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COVARIANT BRST QUANTIZATION OF THE FOUR-DIMENSIONAL SUPERPARTICLE

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ABSTRACT

We construct a generalized covariant BRST operator defining the quantum states of the $D = 4$ Brink-Schwarz superparticle. It is related to a theory with a local non-Abelian BRST-like invariance. The results are extended to the case of the spinning superparticle.

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1. Recent developments in mechanics and field theory [1-8] have led to the recognition that the spectrum of a physical model can be found in one of three different, but usually equivalent ways:

- i) by computing the path-integral, starting from an action principle;
- ii) by solving the Schrödinger equation and diagonalizing the Hamiltonian;
- iii) by solving for the cohomology classes of some self-adjoint nilpotent operator, as in the BRST approach [9-11].

Each of these methods has its counterpart in classical mechanics [12,13] but here we focus on quantum theories. In spite of its widely recognized usefulness as a tool in quantum theory, the cohomology approach is most often applied in the context of models that were first described in one of the two earlier formalisms (Lagrangian or Hamiltonian). In this paper we present an example of a relativistic system for which a covariant and nilpotent operator can be constructed and its cohomology solved explicitly, defining in this way a complete space of quantum states, without having a covariant quantum action or Hamiltonian to start from. The system is the superparticle in 4(mod 8) space-time dimensions, for which a classical Lagrangian has been proposed by Brink and Schwarz [14], but for which a covariant quantum action is lacking so far. Therefore this system offers an instance of a theory for which the BRST approach seems to provide a more tractable and convenient procedure than one of the older formalisms.

Our aim is to construct a nilpotent operator Ω , the cohomology classes of which are the equivalence classes of solutions of the equations

$$D_- \phi = 0, \quad \not{P} D_+ \phi = 0, \quad (1)$$

or their complex conjugates. Here $D_{\pm}$ denote the chiral supercovariant derivatives

$$D_{\pm} = i \left(\frac{\partial}{\partial \bar{\theta}_{\pm}} + \not{P} \theta_{\mp} \right), \quad (2)$$

satisfying the algebra

$$\{D_{\pm}, D_{\mp}\} = (1 \pm \gamma_5) \not{P}, \quad \{D_{\pm}, D_{\pm}\} = 0, \quad (3)$$

whilst P_{μ} is the particle's four-momentum; θ is a Majorana spinor representing the fermionic superspace co-ordinate^{*)}.

*) The chiral components of a spinor are defined by $\theta_{\pm} = \frac{1}{2}(1 \pm \gamma_5)\theta$; obviously, they are eigenstates of γ_5 with eigenvalue ± 1 . Similarly, $\bar{\theta}_{\pm} = \theta_{\pm}^{\dagger} \gamma_4$. For Majorana spinors θ , charge conjugation gives $\theta_{\pm} = C \bar{\theta}_{\pm}^T$.

It is easy to see that Eqs. (1) define an irreducible (minimal) representation of massless Poincaré supersymmetry, with one helicity $\lambda = 0$ state, and one helicity $\lambda = \frac{1}{2}$ state. The first equation merely states that we are dealing with a chiral superfield $\Phi(x, \theta)$ having non-trivial dependence on θ_+ only. The second equation describes its dynamical content. In components one finds

$$\begin{aligned} \Phi(x, \theta) &= e^{-\bar{\theta}_+ \not{\partial} \theta_+} (A + \bar{\theta}_+ \psi_+ + \frac{1}{2} \bar{\theta}_+ \theta_+ F), \\ \square A &= 0, \quad \not{\partial} \psi_+ = 0, \quad \partial_\mu F = 0. \end{aligned} \quad (4)$$

Being wave functions, the components (A, ψ_+, F) are complex. However, they describe only one massless scalar ($\lambda = 0$), one chiral fermion ($\lambda = \frac{1}{2}$) and a non-dynamical (auxiliary) quantity F which is included only to complete the linear representation of $N = 1$ supersymmetry in Hilbert space. This counting of states is to be contrasted with counting in quantum field theory, in which a complex scalar describes two spin-0 particles, and a chiral fermion field represents both helicities $\lambda = \pm \frac{1}{2}$.

2. We now present a nilpotent operator Ω with the required cohomology. Its explicit form is

$$\Omega = \bar{s} Q + \bar{Q} s + \gamma Z - 2\bar{s}s\beta. \quad (5)$$

Here Q , $\bar{Q}$ and Z denote the following operator expressions:

$$\begin{aligned} Q &= \bar{\omega}_- D_- + \bar{D}_+ \not{\partial} \phi_- + c \not{P}_\mu^2 - 2\bar{\omega}_- \phi_- b, \\ \bar{Q} &= \bar{D}_+ \omega_+ + \bar{\phi}_+ \not{\partial} D_- + c \not{P}_\mu^2 - 2\bar{\phi}_+ \omega_+ b, \\ Z &= (\bar{\omega}_- - \bar{\phi}_+ \not{\partial}) \not{\partial} (\omega_+ - \not{\partial} \phi_-). \end{aligned} \quad (6)$$

$D_\pm$, $\bar{D}_\pm$ and P_μ have been defined before, $(\omega_\pm, \phi_\pm)$ are two sets of (complex) commuting chiral spinor ghosts, $(s, \bar{s})$ a pair of complex commuting scalars whilst (b, c) and (β, γ) denote two pairs of canonically conjugate anticommuting scalar ghosts:

$$\{b, c\} = 1, \quad \{\beta, \gamma\} = 1. \quad (7)$$

The canonically conjugate momenta of the commuting ghost variables do not appear in Ω . Note that Ω is real and a Lorentz scalar.

The scalar operators Q , $\bar{Q}$ and Z define a simple graded Lie algebra with Q and $\bar{Q}$ nilpotent and Z a central charge:

$$\begin{aligned} Q^2 &= 0, & \bar{Q}^2 &= 0, \\ \{Q, \bar{Q}\} &= 2Z, & [Q, Z] &= [\bar{Q}, Z] = 0. \end{aligned} \quad (8)$$

To check this, one only needs the algebra of the supercovariant derivatives (3) and the anticommutation relations (7). It is easily recognized that Ω is precisely the BRST charge corresponding to the superalgebra (8) with $(s, \bar{s})$ playing the role of ghost variables for the generators $(Q, \bar{Q})$ and (γ, β) the ghost/antighost for Z . Thus Ω is nilpotent by construction:

$$\Omega^2 = 0. \quad (9)$$

We are now in a position to prove the following statement:

the cohomology classes of Ω in the space of states satisfying

$$s|\psi\rangle = 0, \quad \gamma|\psi\rangle = 0, \quad (10)$$

are the Ω -equivalence classes of solutions of

$$(\bar{\omega}_- D_- + \bar{\phi}_- \not{D}_+) |\psi\rangle = 0. \quad (11)$$

This result is obtained as follows. By Eq. (10)

$$\Omega|\psi\rangle = 0 \iff Q|\psi\rangle = 0. \quad (12)$$

Hence we only have to study the solutions of the simpler second equation. Decompose states with respect to the anticommuting variable c :

$$\begin{aligned} |\psi\rangle &= |\varphi_0\rangle + c|\varphi_1\rangle, \\ b|\varphi_{0,1}\rangle &= 0. \end{aligned} \quad (13)$$

The state $|\psi\rangle$ is annihilated by Q if and only if

$$\begin{aligned} (\bar{\omega}_- D_- + \bar{\phi}_- \not{D}_+) |\varphi_0\rangle &= 2\bar{\omega}_- \phi_- |\varphi_1\rangle, \\ (\bar{\omega}_- D_- + \bar{\phi}_- \not{D}_+) |\varphi_1\rangle &= p_\mu^2 |\varphi_0\rangle. \end{aligned} \quad (14)$$

The second of these equations follows from the first, as may be seen by solving the first equation:

$$|\varphi_1\rangle = \frac{1}{2\bar{\omega}_-\phi_-} (\bar{\omega}_- D_- + \bar{\phi}_- \not{D}_+) |\varphi_0\rangle, \quad (15)$$

substituting this in the second equation and using the anticommutation properties of the supercovariant derivatives Eq. (3). However, if we require the states to be normalizable or at least regular in their dependence on (ω_-, ϕ_-) , the solution (15) does not always exist. In particular, for states which have vanishing ghost number, $|\varphi_0\rangle$ is independent of (ω_-, ϕ_-) and hence the solution (15) acquires singular terms:

$$|\varphi_1\rangle \sim \frac{1}{\bar{\omega}_-\phi_-}. \quad (16)$$

Now note that any solution of the form (15) is BRST exact. Indeed it can be written as

$$|\psi\rangle = \Omega \left(\frac{-c}{2\bar{\omega}_-\phi_-} |\varphi_0\rangle \right) = Q \left(\frac{-c}{2\bar{\omega}_-\phi_-} |\varphi_0\rangle \right). \quad (17)$$

Thus we conclude that in order to obtain non-trivial solutions of the BRST condition (12), the ghost dependence of the states must be restricted in such a way that the solution (15) does not exist. Equivalently, if it exists $|\varphi_1\rangle$ may be gauged away by adding a BRST exact state to $|\psi\rangle$. It follows that the only non-trivial solutions are those for which

$$(\bar{\omega}_- D_- + \bar{\phi}_- \not{D}_+) |\varphi_0\rangle = 0. \quad (18)$$

This proves Eq. (11). Now, since (ω_-, ϕ_-) are independent, Eq. (11) implies the conditions (1). We conclude that the cohomology classes of Ω in the space of states specified by Eqs. (10) and (12) form the irreducible multiplet of $N = 1$ supersymmetry expected as solutions to the first-quantized Brink-Schwartz superparticle model.

3. Some remarks on completeness, interpretation and possible extensions of the above construction are in order.

- 1) It is not possible to construct a second-quantized field theory based only on the nilpotent operator Q . In contrast to Ω , the operator Q is complex and does not anticommute with $\bar{Q}$. Hence states which are Q -exact do not have vanishing norm and are not spurious. Since Ω is real, this problem does not arise for Ω -exact states, which do have zero norm.

ii) Instead of choosing the conditions (10), one can also impose

$$\bar{s}|\psi\rangle = 0, \quad \gamma|\psi\rangle = 0. \quad (19)$$

This reduces the cohomology classes of Ω to those of $\bar{Q}$, describing a supermultiplet with helicities opposite to those of the states (4). Again, the Ω -exact states subject to Eqs. (19) are spurious and decouple from physical observables. It should be pointed out that by not submitting the states to conditions (10) or (19), we get both sets of states as solutions to the Ω -cohomology problem [16]. These solutions then no longer describe an irreducible $N = 1$ supermultiplet, but have the virtue of including with every state its CPT-conjugate.

iii) We remind the reader that the constraints (1), describing the $d = 4$ Brink-Schwarz superparticle, are reducible due to the identity

$$\not{D}(\not{D}D_+) - D_+ P_\mu^2 = 0. \quad (20)$$

As in the case of the $d = 10$ superparticle [17,18], the $d = 4$ superparticle is an infinitely reducible theory. Therefore we expect a complete expression for $(Q, \bar{Q})$ to include an infinite sequence of terms involving ghosts for ghosts [7,8,10,19]. However, this should not change the cohomology properties of $(Q, \bar{Q})$ and Ω since, as shown above, the cohomology classes of our "short" Ω already describe the correct sets of states.

iv) It is clear from Eq. (5) that our operator Ω is obtained by promoting the symmetries generated by $(Q, \bar{Q}, Z)$, Eq. (8), to local gauge symmetries, with Ω the BRST-operator of this local non-Abelian BRST-like invariance. However, in this context it is preferable to use modified generators $(\tilde{Q}, \tilde{\bar{Q}}, Z)$, invariant under the second level of BRST transformations generated by Ω . This is achieved by defining

$$\begin{aligned} \tilde{Q} &= -i[\Omega, \bar{p}] = Q - 2s\beta, \\ \tilde{\bar{Q}} &= -i[\Omega, p] = \bar{Q} - 2\bar{s}\beta, \\ \tilde{Z} &= \{\Omega, \beta\} = Z, \end{aligned} \quad (21)$$

where $(p, \bar{p})$ are the momenta conjugate to $(s, \bar{s})$. The origin of such a local BRST invariance will be explained elsewhere [20].

v) The construction of Ω above can easily be extended to the case of the spinning superparticle [21]. This model describes a superparticle with an additional set of anticommuting co-ordinates Γ^μ which are $d = 4$ vectors, rather than spinors. They form a Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu}. \quad (22)$$

The complex nilpotent BRST charges at the first level read

$$\begin{aligned} Q &= \bar{\omega}_- D_- + \bar{D}_+ \not{x} \phi_- + c P_\mu^2 - 2b(\bar{\omega}_- \phi_- + \eta^2) \\ &\quad + 2\eta \Gamma \cdot P, \\ \bar{Q} &= \bar{D}_+ \omega_+ + \bar{\phi}_+ \not{x} D_- + c P_\mu^2 - 2b(\bar{\phi}_+ \omega_+ + \eta^2) \\ &\quad + 2\eta \Gamma \cdot P, \end{aligned} \quad (23)$$

whilst the central charge in their algebra is

$$\begin{aligned} \{Q, \bar{Q}\} &= 2\tilde{Z}, \\ \tilde{Z} &= (\bar{\omega}_- - \bar{\phi}_+ \not{x})(\omega_+ - \not{x} \phi_-) + 2\eta^2 P_\mu^2. \end{aligned} \quad (24)$$

Here η is a new bosonic ghost introduced to account for $d = 1$ local supersymmetry. The constraints on the physical states annihilated by Q are

$$\begin{aligned} D_- \phi_\alpha &= 0, & \not{x} D_+ \phi_\alpha &= 0, \\ \not{x} \phi_\alpha \phi_\beta &= 0, & P_\mu^2 \phi_\alpha &= 0, \end{aligned} \quad (25)$$

where $\Phi_\alpha(x^\mu, \theta)$ is a spinor superfield. The solution of these constraints corresponds to a massless $N = 2$ vector multiplet [21], with helicity content $(0, \frac{1}{2}, \frac{1}{2}, 1)$. Again Q is complex and states $Q|\chi\rangle$ are not spurious because of the algebra (24). As before, we construct a second level BRST operator $\mathcal{Q}$ as in Eq. (5), but replacing $(Q, \bar{Q})$ by the expressions (23) and Z by $\tilde{Z}$, Eq. (24). The rest of the construction follows exactly the same steps as in the case of the ordinary superparticle. Clearly, the method described here is quite general. It will be discussed in more detail elsewhere [16].

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