1m_2}{\hbar c} \cdot \gamma_0 \left(\frac{1+\beta_0^2}{\beta_0} \right) \quad (4)$$

and write it in terms of Lorentz invariants:

$$\eta = \frac{G}{\hbar c^5} \left[\frac{2(p_1 p_2)^2 - m_1^2 m_2^2}{\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \right] \quad (5)$$

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^{#2} With the Schwarzschild metric, the factor $(1+\beta^2)$ multiplying γ in Eq.(1) is replaced by $[1 + (r\dot{\phi}/r)^2]$. This complicates the expression for the force, but the result [Eq.(3)] is the same.

where the four-vectors and masses have dimensions of energy. If both masses can be neglected, the square bracket in Eq.(5) is equal to Mandelstam's s . In passing it is worth noting the alternative form,

$$\eta = \frac{GE_1E_2}{\hbar c^5} \left[\frac{(1+\beta_1^2)(1+\beta_2^2) + 4\beta_1\beta_2}{\beta_1 + \beta_2} \right] \quad (6)$$

where β_1 and β_2 are positive for $\mathbf{p}_2$ antiparallel to $\mathbf{p}_1$. The square bracket in Eq.(6) takes on the Newtonian value of $1/\beta_1$ when $\beta_2 = 0$ and $\beta_1 \ll 1$, the Einsteinian value of 2 when $\beta_2 = 0$ and $\beta_1 = 1$, and the recently discovered value of 4 when $\beta_1 = \beta_2 = 1$.

To complete the classical discussion we note that Eq.(3) in its manifestly invariant form is

$$\Delta p = \frac{2\eta\hbar}{b} \quad , \quad (3')$$

that $(\Delta p)^2 = -t$ at small angles, and that $d\sigma = \pi d(b^2)$. The invariant small-angle scattering cross section can thus be written generally as

$$\frac{d\sigma}{dt} = \frac{4\pi\eta^2}{t^2} \quad (7)$$

where η is given by Eq.(5) and we have finally set $\hbar = c = 1$. With the criterion of validity that the kinetic energy be large compared to the effective strength of the interaction at $r = b$, we find that Eq.(7) should be valid for all situations, provided $\theta \ll 1$.

The quantum mechanical problem of the small-angle scattering of the test particle proceeds straightforwardly with the Hamiltonian derived from Eq.(1) to first order in the potential:

$$H = \sqrt{m^2 + p^2} \cdot \left[1 - \left(1 + \frac{p^2}{m^2 + p^2} \right) \phi \right] \quad (8)$$

For the Glauber (semiclassical or eikonal) high-energy/weak potential approximation to the relativistic scattering, the relevant spatially varying wave number is

$$p(r) \approx p_0 + \left[\frac{E_0(1+v_0^2)}{v_0} \right] \phi(z, \mathbf{b}) = p_0 + \frac{\eta}{\sqrt{z^2 + b^2}} \quad (9)$$

where p_0 is the incident wave number, $v_0 = p_0/E_0$, and η is given by (4) or (5). The Glauber WKB wave function is

$$\psi(z,b) = \exp\left(ip_0 z + i\eta \int_{-\infty}^z \frac{dz}{\sqrt{z^2 + b^2}} \right) \quad (10)$$

from which the phase shift at fixed impact parameter can be inferred. It is well known [7] that one obtains in this way the form of the exact Coulomb amplitude of the Schroedinger equation [3] (with η replacing $-\eta_C$ because of our attractive interaction),

$$f_c = \frac{\Gamma(1-i\eta)}{\Gamma(-i\eta)} \cdot \left(\frac{2p_0}{-t} \right)^{1-i\eta}, \quad (11)$$

times an angular independent phase factor whose exact form depends on how one cuts off the Coulomb potential. This Glauber result is precisely 't Hooft's Eq.(14) when allowance is made for different normalisations and η is approximated by G_s . It is a peculiarity of the $1/r$ radial dependence that the Glauber approximation, with its first order expansion (9), yields the exact (spinless) Coulomb amplitude, which represents multiple photon (graviton) exchange^{#3}.

Since the magnitude of (11) is $|f_c| = |2\eta p_0/t|$, the quantum-mechanical gravitational scattering cross section is still given by Eq.(7).

Though the analogy between small-angle Coulomb scattering and gravitational scattering is understandably close, there are distinct differences. The parameters η_C and η , entering the formulas in the same way, behave very differently with energy. Both have a $1/v$ dependence in the nonrelativistic limit, but η_C decreases monotonically with energy to the constant value $z_1 z_2 / 137$ at relativistic speeds, while η grows without bound as G_s . The absolute minimum of η occurs when $s = m_1^2 + m_2^2 + \sqrt{6} m_1 m_2$ (or $\beta_0 = 1/\sqrt{3}$ in Eq.(4)) and is equal to

$$\eta_{\min} = 2\sqrt{2} \frac{G m_1 m_2}{\hbar c} \quad (12)$$

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^{#3} The amplitude (11) is exact for electromagnetic scattering of point particles at all angles nonrelativistically. Its absolute square, Eq.(7), has been demonstrated to be valid relativistically at small enough angles for spins 0, 1/2, and 1, and is presumably valid within a restricted angular range for arbitrary spins and systems of non-zero extent.

At ultrahigh energies η can become large compared to unity. Then, as stressed by 't Hooft [1], the small-angle scattering of particles is dominated, because of its long range, by the calculable gravitational amplitude.

We end with one or two comments. The path of our discussion was via the interaction of a test particle of mass m_1 with the gravitational field of a point mass m_2 . Implicit in Eqs.(1) and (8) is the idea that $m_2 \gg m_1$. Yet the results are symmetric in m_1 and m_2 . The reason lies in the restriction to small-angle scattering. If we begin the treatment in the rest frame of one mass and the impulse is small enough, we end up after the collision with that same mass (approximately) at rest. For close collisions, higher order than the first in the potential will be necessary; the gravitational metric tensor must reflect the presence of both masses; etc. The large-angle scattering will depart from Rutherford's law; other interactions may contribute; inelastic effects are presumably important.

It is hoped that for the small-angle scattering the above discussion demystifies and unifies the ultrarelativistic collisions with the more familiar Newtonian scattering and the deflection of light. They are each a manifestation of the same basic scattering law with its energy-dependent effective strength parameter η .

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