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INDUCED TWO-DIMENSIONAL QUANTUM GRAVITY
AND $SL(2, \mathbb{R})$ KAC-MOODY CURRENT ALGEBRA

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ABSTRACT

We start from the covariant action for the induced two-dimensional gravity and derive the equation of motion. We show that the underlying symmetry for the equation of motion is an $SL(2, \mathbb{R})$ Kac-Moody current algebra, and give the explicit connection with the Wess-Zumino-Witten model. We also construct the energy-momentum tensor and work out the operator product expansion.

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1. - INTRODUCTION

Recently, Polyakov [1] and Knizhnik, Polyakov and Zamolodchikov [2] discussed the quantization of the induced gravity action in two dimensions. They thereby discovered an unexpected connection with $SL(2, R)$ current algebra. The action considered arises in any conformal field theory model coupled to gravity upon integrating out the matter field vacuum fluctuations. Apart from being a toy model for realistic four-dimensional quantum gravity, the understanding of this system is essential for the quantization of strings in non-critical dimensions. It is also related to the theory of random surfaces in statistical mechanics.

In the traditional Polyakov path integral approach to the quantization of strings [3], the world sheet metric $h_{\alpha\beta}$ is quantized in the conformal gauge. There the conformal factor of the metric is the only dynamical degree of freedom and as is well known, it decouples in the critical dimension. On the other hand, quantizing in subcritical dimensions requires the solution of a non-trivial quantum gravity theory on the world sheet. To regularize the action one has to introduce a regulator which preserves general covariance. In the conformal gauge this turns out to be difficult. In the light-cone gauge, however, it was shown in Ref. [2] that by adding the covariant regulator $M^{-2} \int d^2x \sqrt{-h} R^2$ to the action makes the theory convergent. Here R is the curvature scalar and M is a cut-off mass which finally will be sent to infinity.

The complete action is given by

$$S = \frac{d}{96\pi} \int d^2x \sqrt{-h} [R \square^{-1} R + \frac{1}{M^2} R^2 + \Lambda] \quad (1.1)$$

where $\square$ denotes the scalar Laplacian constructed from $h_{\alpha\beta}$ and Λ is a cosmological constant. The first term in Eq. (1.1) is the well-known effective action of any massless two-dimensional field coupling to gravity. The coefficient d is equal to the central charge of the associated Kac-Moody algebra. The term $R \square^{-1} R$ is most easily obtained by functionally integrating the trace anomaly equation

$$T^\alpha_\alpha = \frac{d}{24\pi} R \quad (1.2)$$

Actually in Refs. [1] and [2] they did not use the covariant action in Eq. (1.1) but started directly from Eq. (1.2). Writing the light-cone gauge metric in the form^{*)}

^{*)} Our conventions for the light-cone co-ordinates are $x^\pm = x^0 \pm x^1$.

$$ds^2 = dx^+ dx^- + h_{++}(x^+, x^-) dx^+ dx^+ , \quad (1.3)$$

they showed that the equation of motion for h_{++} is simply

$$\partial_-^3 h_{++} = 0 . \quad (1.4)$$

Equation (1.4) is the starting point for the derivation of Ward identities associated with the residual gauge transformations leaving the form of the metric in Eq. (1.3) invariant. The Ward identities are then integrated to yield the recursive correlation functions of an arbitrary number of h_{++} [1]

$$\begin{aligned} \langle h_{++}(z) h_{++}(x_1) \dots h_{++}(x_n) \rangle &= -\frac{k}{6} \sum_j \frac{(z^- - x_j^-)^2}{(z^+ - x_j^+)^2} \langle h_{++}(x_1) \dots \\ &\dots \cancel{h_{++}(x_j)} \dots h_{++}(x_n) \rangle + \sum_j \left[\frac{(z^- - x_j^-)^2}{z^+ - x_j^+} \frac{\partial}{\partial x_j^-} + 2 \frac{z^- - x_j^-}{z^+ - x_j^+} \right] \langle h_{++}(x_1) \dots h_{++}(x_n) \rangle. \end{aligned} \quad (1.5)$$

Parametrizing the most general solution of Eq. (1.4) as

$$h_{++}(x^+, x^-) = J^+(x^+) - 2J^0(x^+)x^- + J^-(x^+)(x^-)^2 , \quad (1.6)$$

one easily verifies the remarkable fact that the $J^a(x^+)$ ($a = 0, +, -$) are the currents of an $SL(2, R)$ current algebra. The operator product expansion of two currents is given by

$$J^a(x^+) J^b(x^+) = -\frac{k}{2} \frac{\eta^{ab}}{(x^+ - x'^+)^2} + f_c^{ab} \frac{J^c(x'^+)}{x^+ - x'^+} + \dots , \quad (1.7)$$

where η^{ab} is the Killing metric tensor for $SL(2, R)$ and f_c^{ab} are the structure constants.

The main purpose of this paper is to determine the origin of this $SL(2, R)$ Kac-Moody algebra. Contrary to Refs. [1] and [2] who did not use the action in Eq. (1.1) in its general form but rather considered an explicit matter system (a Majorana fermion) and evaluated the gauge-fixed one-loop determinant in a very indirect way, we start from the complete (covariant) energy momentum tensor corresponding to Eq. (1.1). Since the equation of motion for h_{++} is simply $T_{--} = 0$, transformations leaving T_{--} invariant should lead to conserved currents. We show that the group of these transformations is the Kac-Moody extension of $SL(2, R)$ and that it arises in a natural way. We show explicitly the connection with the Wess-

Zumino-Witten model. We derive the transformation law of the light-cone components of the energy-momentum tensor T_{++} , T_{+-} and T_{--} under the group of residual gauge transformations compatible with the form in Eq. (1.3) of the metric. This group is the analogue of the conformal co-ordinate transformations in the conformal gauge.

Another aim is to give a systematic comparison between the quantization in the conformal and the light-cone gauges. (This does not include questions of regularization and renormalization.) To this end we review in Section 2 some basic facts of conformal field theory in a language that can be generalized to the light-cone case. Rather than starting from the operator product expansion, the discussion will be based on the induced gravity action. In Section 3 we give the details of the corresponding analysis in the light-cone gauge, uncover the origin of the $SL(2, R)$ Kac-Moody algebra and derive the operator product expansion of the energy momentum tensor. Section 4 is the conclusion.

2. - CONFORMAL GAUGE

The main purpose of this section is to prepare ourselves to deal with the complications of the light-cone gauge by first studying the familiar conformal gauge. However, the regulator term R^2/M^2 in Eq. (1.1) will give a non-linear contribution and is the main obstacle to the quantization of the theory in the conformal gauge. Since this section is mainly for illustration and the regulator term does not affect the analysis in the light-cone gauge, we shall omit this term and use the action:

$$S = \frac{\alpha}{96\pi} \int d^2x \sqrt{-h} [R \square^{-1} R + \Lambda] . \quad (2.1)$$

The induced energy momentum tensor is given by the functional derivative

$$T^{\alpha\beta}(x) = - \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{\alpha\beta}(x)} . \quad (2.2)$$

Standard but lengthy manipulations give^{*)}

$$T^{\alpha\beta} = - \frac{\alpha}{48\pi} \left[2 \nabla^\alpha \nabla^\beta (\square^{-1} R) - \nabla^\alpha (\square^{-1} R) \nabla^\beta (\square^{-1} R) \right. \\ \left. - h^{\alpha\beta} \left(2R - \frac{1}{2} \nabla_\gamma (\square^{-1} R) \nabla^\gamma (\square^{-1} R) - \frac{1}{2} \Lambda \right) \right] . \quad (2.3)$$

*) Our conventions are:
 $R^\delta_{\gamma\alpha\beta} = \partial_\alpha \Gamma^\delta_{\beta\gamma} + \Gamma^\delta_{\alpha\beta} \Gamma^\delta_{\gamma\beta} - \alpha \leftrightarrow \beta$, $R = h^{\beta\gamma} R^\alpha_{\gamma\alpha\beta}$.

[The regulator term would contribute the additional piece $(d/48\pi M^2)(2\Box R + \frac{1}{2}R^2)h^{\alpha\beta}$.]

We now specialize to the conformal gauge. The only non-vanishing component of the metric is $h_{+-} \equiv \psi \equiv \exp(\phi)$:

$$ds^2 = 2 \psi(x^+, x^-) dx^+ dx^- \quad . \quad (2.4)$$

The residual gauge transformations consistent with Eq. (2.4) are the conformal coordinate transformations $x^+ \rightarrow y^+(x^+)$, $x^- \rightarrow y^-(x^-)$. The curvature scalar reads in this gauge

$$R = -2e^{-\phi} \partial_+ \partial_- \phi = -\Box \phi \quad . \quad (2.5)$$

Equation (2.5) is then inverted to give

$$\Box^{-1} R = -\phi + f(x^+) + g(x^-) \quad . \quad (2.6)$$

We use the arbitrariness in defining ϕ to absorb $f(x^+)$ and $g(x^-)$, or equivalently we set these to zero:

$$\Box^{-1} R = -\phi \quad . \quad (2.7)$$

Inserting Eq. (2.7) into Eq. (2.3) we obtain for the components of the energy-momentum tensor

$$T_{++} = -\frac{c}{24\pi} \left[\frac{\partial_+^2 \psi}{\psi} - \frac{3}{2} \left(\frac{\partial_+ \psi}{\psi} \right)^2 \right] \quad , \quad (2.8a)$$

$$T_{--} = -\frac{c}{24\pi} \left[\frac{\partial_-^2 \psi}{\psi} - \frac{3}{2} \left(\frac{\partial_- \psi}{\psi} \right)^2 \right] \quad , \quad (2.8b)$$

$$\begin{aligned} T_{+-} &= \frac{c}{48\pi} \left[2 \partial_+ \partial_- \phi + \frac{1}{2} \Lambda e^\phi \right] \\ &= -\frac{c}{48\pi} e^\phi \left[R - \frac{1}{2} \Lambda \right] \quad . \end{aligned} \quad (2.8c)$$

In going from Eq. (2.3) to Eq. (2.8) we have changed d into $d-26 \equiv -c$ to account for the Fadeev-Popov determinant associated with the conformal gauge fixing.

In the quantum gravity theory the components T_{++} , T_{--} and T_{+-} have a very different status. Putting $T_{+-} = 0$ or

$$R = \frac{1}{2} \Lambda \quad (2.9)$$

yields the (operator) equations of motion for h_{+-} . On the other hand, T_{++} and T_{--} are the generators of the residual gauge transformations. They define physical states by the requirement that the T_{++} and T_{--} matrix elements between them must vanish.

Now let us briefly look at Eq. (2.9) as a classical equation. It describes spaces of constant curvature $\Lambda/2$. Inserting Eq. (2.5) one finds the Liouville equation [4]

$$\partial_+ \partial_- \phi + \frac{1}{4} \Lambda e^\phi = 0. \quad (2.10)$$

The most general solution reads $\phi = \ln \psi$ with

$$\psi(x^+, x^-) = \frac{F_+'(x^+) F_-'(x^-)}{\left[1 + \frac{\Lambda}{8} F_+(x^+) F_-(x^-)\right]^2}. \quad (2.11)$$

It is parametrized by two arbitrary functions $F_+(x^+)$ and $F_-(x^-)$. Inserting Eq. (2.11) into Eq. (2.4), we observe that the resulting metric can be brought to the form

$$ds^2 = \frac{2 dx^+ dx^-}{\left[1 + \frac{\Lambda}{8} x^+ x^-\right]^2} \quad (2.12)$$

by means of a conformal co-ordinate transformation

$$x^+ \rightarrow F_+^{-1}(x^+), \quad x^- \rightarrow F_-^{-1}(x^-) \quad (2.13)$$

and thus coincides with the group of residual gauge transformations leaving the form of the metric in Eq. (2.4) invariant. This will not be the case in the light-cone gauge. Evaluating T_{++} and T_{--} for metrics of constant curvature as given in Eq. (2.11) we find that they are independent of the value of Λ :

$$\begin{aligned} T_{++} &= -\frac{c}{24\pi} \{F_+; x^+\}, \\ T_{--} &= -\frac{c}{24\pi} \{F_-; x^-\}. \end{aligned} \quad (2.14)$$

Here $\{F; x\}$ denotes the Schwarzian derivative:

$$\{F; x\} = \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2} \left(\frac{\partial_x^2 F}{\partial_x F} \right)^2. \quad (2.15)$$

Obviously T_{++} and T_{--} vanish for the standard form of the metric as given in Eq. (2.12).

Now we find how T_{++} and T_{--} transform under a conformal co-ordinate transformation $x^+ \rightarrow y^+(x^+)$. This will change $F_+(x^+)$ to $F_+(y^+(x^+))$ and leaves $F_-(x^-)$, and hence T_{--} , unchanged. The transformation law for T_{++} is most easily found by using the "chain rule" for the Schwarzian derivative:

$$\{\bar{F}_+; x^+\} = \left(\frac{dy^+}{dx^+}\right)^2 \{\bar{F}_+; y^+\} + \{y^+; x^+\} . \quad (2.17)$$

This identity implies the transformation [5]

$$T_{++}(x^+) = \left(\frac{dy^+}{dx^+}\right)^2 T'_{++}(y^+) - \frac{c}{24\pi} \{y^+; x^+\} . \quad (2.18)$$

A similar formula can be obtained for the change of T_{--} under $x^- \rightarrow y^-(x^-)$. The first term on the right-hand side of Eq. (2.15) corresponds to the usual tensor transformation law, while the second is a Schwinger term. From the infinitesimal version of Eq. (2.18) one can easily extract the Virasoro algebra for T_{++} [5]. The tensor components T_{++} and T_{--} are invariant under the $SL(2, R) \times SL(2, R)$ group transformations

$$\bar{F}_\pm \longrightarrow \frac{A_\pm \bar{F}_\pm + B_\pm}{C_\pm \bar{F}_\pm + D_\pm} , \quad A_\pm D_\pm - B_\pm C_\pm = 1 .$$

They correspond to the anomaly-free subalgebra of the Virasoro algebra. The isometry group of spaces of constant curvature acting on $F_\pm$ of Eq. (2.11) is the $SL(2, R)$ which is the diagonal subgroup of $SL(2, R) \times SL(2, R)$. These, however, should not be confused with the infinite dimensional Kac-Moody $SL(2, R)$ algebra we will be dealing with in the next section.

3. - LIGHT-CONE GAUGE

Having illustrated our strategy for the simpler conformal gauge we shall repeat the same steps, as far as possible, for the light-cone gauge with the metric given in Eq. (1.3). If we denote the invariant length by $h_{\alpha\beta} dx^\alpha dx^\beta$ where $\alpha, \beta = +, -$, then the non-vanishing Christoffel connections are

$$\Gamma_{++}^+ = -2h , \quad \Gamma_{++}^- = 2h + 2h \partial h , \quad \Gamma_{+-}^- = 2h , \quad (3.1)$$

where we have abbreviated h_{++} by h . The curvature scalar is then easily evaluated to be

$$R = 4 \partial^2 h . \quad (3.2)$$

To evaluate the components of the energy-momentum tensor from Eq. (2.3) we must first determine $\square^{-1}R$. To do this, we first evaluate the action of the Laplacian $\square$ on a scalar ϕ :

$$\square \phi \equiv h^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 4 (\partial_-)(\partial_+ - h \partial_-) \phi . \quad (3.3)$$

This implies that the action of $\square^{-1}$ on a scalar is given by

$$\square^{-1} \phi = \frac{1}{4} (\partial_+ - h \partial_-)^{-1} (\partial_-)^{-1} \phi \quad (3.4)$$

allowing us to simplify the function

$$g \equiv \square^{-1}R = (\partial_+ - h \partial_-)^{-1} \partial_- h . \quad (3.5)$$

Thus the problem reduces to finding a function g such that

$$(\partial_+ - h \partial_-)g = \partial_- h . \quad (3.6)$$

To find a solution of this non-trivial equation, we first define a function $f(x^+, x^-)$ that satisfies a homogeneous form of Eq. (3.6):

$$\partial_+ f = h \partial_- f . \quad (3.7)$$

Such a function was shown [6] to correspond to the transformation from the conformal co-ordinates $(\tilde{x})$ to the light-cone ones (x) in the form

$$\tilde{x}^+ = x^+ , \quad \tilde{x}^- = f(x^+, x^-) . \quad (3.8)$$

Then an explicit solution for g is given by

$$g = A(f) + \ln(\partial_- f) , \quad (3.9)$$

where $A(f)$ is an arbitrary function of f , and reflects the freedom in defining the function f [compare with Eq. (2.6)]. For simplicity, since we have the freedom, we shall set $A(f)$ to zero:

$$g = \ln(\partial_- f) . \quad (3.10)$$

Using Eqs. (3.5) and (3.6) we can evaluate the components of $T_{\alpha\beta}$. We immediately find

$$T_{--} = - \frac{d}{48\pi} [2 \partial_-^2 g - (\partial_- g)^2] \quad (3.11)$$

and since $T^{++} = 4T_{--}$ is the variation with respect to the dynamical variable h_{++} , setting it to zero will give the (operator) equation of motion

$$2 \partial_-^2 g = (\partial_- g)^2 . \quad (3.12)$$

Differentiating Eq. (3.12) with respect to ∂_+ and using the definition of Eq. (3.6) for g , we arrive at the equation

$$\partial_-^3 h = 0 , \quad (3.13)$$

which is the same as Eq. (1.4) obtained in Ref. [1].

If we now insert Eq. (3.10) into Eq. (3.11), T_{--} takes the useful Schwarzian form

$$T_{--} = - \frac{d}{24\pi} \{f; x^-\} . \quad (3.14)$$

Using the known property of the Schwarzian derivative

$$\{f; x^-\} = \left\{ \frac{A(x^+)f + B(x^+)}{C(x^+)f + D(x^+)} ; x^- \right\} , \quad AD - BC = 1 , \quad (3.15)$$

implies that T_{--} is invariant under the $SL(2, R)$ transformations with continuous parameters. Imposing the equation of motion then yields, after using another property of the Schwarzian derivative,

$$\{f; x^-\} = 0 \iff f(x^+, x^-) = \frac{a(x^+)x^- + b(x^+)}{c(x^+)x^- + d(x^+)} , \quad ad - bc = 1 , \quad (3.16)$$

where $a(x^+)$, $b(x^+)$, $c(x^+)$ and $d(x^+)$ are, apart from the constraint $ad - bc = 1$, arbitrary. This arbitrariness is a reflection of the $SL(2, R)$ symmetry in Eq. (3.15). To make contact with the current algebra of the Wess-Zumino-Witten model treated in Ref. [7], it is useful to use 2×2 matrix representations. If we denote the action of f on x^- by

$$f : x^- \longrightarrow \frac{a(x^+)x^- + b(x^+)}{c(x^+)x^- + d(x^+)} , \quad (3.17)$$

we can represent it by the 2×2 matrix $F(x^+)$:

$$F(x^+) = \begin{pmatrix} a(x^+) & b(x^+) \\ c(x^+) & d(x^+) \end{pmatrix} . \quad (3.18)$$

Then acting on x^- by the $SL(2, R)$ transformation $\Omega^{-1}(x^+)$:

$$\omega : x^- \longrightarrow \frac{\alpha(x^+)x^- + \beta(x^+)}{\gamma(x^+)x^- + \delta(x^+)} , \quad \alpha\delta - \beta\gamma = 1 , \quad (3.19)$$

f will transform as $f \cdot \omega^{-1}$, or in matrix representation

$$F(x^+) \longrightarrow F(x^+) \Omega^{-1}(x^+) , \quad (3.20)$$

where

$$\Omega(x^+) = \begin{pmatrix} \alpha(x^+) & \beta(x^+) \\ \gamma(x^+) & \delta(x^+) \end{pmatrix} .$$

We will come back to this equation shortly.

Inserting the solution given in Eq. (3.16) for f into Eq. (3.7), we arrive at the solution for the metric h

$$h = (\dot{b}\dot{d} - b\dot{d}) + [(\dot{a}\dot{d} - a\dot{d}) + (\dot{b}\dot{c} - b\dot{c})]x^- + (\dot{a}\dot{c} - a\dot{c})(x^-)^2 \quad (3.21)$$

and this trivially satisfies Eq. (3.13). Comparing with the parametrization for h in Eq. (1.6) which was used in Refs. [1] and [2], we arrive at the identification

$$\begin{aligned} J^+ &= \dot{b}\dot{d} - b\dot{d} , \quad J^- = \dot{a}\dot{c} - a\dot{c} , \\ J^0 &= \frac{1}{2} [(\dot{a}\dot{d} - a\dot{d}) + (\dot{b}\dot{c} - b\dot{c})] . \end{aligned} \quad (3.22)$$

The form of the J^a 's suggests that they are bosonic currents. It should not be surprising that we are able to extract more information on the form of h as given in Eq. (3.21) since this was derived by solving Eq. (3.12), while Eq. (3.13) is obtained by differentiating Eq. (3.12).

Now we are ready to make the connection with the (non-compact) version of the $SU(2)$ current algebra for the Wess-Zumino model as developed in Refs. [7] and [8]. We can easily verify that Eq. (3.22) can be written as

$$2J^a t_a = F^{-1} \partial_+ F - \frac{1}{2} \text{tr}(F^{-1} \partial_+ F) , \quad (3.23)$$

where t^a are the $SL(2, \mathbb{R})$ matrices

$$t^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad t^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad t^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.24)$$

and

$$t_a = \eta_{ab} t^b , \quad \eta_{+-} = \frac{1}{2} , \quad \eta_{00} = -1 ,$$

$$[t_a, t_b] = f^c_{ab} t_c .$$

Under the $SL(2, R)$ transformation in Eq. (3.20), and after rescaling $J^a \rightarrow (1/k)J^a$, the currents $J^a t_a$ transform as

$$J^a t_a \rightarrow \Omega J^a t_a \Omega^{-1} + \frac{k}{2} [\Omega \dot{\Omega}^{-1} - \frac{1}{2} \text{tr}(\Omega \dot{\Omega}^{-1})] . \quad (3.25)$$

To obtain the infinitesimal form of Eq. (3.25) we write

$$\Omega(x^+) = 1 + \omega^a(x^+) t_a + O(\omega^2) . \quad (3.26)$$

We find that the $J^a(x^+)$ are the generators of the Kac-Moody $SL(2, R)$ group with central charge k

$$\delta J^a(x^+) = f^a_{bc} \omega^b(x^+) J^c(x^+) - \frac{k}{2} \dot{\omega}^a(x^+) . \quad (3.27)$$

Comparing Eq. (3.27) with Eq. (1.14) of Ref. [7] done for the $SU(2)$ Wess-Zumino-Witten model, we immediately see that they are identical (apart from the non-compactness of the group). We can then use the results of Refs. [7] and [8] to write the operator product expansion of the $SL(2, R)$ currents

$$J^a_{(x^+)} J^b_{(x'^+)} = -\frac{k}{2} \frac{\eta^{ab}}{(x^+ - x'^+)^2} + \frac{f_c^{ab} J^c_{(x'^+)}}{x^+ - x'^+} + \dots \quad (3.28)$$

This is the same as Eq. (1.7) derived from the Ward identities in Ref. [1]. We thus see that the underlying symmetry in the light-cone gauge is the $SL(2, R)$ Kac-Moody algebra which is larger than the Virasoro algebra present in the conformal gauge. Going back to the energy-momentum tensor we can use Eqs. (3.5) and (3.6) to evaluate the T_{++} and T_{--} components:

$$T_{+-} = h T_{--} - \frac{\alpha}{48\pi} [-2 \partial_-^2 h + \frac{1}{4} \Lambda] , \quad (3.29)$$

$$T_{++} = h^2 T_{--} - \frac{\alpha}{48\pi} [-6 h \partial_-^2 h + (\partial_- h)^2 + 2 \partial_+ \partial_- h - \frac{1}{2} \Lambda] . \quad (3.30)$$

Unlike T_{--} which vanishes to give the dynamical equation of motion, only the matrix elements of T_{+-} and T_{++} between physical states are required to vanish (after the matter and ghost field contributions have been included).

Next, and as was done in the last section for the conformal gauge, we consider the residual gauge transformations preserving the form of the metric in Eq. (1.3). Denoting the transformations by $y^\alpha = y^\alpha(x^\beta)$, the metric $h_{\alpha\beta}(x)$ transforms to $h'_{\alpha\beta}(y)$ where

$$h_{\alpha\beta}(x) = \frac{\partial y^\gamma}{\partial x^\alpha} \frac{\partial y^\delta}{\partial x^\beta} h'_{\gamma\delta}(y). \quad (3.31)$$

Using the information that $h_{+-}(x) = h'_{+-}(y) = \frac{1}{2}$ and $h_{--}(x)$ and $h'_{--}(y) = 0$ we deduce that the transformation y^α must have the form

$$\begin{aligned} y^+ &= a(x^+) , \\ y^- &= \frac{x^-}{\dot{a}(x^+)} + b(x^+) . \end{aligned} \quad (3.32)$$

Thus, unlike the conformal gauge, the residual gauge transformations of the metric and the symmetries of the equation of motion do not coincide. From Eqs. (3.31) and (3.32) we find that the component h_{++} of the metric (denoted by h) transforms as

$$h(x) = \dot{a}^2 h'(y) + \dot{a} \dot{b} - \frac{\ddot{a}}{\dot{a}} x^- . \quad (3.33)$$

This will allow us to determine the behaviour of the currents J^a under the transformations in Eq. (3.32). We obtain

$$\begin{aligned} J^+(x^+) &= \dot{a}^2 [J'^+(y^+) - 2b(x^+) J'^0(y^+) + b^2(x^+) J'^-(y^+)] + \dot{a} \dot{b} , \\ J^0(x^+) &= \dot{a} [J'^0(y^+) - b(x^+) J'^-(y^+)] + \frac{1}{2} \frac{\ddot{a}}{\dot{a}} , \\ J^-(x^+) &= J'^-(y^+) . \end{aligned} \quad (3.34)$$

We shall also need to compute the transformations of $T_{\alpha\beta}$. First we find that T_{--} transforms into itself

$$T_{--}(x) = \frac{1}{\dot{a}^2} T'_{--}(y) , \quad (3.35)$$

and as expected, since it is an equation of motion, can be set to zero consistently. After setting T_{--} to zero, we find that T_{+-} transforms into itself but T_{++} gets, besides a term proportional to T_{+-} , also a Schwinger term

$$\begin{aligned} T_{+-}(x) &= T'_{+-}(y) , \\ T_{++}(x) &= \dot{a}^2 T'_{++}(y) + 2\left(\dot{a} \dot{b} - \frac{\ddot{a}}{\dot{a}} x^-\right) T'_{+-}(y) + \frac{c}{12} \{a(x^+); x^+\} , \end{aligned} \quad (3.36)$$

where we have replaced $d/2\pi$ by c . From T_{++} and T_{+-} we can construct a new tensor

T_{++}^N that transforms diagonally. This is given by

$$\begin{aligned} T_{++}^N &\equiv T_{++} - 2h T_{+-} \\ &= -\frac{d}{48\pi} \left[-2h \partial_-^2 h + (\partial_- h)^2 + 2\partial_+ \partial_- h \right] , \end{aligned} \quad (3.37)$$

and is exactly the form obtained in Ref. [2] for reasons that will become clear shortly. Its transformation is simple and given by

$$T_{++}^N(x) = \dot{\alpha}^2 T_{++}^N(y) + \frac{c}{12} \{a(x^+); x^+\} . \quad (3.38)$$

It is useful to write the infinitesimal form of all the above transformations. Following Ref. [2], we first write

$$a(x^+) = x^+ + \varepsilon(x^+) , \quad b(x^+) = \gamma(x^+) , \quad (3.39)$$

which gives for T_{++}^N and T_{+-} :

$$\begin{aligned} \delta T_{++}^N &= 2 \dot{\varepsilon} T_{++}^N + \varepsilon \partial_+ T_{++}^N + (\gamma - \dot{\varepsilon} x^-) \partial_- T_{++}^N + \frac{c}{12} \ddot{\varepsilon} , \\ \delta T_{+-} &= \varepsilon \partial_+ T_{+-} . \end{aligned} \quad (3.40)$$

From Eq. (3.33) we can find the infinitesimal transformation of the field h :

$$\delta h = 2 \dot{\varepsilon} h + \varepsilon \partial_+ h + (\gamma - \dot{\varepsilon} x^-) \partial_- h + (\dot{\gamma} - \ddot{\varepsilon} x^-) , \quad (3.41)$$

but remarkably this can be expressed in terms of one parameter

$$\delta h = (\partial_+ + \partial_- h - h \partial_-) \kappa \quad (3.42)$$

where

$$\kappa \equiv \gamma - \dot{\varepsilon} x^- + \varepsilon h . \quad (3.43)$$

(κ is called ε_+ in Refs. [1] and [2].) From Eqs. (3.7) and (3.42) we also deduce that

$$\delta f = \kappa \partial_- f . \quad (3.44)$$

These two equations (3.42) and (3.44) were the starting point in Refs. [1] and [2] as a required invariance when coupling the gauge-fixed action to external sources h_{+-} and h_{--} . This explains why they arrived directly at the diagonalized T_{++}^N . At this point we can either derive the Ward identities associated with the invariance of the action under Eq. (3.42), and obtain the recursive correlation functions

pointed to in Eq. (1.5), or take Eq. (3.28) as our starting point. From it we can deduce, by substituting in the parametrization of Eq. (1.6) for h :

$$h(x) h(x') = -\left(\frac{x^- - x'^-}{x^+ - x'^+}\right)^2 + 2\left(\frac{x^- - x'^-}{x^+ - x'^+}\right)h(x') + \frac{(x^- - x'^-)^2}{x^+ - x'^+} \partial_- h(x') + \dots \quad (3.45)$$

which can be easily seen as contained in Eq. (1.5). To determine the operator product expansions of T_{++}^N and T_{+-} , we first express them in terms of the currents J^a . We find from Eqs. (3.29) and (3.37) that

$$T_{++}^N = T_{++}^{SUG} + \partial_+ J^0 - (\partial_+ J^-) x^- \quad (3.46)$$

$$T_{+-} = J^-$$

where [8]

$$T_{++}^{SUG} = \frac{1}{k+2} \eta_{ab} : J^a J^b : \quad , \quad (3.47)$$

and we have rescaled the field h in T_{++}^N and T_{+-}^N . After some manipulations we obtain the following operator product expansions

$$\begin{aligned} T_{++}^N(x) T_{++}^N(x') &= \frac{1}{2} \frac{\left(\frac{3k}{k+2} - 6k\right)}{(x^+ - x'^+)^4} + \frac{2 T_{++}^N(x')}{(x^+ - x'^+)^2} + \frac{\partial_+ T_{++}^N(x')}{x^+ - x'^+} \\ &\quad - \frac{x^- - x'^-}{x^+ - x'^+} \left(\frac{2 \partial_- T_{++}^N(x')}{(x^+ - x'^+)^2} + \frac{\partial_+ \partial_- T_{++}^N(x')}{x^+ - x'^+} \right) + \dots \end{aligned} \quad (3.48)$$

$$T_{++}^N(x) T_{+-}^N(x') = \frac{2 T_{+-}^N(x')}{(x^+ - x'^+)^2} + \frac{\partial_+ T_{+-}^N(x')}{x^+ - x'^+} + \dots$$

$$T_{+-}(x) T_{+-}(x') = 0 + \dots$$

where the $+\dots$ means the non-singular pieces. From this it is seen that T_{++}^N and T_{+-} are the generators of the residual gauge symmetry. The physical requirement of the vanishing of the anomalous term in the operator product expansion in (3.48), after taking the matter and ghost contributions, gives

$$d - 28 + \frac{3k}{k+2} - 6k = 0 \quad . \quad (3.49)$$

The above equation has been extensively studied in Ref. [2] and can lead in particular to non-critical strings in the domains $d \leq 1$ and $d > 25$. Inside the domain $1 < d < 25$ the central charge k will only have complex solutions from Eq. (3.49).

It was suggested in Ref. [2] and verified, within the context of random surface simulations on the lattice, in Ref. [9] that a phase transition must take place due to the strong gravity effects. This of course remains an open problem, and we hope that by properly analyzing the complete system we may gain some insight into this phenomenon.

4. - CONCLUSION

Starting from a covariant action for the induced two-dimensional gravity, we have derived the equation of motion in the light-cone gauge and determined the origin of the $SL(2,R)$ Kac-Moody algebra as the symmetry for the equations of motion. We have shown explicitly how the currents emerge and made the connection with the Wess-Zumino-Witten current algebra. We also investigated the residual gauge symmetry and determined the energy-momentum tensors T_{++}^N and T_{+-} and gave their operator product expansion.

Since this analysis has drastic consequences for statistical models and string models, much work remains to be done in order to understand the effects of the induced quantum gravity on some of the string results.

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