CERN-TH.5106/88

DUALITY IN STRING BACKGROUND SPACE

A. Giveon*, E. Rabinovici*

The Racah Institute of Physics, Jerusalem, Israel

and

G. Veneziano

CERN, Geneva

ABSTRACT

The (so-called) duality symmetry of closed strings with d compactified dimensions is reformulated in geometrical terms as an invariance under a particular Z_2 transformation acting on (constant) metric, torsion and gauge background fields. Fixed points of this transformation - moded by $SL_d(Z)$ - yield enlarged symmetries based on simply-laced gauge groups. Exact duality requires revision of naive compactification prescriptions in the effective low energy action. It could also prove important for dynamical compactification in string theories.

*Supported in part by the Israeli Academy of Science and by the BSF-American Israeli Bi-National Science Foundation

CERN-TH.5106/88

July 1988

1. INTRODUCTION

The geography of the space of all conformal two-dimensional field theories is just starting to unfold.

For the allowed values of the central charge c , in the range $0 < c < 1$ [1], there exist, generally speaking, two such field theories, i.e., the conformal space consists of just 2 points. For six special values of c the space enlarges to three points [2,3].

For $c > 1$ simplicity gives place to increasing complexity manifested on a multidimensional space filled with multicritical points and surfaces. For integer values of c there are many points in the conformal space, which can be described as internal space backgrounds on which strings or superstrings propagate.

For example, all known $c = 1$ theories [3-6] can be described either as strings moving on a circle of radius R_c (described by an even, self dual Lorentzian lattice) or an orbicircle of radius R_0 (not described, in general, by a Lorentzian lattice). In addition there are three theories describing a string moving on orbifold versions of the group manifold $SU(2)$ [5].

The theories can be described in terms of background fields. When interpreted this way, different conformal theories can be regarded as representing one and the same theory around each one of its possible candidate vacua. The parameters that label the possible consistent theories are reinterpreted as those parametrizing the vacuum manifold of the string theory effective action. The distinction could turn out to be more than academic, in the sense that some dynamics, possibly a non-perturbative one, might eventually decide which among those candidates is the true vacuum.

A related point is that, studying a string theory as a function of the backgrounds can reveal symmetries which are usually (spontaneously) broken in any given vacuum: presumably the true symmetry of string theory is infinitely larger than the symmetry (stability group) of any one of its possible vacua.

It has also been known for some time that a string cannot distinguish whether it is moving on a circle [7] (orbicircle) [6] of a radius $R_c(R_0)$ or on one of "reciprocal" radius $\alpha'/R_c(\alpha'/R_0)$. This "duality" introduces an extra Z_2 structure in the already complicated space of conformal field theories (see fig. 1 for a picture of the $c = 1$ theories connected by this transformation). Duality

was formulated in reference [8] in Lorentzian lattice terms. Again, we shall see how to convert this pairwise equivalence of string theories into a (possibly spontaneously broken) Z_2 invariance in string-background space, in our case that of a constant metric G and torsion B (and gauge fields in the heterotic case).

We start by reviewing and formulating duality for a compactified closed bosonic string with identified right and left moving center of mass coordinates. We find that duality has a particularly simple form in terms of a certain combination of the background fields. We next show that though the fixed points of the duality transformations have only $SU(2)$ properties, once moded by a discrete symmetry, all simply laced groups emerge. After extending our considerations to the heterotic case, we discuss compactification effects as well as some more adventurous ideas on the concept of an invariant volume in string theories.

2. BOSONIC NARAIN MODELS AND BACKGROUND FIELDS

As first pointed out by Narain [9], a closed bosonic string living in d compact (and $26-d$ non-compact) dimensions can be characterized in terms of the Lorentian lattice $\Gamma(d,d)$ of its allowed left and right momenta (p_L, p_R) under the restriction (due to σ -translational invariance):

$$p_L^2 - p_R^2 = 0 \tag{1}$$

such a condition is left invariant by an $O(d,d)$ Lorentz transformation but the spectrum usually is not, since

$$2M^2 = p_L^2 + p_R^2 + \text{oscillators} \tag{2}$$

is only invariant under the appropriate $O(d) \times O(d)$ subgroup of $O(d)$ subgroup of $O(d,d)$. Thus, the physically inequivalent models are labelled by the d^2 parameters of the coset $O(d,d)/O(d) \times O(d)$.

Later on, Narain, Sarmadi and Witten [10] have pointed out that all these models can be equally described by letting the string propagate in a constant metric G_{ij} and torsion B_{ij} , for a total, again, of d^2 parameters.

In order to describe our approach to duality transformations, we shall start by giving a slightly more general formulation of Narain's models, encompassing both his original approach and the one of ref. [10].

The compact coordinate part of the string action takes the (bosonic) form

$$S/\hbar = -1/4\pi \int_0^{2\pi} d\sigma \int d\tau (\sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j B_{ij}) \quad (3)$$

where with our definition G and B have dimensions of $[\text{length}]^{-2}$ and thus, include factors of α' and $\hbar$. By the compactification assumption,

$$\begin{aligned} X^i &\equiv X^i + 2\pi m_a e_a^i \quad m_a \in \mathbb{Z} \\ X &\equiv X + 2\pi \wedge \end{aligned} \quad (4)$$

As a result X^i has a constant mode:

$$X^i = L/2\pi + \text{oscillators} \quad L \in 2\pi\Lambda. \quad (5)$$

The canonical momentum P_i associated with X^i is given by

$$2\pi P_i = G_{ij} \dot{X}^j + B_{ij} X'^j = p_i + \text{oscillators} \quad (6)$$

where p belongs, as usual, to the dual lattice

$$\begin{aligned} p &\in \wedge^* \quad , \quad p_i = n_a e_{ai}^* \quad , \quad n_a \in \mathbb{Z} \\ e_{ai}^* e_b^i &= \delta_{ab} \end{aligned} \quad (7)$$

The Hamiltonian (in the ON gauge) and σ -translation constraints take the form:

$$\begin{aligned}
2H &= 2(L_+ + L_-) = (2\pi)^2 (P_i G^{ij} P_j) + X'^i (G - BG^{-1}B)_{ij} X'^j \\
&\quad + 4\pi X'^i B_{ik} G^{kj} P_j = 1/2 (P_{Li} G^{ij} P_{Lj} + P_{Ri} G^{ij} P_{Rj}) \\
P_{Li} &= 2\pi P_i + (G - B)_{ij} X'^j \quad P_{Ri} = 2\pi P_i - (G + B)_{ij} X'^j
\end{aligned} \tag{8a}$$

$$0 = L_+ - L_- = 2\pi PX' \tag{8b}$$

We deduce from (8a) that P_L and P_R , even in the presence of the backgrounds, still decouple and that they describe left and right movers respectively. The zero mode part of the Hamiltonian reads:

$$\begin{aligned}
H &= L_+ + L_- = 1/2 [n_a e_{ai}^* G^{ij} e_{bj}^* n_b + m_a e_a^i (G - BG^{-1}B)_{ij} e_b^j m_b \\
&\quad + 2 m_a e_a^i B_{ik} G^{kj} e_{bj}^* n_b]
\end{aligned} \tag{9}$$

We notice that (8a,b) are invariant under the simultaneous change of lattices and backgrounds defined by:

$$\begin{aligned}
e_a^i &\rightarrow \Omega_j^i e_a^j \quad ; \quad e_{ai}^* \rightarrow (\Omega^{-1})_i^j e_{aj}^* \\
G_{ij} &\rightarrow (\Omega^{-1})_i^k G_{kl} (\Omega^{-1})_j^l \quad ; \quad B_{ij} \rightarrow (\Omega^{-1})_i^k B_{kl} (\Omega^{-1})_j^l
\end{aligned} \tag{10}$$

Thus, only "gauge invariant" (and dimensionless) combinations such as

$$\hat{G}_{ab} = e_a^i G_{ij} e_b^j \quad , \quad \hat{B}_{ab} = e_a^i B_{ij} e_b^j \quad \text{etc.} \tag{11}$$

do really matter physically. Also, by a suitable choice of Ω , we can always go to the "gauge" $G = 1$ corresponding to Narain's description [9]. Alternatively one can go to the "gauge"

$$e_a^i = \delta_a^i R \quad , \quad e_{ai}^* = \delta_{ai} R^{-1} \tag{12}$$

where R can be taken to be a common compactification radius. This would correspond to the choice of ref.[10]. Of course, what will matter in the end is not the value of R but of GR^2 as it appears in eq. (11), but we would like to see this as a dependence upon (and eventually a determination of) the background G once R is given rather than the other way around. This is why we shall also stick, from now on, to the choice (12).

3. DUALITY

We are now ready to define a duality relation in G - B space leaving the physical content of the theory unchanged. This is simply given by

$$\begin{aligned} (\hat{G} + \hat{B})_{ab} &\rightarrow (\hat{G} + \hat{B})_{ab}^{-1} \\ (\hat{G} - \hat{B})_{ab} &\rightarrow (\hat{G} - \hat{B})_{ab}^{-1} \quad \text{i.e.} \quad (G \pm B)_{ij} \rightarrow (G \pm B)_{ij}^{-1} R^{-4} \end{aligned} \quad (13)$$

which can also be rewritten in several equivalent forms (those containing B^{-1} require of course a regular B):

$$\begin{aligned} \hat{G} &\rightarrow \hat{G}' = (\hat{G} + \hat{B}\hat{G}^{-1}\hat{B}^T)^{-1}, \quad \hat{B}^T = -\hat{B} \\ \hat{B} &\rightarrow \hat{B}' = (\hat{B} - \hat{G}\hat{B}^{-1}\hat{G})^{-1}, \quad \hat{G}^{-1}\hat{B} \rightarrow -\hat{B}\hat{G}^{-1} \end{aligned} \quad (14)$$

showing that the transformation maintains the symmetry (antisymmetry) of $G(B)$. Under this transformation we have, for instance:

$$P_L \rightarrow \tilde{P}_L = 2\pi P + R^{-4} (G + B)^{-1} X' \quad (15)$$

Hence:

$$\begin{aligned} 2L_+ &\rightarrow 1/2 (2\pi P + X' (G + B)^{-1} R^{-1}) (G + B) G^{-1} (G - B) R^4 (2\pi P + R^{-4} (G - B)^{-1} X') \\ &= 1/2 (P(G + B) (2\pi) R^2 + X' R^{-2}) G^{-1} ((G - B) P (2\pi) R^2 + X' R^{-2}) \end{aligned} \quad (16)$$

which is again $2L_+$, except for the interchange $(2\pi)R^2P \leftrightarrow X'$.

It is a simple matter to check that such an interchange is inessential: for the zero mode part it just amounts to interchanging pR^2 with $L/2\pi$ which is the same as exchanging the labels m and n , while for the oscillator part it amounts to a trivial redefinition of the creation and destruction operators.

Another way to argue that the interchange of P with X' is irrelevant is to notice that there exists a canonical transformation doing precisely that. It is defined by the generating function of the old and new coordinates: (denoted, the latter, by a tilde):

$$F = -R^{-2} \int_0^{2\pi} X'(\sigma) \tilde{X}(\sigma) d\sigma/2\pi ;$$

$$P(\sigma) = \delta F/\delta X(\sigma) = \frac{X'(\sigma)}{2\pi R^2} ; \quad \tilde{P}(\sigma) = -\delta F/\delta \tilde{X}(\sigma) = \frac{X'(\sigma)}{2\pi R^2} \quad (17)$$

Note that the canonical transformation (17) is equivalent to the simple change of backgrounds (13,14) only when these are constant. It could be amusing to check if it leads to a more general symmetry in the case of arbitrarily varying G and B .

The existence of a duality transformation in the lattice notation appears in reference [8], where the following construction was invoked. Starting with the lattice associated with the background $G=I$ and $B=0$, one performs Lorentz transformations, L and $(L^\dagger)^{-1}$ on it. The two new resulting lattices are related by a duality transformation.

In the absence of torsion ($B = 0$) one obtains the familiar transformation:

$$\hat{G} \rightarrow \hat{G}^{-1} \quad \text{or} \quad G \rightarrow G^{-1} R^{-4} \quad (18)$$

Consider the case of $c = 1$, i.e. one compact dimension. Recall that, by our starting point (3):

$$G_{ij} = (\hbar\alpha')^{-1} g_{ij} \quad (19)$$

where we have reinserted, for a moment, Planck's constant, g_{ij} is the conventional metric and $(\hbar\alpha')^{1/2}$ is the string length parameter. We see that the transformation (18) corresponds to either $\alpha' \rightarrow R^4 (\alpha')^{-1}$ at fixed R or, more conventionally, to $R \rightarrow \alpha'/R$ at fixed α' . Conventionally, one says that the theory is invariant under a change in R (at fixed α'). We rather prefer to say that, given a string moving on a circle of radius R , the theory has to decide what is the best value of the quantum length $(\hbar\alpha')^{1/2}$ and such a choice has an intrinsic Z_2 ambiguity given by (18). If the value $\alpha' = R^2$ is (is not) picked up the "vacuum" does not (does) break spontaneously Z_2 .

It is known that, when the symmetric point is picked up, an enlarged gauge symmetry also follows corresponding to a level one $SU(2)_R \times SU(2)_L$ representation of the Kac-Moody algebra. Thus a Z_2 preserving vacuum appears to preserve enlarged gauge symmetries as well. In the orbifold case the fixed point corresponds to a representation of the twisted $SU(2)$ KM algebra whose residual symmetry is $U(1) \times U(1)$.

For the more general case, the duality transformation still has strictly speaking a single fixed point given by:

$$G = 1 \quad , \quad B = 0 \quad \text{i.e.} \quad G_{ij} = R^{-2} \delta_{ij} \quad (20)$$

This can be shown as follows :

The fixed point condition leads to the relations

$$B^2 + G^2 \pm \{B, G\} = I \quad (21)$$

That is

$$\begin{aligned} \{B, G\} &= 0 \\ B^2 + G^2 &= I \end{aligned} \quad (22)$$

For a positive G with eigenvalues λ_i , diagonalize G by an orthogonal matrix, the relation is

invariant under a change of basis and thus one has at the fixed point

$$B_{ik} (\lambda_i + \lambda_k) = 0 \quad (23)$$

At the fixed point $B = 0$ and $G = I$. At the single fixed point the string form an $SU(2)_R^d \times SU(2)_L^d$ level one representation. This structure is enriched by noting that if (G', B') are related to (G, B) by

$$B' = -M^t B M \quad \text{and} \quad G' = M^t G M \quad (24)$$

where M is a matrix belonging to $SL_d(\mathbb{Z})$, then the spectra of both systems is identical. The two backgrounds are identical up to a basis rearrangement.

This is shown by considering

$$Z(G, B) = \text{tr } q^{L_0(G, B)} = \sum_{(n) (m)} q^{1/2 \{n^t G^{-1} n + m^t (G - B G^{-1} B) m - 2n^t G^{-1} B m\}} \quad (25)$$

Thus:

$$Z(G', B') = \sum_{(n) (m)} q^{1/2 \{n^t M^{-1} G^{-1} (M^{-1})^t n + m^t M^t (G - B G^{-1} B) M m - 2n^t M^{-1} (G^{-1} B) M m\}} \quad (26)$$

by defining

$$\begin{aligned} m' &= M m \\ n' &= (M^{-1})^t n \end{aligned} \quad (27)$$

one obtains that indeed

$$Z(G', B') = Z(G, B) \quad (28)$$

These points will become fixed points if one considers the conformal space module $SL_d(\mathbb{Z})$. Instead we term these points "physical fixed points". They include all cases in which the string moves on a Cartan subalgebra of a general group g . In those cases the half Cartan matrix valued background metric and torsion [6] are known to obey in addition, the relation

$$B \pm G \in SL_d(\mathbb{Z}) \tag{29}$$

in this case $B' = -M^T B M$, $G' = M^T G M$ where $M = (B + G)^{-1}$. We have not shown that the condition $B \pm G \in SL_d(\mathbb{Z})$ either implies an extended symmetry or exhausts all solutions. If that is the case these are a discrete set of points.

At each such background there appears a massless scalar field in the adjoint representation of $g_L \times g_R$. This marginal family of operators are responsible for the formation of a Higgs like breaking of $g_L \times g_R$ away from the physical fixed points.

In addition to the physical fixed points it is possible that the matrices G and B factorize in such a manner that a submatrix corresponds to a lower symmetry than at the fixed point. However, even though the symmetry is lower, these points may reside in entire submanifolds of the conformal space, the lower the symmetry the larger the possible dimension of the submanifold supporting these backgrounds.

The physical fixed points are situated on a special critical surface which separates equivalent sections of the conformal space. We define the D matrix by $D_{ij} = G_{ij} + B_{ij}$. The duality transformation maps models with a given $\det D$ into model with $1/\det D$. Thus in a given sector (such as a torus or orbifold) all different models are contained in a non-compact region for which $\det (B + G) \leq 1$. Models for which $\det (B + G) < 1$ are mapped into equivalent models for which $\det (B + G) > 1$. The surface of models for which $\det (B + G) = 1$ are mapped into each other, include of course, the single fixed point and the physically fixed points as well. Each "section" in the conformal space contains such a surface. Note also that the B terms serve as two dimensional θ parameters [10] and as such they are themselves compact variables. As a simple illustration we describe a section of the $c = 2$ space described by the backgrounds:

$$G = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad (30)$$

Under the dual transformation

$$g'(g,b) = \frac{g}{g^2 + b^2} \quad ; \quad b'(g,b) = -\frac{b}{g^2 + b^2} \quad (31)$$

In polar coordinates denote

$$(g,b) = (r \cos \theta, r \sin \theta) \quad (32)$$

then

$$(g',b') = \left(\frac{1}{r} \cos \theta, -\frac{1}{r} \sin \theta \right) \quad (33)$$

That is under duality

$$(r',\theta') = \left(\frac{1}{r}, -\theta \right) \quad (34)$$

In figure 2 we show the section. $\det D$ is r^2 in this case. Models with $r = 1$ are mapped into themselves. This section contains the fixed point $g = 1 \quad b = 0$ but not the physical fixed point

$$2G = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad ; \quad 2B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{representing SU(3)}$$

In the case of $B = 0$ the value of $(\det D)^{1/2}$ is equivalent to that of the volume of the target space occupied by the string. This simple geometrical quantity enters the dynamics in a surprising way. String backgrounds with a volume $\det G$ are equivalent to those with volume $1/\det G$ and minima of the zero point function appear at the fixed point. It was shown [11] that minima of the zero point function appear at points of extended symmetry. It is as if there was a repulsion at scales of order α' . Volumes if measured in yardsticks such that our universe has 4 large dimensions, do not really shrink beyond a certain point. Renormalization group equations would switch their behaviour once approaching the fixed point.

We wish to point out that it is $\det(B + G)$ which actually emerges as the object of physical interest. The various correlation functions are constrained to be invariant under the duality of $\det(B + G)$.

We believe that in some sense such a function should appear in the correct measure even of an effective theory valid for the non compact dimensions (which themselves can be viewed) to have very large or very small values of $\det D$. The internal dimensions appear in the effective action describing the scattering in the uncompactified dimensions. This effective lagrangian [12] contains the Einstein term and gauge interactions. The compactified dimensions are usually factored out of the 10 (26) field theory limit as an internal volume factor. This is explicitly violating the duality. We shall return to this issue in Section 6.

One can show that no low energy excitations that are B dependent survive in the limit of large $\det G$. Nevertheless, we wish to find a direct geometrical meaning of $\det D$. In this section we give one such indication, we shall return also to this matter in greater detail, in the last section.

The ratio of the volumes in space time described by a string moving in (B, G) and its dual background (B', G') is $\det(B + G)$. The relation $G' = (D^{-1})^t G D^{-1}$ leads to

$$\frac{\sqrt{\det G'}}{\sqrt{\det G}} = \frac{1}{\det(B + G)} \quad (35)$$

4. DUALITY OF THE HETEROTIC-STRING SPECTRUM

The Heterotic-string action, including the coupling to a background gauge field A_μ^α and a background antisymmetric tensor $B_{\mu\nu}$, $\mu, \nu = 1, \dots, d$, $\alpha = 1, \dots, 16$ is

$$S = \frac{1}{2\pi} \int (G^{\mu\nu} + B^{\mu\nu}) \partial X_\mu \bar{\partial} X_\nu + A^{\alpha\mu} \partial X_\alpha \bar{\partial} X_\mu + (G^{\alpha\beta} + B^{\alpha\beta}) \partial X_\alpha \bar{\partial} X_\beta + (\text{irrelevant terms}) \quad (36)$$

with the constraint $C^\alpha \equiv \partial X^\alpha = 0$

The index α is an internal index. The internal space lives on the weight lattice of $E_8 \times E_8$ or $\text{spin}(32)/\mathbb{Z}_2$. The indices (μ, α) define a $16 + d$ dimensional orthonormal basis : $G_{\mu\nu} \equiv \delta_{\mu\nu}$, $G_{\alpha\beta} \equiv \delta_{\alpha\beta}$.

Due to chirality the geometrical interpretation of the background fields is not manifest. Therefore, we will "embed" the compactified section of the heterotic string (not including the world-sheet fermions) in $16 + d$ space-coordinates of bosonic string. The Lorentzian lattice of the bosonic string is of the form:

$$\begin{pmatrix} P_R \\ P_L \end{pmatrix} = \begin{pmatrix} P_R^i & P_R^I \\ P_L^j & P_L^J \end{pmatrix} \quad i, j = 1, \dots, d \quad I, J = 1, \dots, 16 \quad (37)$$

The embedding is such that the lattice generated by

$$\begin{pmatrix} P_R^i & 0 \\ P_L^j & P_L^J \end{pmatrix}$$

is a sub-lattice of the original one, and gives the spectrum of the heterotic string. This will allow to get the spectrum of the heterotic string from that of the bosonic string by truncation. The sub-spectrum of the dual spectrum is given by a duality transformation of the original sub-spectrum. Thus, all one has to find is the D-matrix of parameters of the relevant bosonic coordinates.

Let $\{E_I^\alpha\}_{I=1,\dots,16}$ $\{e_i^\mu\}_{i=1,\dots,d}$ satisfy

$$\begin{aligned} E_I \cdot E_J &= 2G_{IJ} & e_i \cdot E_I &= 0 \\ e_i \cdot e_j &= 2G_{ij} \end{aligned} \quad (38)$$

where

$$\begin{aligned} G^{\mu\nu} &= 2G_{ij} (e_i^\star)^\mu (e_j^\star)^\nu & G^{\alpha\beta} &= 2G_{IJ} (E_I^\star)^\alpha (E_J^\star)^\beta \\ B^{\mu\nu} &= 2B_{ij} (e_i^\star)^\mu (e_j^\star)^\nu & B^{\alpha\beta} &= 2B_{IJ} (E_I^\star)^\alpha (E_J^\star)^\beta \\ A^{\mu\alpha} &= 2A_{iI} (e_i^\star)^\mu (E_I^\star)^\alpha \end{aligned} \quad (39)$$

$E_I^\star (e_i^\star)$ is the dual basis of $E_I (e_i)$.

The A-field stands for an asymmetric-tensor and a metric in the coefficients D_{iI}, D_{jJ} of the parameter matrix. It is also impossible for D_{IJ} to depend on A, as the gauge field is a continuous parameter, while there are only discrete solutions to D_{IJ} which allow truncation of the spectrum of the $(16 + d, 16 + d)$ model to a chiral $(16 + d, d)$ model.

The operator D of the embedding bosonic string has the form:

$$D \equiv B + G = \begin{pmatrix} (G + B + 1/4 A^K A_K)_{ij} & A_{iI} \\ 0 & (G + B)_{IJ} \end{pmatrix} \quad (40)$$

To prove it, let us represent the basis $\alpha \equiv (\alpha_i, \alpha_I)$ of a lattice Λ_{d+16} with the metric $2G$:

$$\begin{aligned} \alpha_i &= \left(e_i, \frac{1}{2} A_i^K E_K \right) \\ \alpha_I &= (0, E_I) \end{aligned} \quad (41)$$

The dual basis $\alpha^\star$ is

$$\begin{aligned} \alpha_i^\star &= (e_i^\star, 0) \\ \alpha_I^\star &= \left(-\frac{1}{2} A_i^I e_i^\star, E_I^\star \right) \end{aligned} \quad (42)$$

thus, the bosonic lorentzian lattice (P_L, P_R) has the form:

$$\begin{aligned}
 P_R &= [(n)^t + (m)^t (B - G)] (\alpha^*) \\
 &= n_i \alpha_i^* + n_I \alpha_I^* + (m_i, m_I) \begin{pmatrix} (B - G - 1/4 A^K A_K)_{ij} & 0 \\ -A_{jI} & (B - G)_{IJ} \end{pmatrix} \begin{pmatrix} \alpha_j^* \\ \alpha_J^* \end{pmatrix} \\
 &= \left(\left[n_i + m_j (B - G)_{ji} - \frac{1}{4} m_j A_j^K A_{Ki} - \frac{1}{2} (n_J + m_I (B + G)_{IJ}) A_i^J \right] e_i^*, [n_J + m_I (B - G)_{IJ}] E_J^* \right) \quad (43)
 \end{aligned}$$

$(G \pm B)_{IJ}$ is a half cartan matrix valued background metric and torsion of $E_8 \times E_8$, and thus

$(V_J, W_I) \in E_8 \times E_8$ root lattice

where

$$\begin{aligned}
 n_J + m_I (B + G)_{IJ} &\equiv V_J \\
 n_J + m_I (B - G)_{IJ} &\equiv W_J \quad (44)
 \end{aligned}$$

The truncation to a sub-lattice describing the heterotic string spectrum is the set of (n_I, m_I) such that $W_J = 0$.

Recall that

$$\begin{aligned}
 n_i e_i^* &\equiv \frac{1}{2} P \\
 m_j (B - G)_{ji} e_i^* &\equiv -L - BL
 \end{aligned}$$

we get

$$P_R = \left(\frac{1}{2} P - L - BL - \frac{1}{2} VA - \frac{1}{4} A(AL), 0 \right) \quad (45)$$

in the same way

$$P_L = [(n)^t + (m)^t (B + G)] (\alpha^*) = \left(\frac{1}{2} P + L - BL - \frac{1}{2} AV - \frac{1}{4} A(AL), V + AL \right) \quad (46)$$

Thus, the bosonic truncated spectrum coincides with the heterotic model. The A-field stands for a metric

$$\begin{pmatrix} 0 & 1/2 A_{IJ} \\ 1/2 A_{IJ} & 0 \end{pmatrix}$$

and an antisymmetric tensor

$$\begin{pmatrix} 0 & 1/2 A_{IJ} \\ -1/2 A_{IJ} & 0 \end{pmatrix}$$

in the cross directions. It also affects the metric in the compactified space: $G_{ij} \rightarrow G_{ij} + \frac{1}{4} A_i^K A_{Kj}$.

The duality transformation for the bosonic string $D' = D^{-1}$, induces the transformation on the heterotic string background fields:

$$\begin{aligned} A'_{ij} &= - \left(G + B + \frac{1}{4} A^K A_K \right)_{ij}^{-1} A_{jI} (G + B)_{IJ}^{-1} \\ (G' + B')_{ij} &= \left(G + B + \frac{1}{4} A^K A_K \right)_{ij}^{-1} - \frac{1}{4} A_i^K A'_{Kj} \\ (G' + B')_{IJ} &= (G + B)_{IJ}^{-1} \equiv (G + B)_{IJ} \text{ mod } SL_{16}(\mathbb{Z}) \text{ transformation} \end{aligned} \quad (47)$$

A physical fixed point of the dual transformation in the bosonic embedding is such that (G', B') are related to (G, B) by a $SL_{16+d}(\mathbb{Z})$ transformation:

$$G' = M^t G M \quad B' = -M^t B M \quad (48)$$

$$\text{where } M \in SL_{16+d}(\mathbb{Z}) \quad , \quad G \equiv \begin{pmatrix} (G + 1/4 A^K A_K)_{ij} & 1/2 A_{iJ} \\ 1/2 A_{Ij} & G_{IJ} \end{pmatrix} \quad , \quad B \equiv \begin{pmatrix} B_{ij} & 1/2 A_{iJ} \\ -1/2 A_{Ij} & B_{IJ} \end{pmatrix} \quad (49)$$

An extended symmetry in the bosonic embedding is a physical fixed point under duality. It induces an extended gauge symmetry on the embedded heterotic model.

The fact that the gauge field A stands for a metric in the cross terms D_{iI} affects the low energy limit of the model. The antisymmetric tensor does not contribute to low-energy when the compactification radius R is large (G_{ij} is large). Also a non-zero winding excites high-energy states. However, for zero winding, the coupling of the gauge field to the $E_8 \times E_8$ "momenta"

$A_\mu^\alpha V_\alpha$ contributes to the low energy spectrum.

For $R \rightarrow \infty$, $L = 0$:

$$\begin{pmatrix} P_R \\ P_L \end{pmatrix} = \begin{pmatrix} -1/2 VA & , & 0 \\ -1/2 VA & , & V \end{pmatrix} \quad (50)$$

and for $V^2 = 2$:

$$\frac{1}{8} M^2 = \frac{1}{4} (VA)^2 \quad (51)$$

If A is small this is a positive low-energy [13].

5. COMPACTIFICATION FACTORS IN LOW ENERGY EFFECTIVE ACTIONS

There are non-lattice theories that possess duality. Even the three isolated points at $c = 1$ have their own duality-conjugate points. We thus expect duality to be a symmetry of more general validity. In particular the four $c=1$ modular invariant parafermionic models [14] are to be identified with four models in which the string moves in a background of a circle (and its dual) and an orbifold (and its dual) [15]. Backgrounds constructed by using parafermions [16] will thus also possess the duality property. If it will turn out that these are indeed Calabi-Yau spaces [16] then at least some modular invariant models of strings moving on such spaces will have the symmetry. Indeed there is an intimate connection between duality and modular invariance, a symmetry which is believed to be exact.

It is well known [17] that the integrand contains the compactification-dependent factor (we consider $d=1$ for simplicity)

$$f_2(\tau, a) = (\text{Im}\tau)^{1/2} \sum_{n,m} \exp \left[-\pi \text{Im}\tau (a^2 n^2 + m^2/a^2) - 2i\pi \text{Re}\tau n \cdot m \right] \\ a^2 = \alpha'/R^2 \quad (52)$$

for a general G and B background the compactification factor would read, for generic d

$$f_2(\tau, R, G, B) = (\text{Im}\tau)^{d/2} \sum_{n_i, m_j} \left[-\pi \text{Im}\tau \left(\frac{n_i G^{ij} n_j}{R^2} + m_i (G - B G^{-1} B)_{ij} m_j R^2 \right) - 2\pi n_i m_j i \text{Re}\tau (\delta_{ij} + (G^{-1} B)_{ij}) \right] \quad (53)$$

Using the method described in [17] one can prove that this function satisfies:

$$f_2(\tau, R, G, B) = f_2(-1/\tau, R, \tilde{G}, \tilde{B}) \quad (54)$$

but would fail to satisfy modular invariance unless duality holds to give

$$f_2(-1/\tau, R, \tilde{G}, \tilde{B}) = f_2(-1/\tau, R, G, B) \quad (55)$$

There is however a disturbing point still remaining. The complete compactification factor is usually taken to be:

$$F_2 = a^d f_2 \quad (56)$$

The factor a does not affect modular invariance, of course, but spoils duality since it has to be interpreted as a factor:

$$(R^2 \det G)^{-d/2} \quad (57)$$

which is not invariant under the duality transformation. Also, such a factor would imply that, at low energy, the ratio of the loop diagram to the tree is $O(g_{26}^2/V)$ where V is the volume of the compactified space in string units. This makes sense for large volumes, since $g_{26}^2/V = g_{26-d}^2$, but hardly makes sense for small volumes (in string units) because it would imply divergent string loop effects as $V \rightarrow 0$. We expect instead loop effects which are identical under $V \rightarrow V^{-1}$.

Indeed we believe the problem to lie upstream i.e. at the tree level. The internal dimensions usually appear in the effective action describing the low energy phenomena in the uncompactified dimensions as an internal volume factor multiplying the usual Einstein and gauge terms, thus rescaling the Newton and gauge couplings.

Such factor explicitly violates duality, by making the gauge coupling larger and larger as the internal volume is shrunk. It was suggested in ref.[11] that duality could be restored by an appropriate rescaling of the dilaton field but this can be shown not to work either. It is not viable in the presence of torsion and also is inadequate at the torus level (where a constant dilaton field does not enter).

What could then be the correct way of adding compactification effects to the tree and loop effective action? We have no firm answer to this question but can offer an educated guess which does not suffer from the problems mentioned above. At the tree level the factor replacing the usual

$$\begin{aligned} \Phi = \text{tr}(e^{-H}) = \sum \exp [-1/2 [n_a e_{ai}^* G^{ij} e_{bj}^* n_b + m_a e_a^i (G - BG^{-1}B)_{ij} e_b^j m_b \\ + 2m_a e_a^i B_{ik} G^{kj} e_{bj}^* n_b] \end{aligned} \quad (58)$$

which, for large GR^2 , gives back the usual factor

$$\Phi \rightarrow (2\pi R)^d (\det (G/2\pi))^{1/2} = V/(2\pi\alpha')^{d/2} \quad (59)$$

while for small GR^2 gives:

$$\begin{aligned} \Phi \rightarrow (2\pi/R)^d (\det (G - BG^{-1}B) 2\pi)^{-1/2} \\ = (2\pi/R)^d (\det (G/2\pi))^{1/2}/(\det (G + B)) \rightarrow (2\pi\alpha')^{d/2}/R^d \end{aligned} \quad (60)$$

The two expressions coincide of course at the self reciprocal point $R^2 = \alpha'$. Note that the above sphere-compactification factor ϕ is also the zero point function on the torus at a particular value of τ .

The proposed modification of the torus measure is then simply to replace (56) by:

$$F = (2\pi)^{d/2} \Phi^{-1} f_2 \quad (61)$$

which again has the correct large volume limit but obeys duality and has an upper bound reached at the self dual point.

Let us try to give a physical and geometrical interpretation to the proposed modified compactification recipes. The usual compactification factor (59) can be written (apart from numerical coefficients) as:

$$V = \int dx \sqrt{-\det G} \quad ; \quad G = g/\alpha' \quad (62)$$

We may then ask what is in general the meaning of the pure number:

$$V_{R_x} = \int_{R_x} dx \sqrt{-\det G} \quad (63)$$

for any region R_x of space. We shall argue that, for large V , V is just the average number of minimal size strings that can fit into R_x without overlapping.

The above statement follows from an estimate of the minimal volume occupied by a string moving in a constant (or more generally in a slowly-varying) background $G(B = 0$ for the moment). Take as a definition of the average volume:

$$\langle V \rangle^2 = \det_{\mu\nu} \langle : \Delta X^\mu \Delta X^\nu : \rangle \quad (64)$$

where $\Delta X^\mu = X^\mu - q^\mu$, the averaging includes also σ integration and normal ordering removes typical short distance singularities. It is not difficult to see (e.g. by going to a frame where G is diagonal) that either from the form of the action, or from the usual minimum position uncertainty for harmonic oscillators, one gets:

$$\langle : \Delta X^\mu \Delta X^\nu : \rangle \geq (G^{-1})^{\mu\nu} \quad (65)$$

implying:

$$\langle V_{R_x} \rangle \geq (\det G)^{-1/2} \quad (66)$$

Our interpretation of V_{R_x} follows immediately from (66).

What is then the interpretation of $\det(D) = \det(G + B)$? If B is turned on:

$$X'GX' \rightarrow X'(G - BG^{-1}B)X' \quad (67)$$

and thus:

$$\begin{aligned} \langle V_{R_x} \rangle \xrightarrow{B \neq 0} & \geq (\det (G - BG^{-1}B))^{-1/2} = (\det G)^{1/2} (\det (G + B))^{-1/2} \\ & (\det(G - B))^{-1/2} = (\det G)^{1/2} (\det D)^{-1} \end{aligned} \quad (68)$$

We thus see that the quantity D appears naturally in $\langle V_R \rangle$. Yet $\langle V_R \rangle$ is obviously non symmetric under duality:

$$\langle V_{R_x} \rangle \xrightarrow{\text{duality}} (\det G)^{1/2} R^{2d} \quad (69)$$

What is instead invariant under duality is the volume occupied by a string in phase space. Defining $Z = (X', P)$, the form appearing in the ON action is:

$$Z^T M Z \quad ; \quad M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \quad (70)$$

$Z = (P, X')$ has $2d$ components. Calculating the average volume in phase space occupied by a single string, one obtains

$$\langle V_{\text{phase space}} \rangle \approx (\det M)^{-1/2} \quad (71)$$

Since

$$M = \begin{pmatrix} G^{-1} & 0 \\ BG^{-1} & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G - BG^{-1}B \end{pmatrix} \begin{pmatrix} G^{-1} & -G^{-1}B \\ 0 & 1 \end{pmatrix} \quad (72)$$

One finds $\det M = (\det G^{-1})^2 \det (G + B) \det (G - B)$ i.e.

$$\langle V_{\text{phase space}} \rangle \geq (\det G) (\det (G + B))^{-1/2} (\det (G - B))^{-1/2} \quad (73)$$

This form is duality invariant and can be expressed in two additional forms. The first is:

$$\sqrt{\det M} = 2 \det (D) / \det (D + D^\dagger) \quad (74)$$

From this form it is clear that the operator D contains the information on the phase space volume. For $B=0$ this would be a fixed number. The other possibility is

$$\sqrt{\det M} = \sqrt{\det G} \sqrt{\det G'} \quad (75)$$

The phase space volume occupied by the string is the geometrical mean of the target space volume occupied by a string moving in a background G and that of the duality related model. In target space concepts, the target space phase space, is the mean of the appropriate target space volumes. Again for $B=0$ this has a constant value. Imagine that somehow the Z_2 symmetry is broken by giving either G or G' a fixed expectation value: in that case the phase space volume will reduce to the familiar target space volume for the dual value.

Presumably, our compactification factor, or perhaps a suitable improvement of it, should be related to the average number of strings fitting into compactified phase (rather than configuration) space.

Acknowledgment

E.R. wishes to thank the CERN theory division for its hospitality.

References

1. D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575 and in "Vertex operators in Mathematics and Physics", MSRI publication (Springer 1985), comm. Math. Phys. **107** (1986) 535.
2. A. Cappelli, C. Itzykson and J.B. Zuber, Nucl. Phys. **B280** [FS18] (1987) 445.
J.L. Cardy, Nucl. Phys. **B270** [FS16] (1986) 186.
D. Gepner, Nucl. Phys. **B287** (1987) 111.
3. K. Bardakci, E. Rabinovici and B. Sring, Nucl. Phys. **B299** (1988) 151.
4. R. Dijkgraaf, E. Verlinde and H. Verlinde, "C = 1 conformal field theories on riemann surfaces", Utrecht preprint THU-87/17 (1987).
5. P. Ginsparg, Nucl. Phys. **B295** [FS21] (1988) 153.
6. S. Elitzur, E. Gross, E. Rabinovici and N. Seiberg, Nucl. Phys. **B283** (1987) 413.
7. see for example: N. Sakai and I. Senda, Prog. Theor. Phys. **75** (1986) 357.
8. V.P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. **B287** (1987) 402.
9. K.S. Narain, Phys. Lett. **169B** (1986) 41.
10. K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. **B279** (1987) 369.
11. P. Ginsparg and C. Vafa, Nucl. Phys. **B289** (1987) 414.
12. For review: N. Seiberg, preprint WIS-87/42/June-PH.
13. K.S. Narain and M.H. Sarmadi, Phys. Lett. **B184** (1987) 165.
14. D. Gepner and Z. Qiu, Nucl. Phys. **B285** [FS19] 423.
15. E. Rabinovici and B. Sring. unpublished.
16. D. Gepner, Nucl. Phys. **B296** (1988) 732, Phys. Lett. **199B** (1987) 380.
17. See for example : M.B. Green, J.H. Schwartz and E. Witten, Superstring Theory, Cambridge University Press 1987.

FIGURE CAPTIONS

Figure 1. Duality of $c = 1$ conformal models

Figure 2. Duality on the $c = 2$ section : $D = \begin{pmatrix} g & b \\ -b & g \end{pmatrix}$

- Figure 1 -

- Figure 2 -