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FINITENESS AND DECOUPLING
IN OPEN STRING THEORIES

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ABSTRACT

Quantum consistency of open string theories is considered at the one-loop level, using a unified and consistent regularisation of all divergences. Finiteness of all amplitudes with open string states is shown to be possible for any $SO(n)$ or $USp(n)$ gauge group. Two consequences of the non-unitarity of tachyonic theories are discussed; the presence of unphysical cuts in amplitudes and the non-decoupling of null states. If not for contributions of the open string tachyon, decoupling of all null open bosonic string states would be obtained for $SO(2^{D/2})$ independently of the regularisation, corresponding to vanishing tadpoles of unphysical closed string states.

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I. INTRODUCTION

Ironically, the intense research activity in string theory, which followed the discovery of anomaly cancellation in the Type I open superstring with gauge group $SO(32)$ [1], has mostly been concerned with theories of closed strings [2,3,4]. It is widely believed that quantum consistency of those theories, i.e. their finiteness (modulo the problem of mass renormalisation for massive states) and their unitarity (thus also the decoupling of all zero-norm states) is a direct consequence of their modular invariance (to all orders of string perturbation theory) and the requirement of a tachyon free spectrum [2,4,5,6].

Open string theories have not been developed to the same extent. Two general classes are known [7]; in 26 dimensions, the bosonic theories with gauge groups $U(n)$ for oriented strings and $SO(n)$ or $USp(n)$ for unoriented ones, and in 10 dimensions, the Type I supersymmetric theories of unoriented strings with gauge groups $SO(n)$ or $USp(n)$. Only recently have there been attempts at formulating open string theories in less than 10 dimensions [8].

Quantum consistency of those theories however, restricts the choice. Gauge anomaly cancellation (i.e. decoupling of longitudinal gauge bosons) in Type I superstrings holds for the gauge group $SO(2^5)$ only [1,9,10,11]. In open bosonic strings, a necessary condition for decoupling of all zero-norm states is that the gauge group be $SO(2^{13})$ [12,13], but tachyonic states make such a decoupling impossible [14].

Contrary to theories of closed strings, those results do not seem to follow from modular invariance; such a symmetry does not exist in the present formulation of open string theories, where the gauge degrees of freedom are introduced through Chan-Paton factors [7]. The results actually follow from a cancellation mechanism between different world-sheet topologies contributing at the one-loop level, namely the planar and Möbius strip diagrams. In general, each of those contributions are infinite (with possible exceptions for superstrings [10,15]) and have to be regulated, although the condition for the decoupling of zero-norm states is independent of any specific regulator chosen [1,9,10,14,15,16].

A similar type of cancellation mechanism is responsible for the possible finiteness of open string theories [17]. A specific choice of regularisation can be made such that Type I superstrings are (one-loop) finite for $SO(2^5)$ [13,17,18,19,20,21] and bosonic strings are (one-loop) finite for $SO(2^{13})$ [14,22,23,24]. However, with different choices of regulators, open

string theories can be made finite for any $SO(n)$ and even any $USp(n)$ gauge group [14,20,21].

The specific parametrisation and regularisation of amplitudes which make them finite for the same gauge group $SO(2^{D/2})$ as required by decoupling of zero-norm states, points to deeper underlying significance for the appearance of this gauge group than that of mere numerical coincidence [25]. The double covering of the corresponding annulus and Möbius strip topologies lead to the same torus, i.e. a torus with the same complex structure [21,25]. The open string world-sheet may then be viewed as a two dimensional $\mathbb{Z}_2$ - orbifold constructed from a closed string world-sheet, with the open string states emitted from the fixed points of the involution defining the orbifold [26]. It is claimed [26] that such a construction uniquely leads to an open string theory with gauge group $SO(2^{D/2})$ [26,27]. Clearly, such an approach to open string theories offers many advantages. It puts those theories within the very framework which has proved so powerful in the case of closed string theories. Modular invariance (and the absence of tachyons) should suffice to guarantee full quantum consistency of the corresponding open string theories [26]. Moreover, the possibility of constructing new consistent open string theories in less than ten dimensions, using the techniques developed for the corresponding closed string theories, becomes tenable [14].

This paper gives a detailed account of the results presented in Ref.14. Its main motivation is to reanalyse, through a careful discussion of the relevant divergences, the quantum consistency of open string theories, particularly the bosonic ones, for the gauge group $SO(2^{D/2})$ only. The point of view of Ref.26 and its generalisations to lower dimensions should thus lead to all possible consistent open string theories [8,14].

Finiteness and decoupling in Type I superstrings are well understood, at least for amplitudes of massless bosonic states. We shall mostly concentrate on open bosonic string theories. In this case, the presence of tachyons leads to additional divergences which seem not to have been considered properly so far. Here, we propose a method to regularise these divergences, and justify the use of this method in the context of string theory.

With such an understanding at hand, we reconsider the problem of decoupling of all zero-norm open string states, following the usual operator formulation of one-loop amplitudes. This was first discussed in Ref.12, within a different approach to string amplitudes. No special attention, however, was given to the issue of regularisation, and some important points were overlooked.

All regularisations used in this paper stem from the following simple remark [11]. Consider the proper-time representation of the propagator and of the (one-loop) vacuum amplitude for a scalar particle of mass m :

$$\frac{-i}{p^2 + m^2} = -i \int_0^\infty dt e^{-(p^2 + m^2)t} = \frac{-i}{2} \int_0^1 \frac{dw}{w} w^{1/2(p^2 + m^2)}, \quad (1.1)$$

$$(2\pi)^D \delta_{(0)}^{(D)} \int \frac{d^D p}{(2\pi)^D} \frac{-i}{p^2 + m^2} = \delta_{(0)}^{(D)} \frac{-i}{2} \int_0^1 \frac{dw}{w} \left(\frac{-2\pi}{\ln w} \right)^{D/2} w^{1/2 m^2}. \quad (1.2)$$

The r.h.s. of (1.1) is a correct integral representation of the l.h.s. only if $(p^2 + m^2) > 0$. If $(p^2 + m^2) \leq 0$, the proper-time representation diverges at $w \rightarrow 0$ (or $t \rightarrow \infty$, i.e. at large distances) and is thus inappropriate. The correct result can nevertheless be obtained through analytic continuation in $(p^2 + m^2)$. In practice, given a parameter γ such that $\gamma + 1/2(p^2 + m^2) > 0$, a factor w^γ is introduced in the integrand of (1.1), leading to a finite quantity. The correct propagator is obtained through analytic continuation in γ to $\gamma = 0$, but only after the complete evaluation of the proper-time integration (with $\gamma + 1/2(p^2 + m^2) > 0$).

Similarly, the proper-time representation of the vacuum amplitude in (1.2) reproduces the correct result only if $m^2 \geq 0$, with the usual short-distance divergence at $w \rightarrow 1$. If $m^2 < 0$, the additional divergent but unphysical contribution from $w \rightarrow 0$ can be subtracted away by analytic continuation in m^2 . Again, this is most easily done by introducing a factor w^γ in the integrand of (1.2), with $\gamma + 1/2 m^2 \geq 0$, and analytically continuing to $\gamma = 0$ only in the final result of the integration. Only then is the correct result for the l.h.s. of (1.2) obtained.

This type of regularisation of otherwise ill-defined integrals is already used in the context of string theory. In effect, it has been used [6,11,23,28] in the analysis of tadpole divergences, of non-planar diagrams and in the so-called cancelled propagator argument. Here, its use is extended [14] to the regularisation of divergences due to the open string tachyon, which contributes in the loop channel of amplitudes. We shall discuss why such a procedure is indeed justified in string theory.

The outline of the paper is as follows. In Section II, we discuss the construction of one-loop open bosonic string amplitudes within the covariant operator formalism and give their general structure. Section III considers all different sources of divergences of these amplitudes. A unified and consistent treatment of their regularisation is discussed, and then used to analyse the effects due to tachyons in bosonic theories as well as the conditions required for

finiteness of these amplitudes. In Section IV the problem of decoupling of all zero-norm open string states is addressed, with the conclusion that tachyonic states render such a decoupling impossible [14], although, by ignoring their contribution, decoupling would require the gauge group $SO(2^{13})$; a result given in Ref.12. Conclusions are presented in Section V. Whereas Appendix A discusses the construction of one-loop amplitudes using Feynman's tree theorem [29], Appendix B provides examples of and definitions for amplitudes whilst Appendix C establishes the consistency of the regularisation method used in this paper.

II. ONE-LOOP OPEN STRING AMPLITUDES

String amplitudes are given as integrated world-sheet correlation functions of vertex operators which are in a one-to-one correspondence with physical states [30,31]. Let us briefly review this relationship in the case of open bosonic strings [32].

The open string Fock space is the representation space for the algebra of string zero-modes and oscillators:

$$[q^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0} \quad (2.1)$$

(in the notations of Ref.33, with $2\alpha' = 1$). Any state is thus given through some linear combination of the form

$$|\phi, k\rangle = \sum c_{\mu_1 \dots \mu_m} \alpha_{-n_1}^{\mu_1} \dots \alpha_{-n_m}^{\mu_m} |0, k\rangle, \quad (2.2)$$

where the Fock ground-state is defined by

$$p^\mu |0, k\rangle = k^\mu |0, k\rangle, \quad \alpha_n^\mu |0, k\rangle = 0 \quad \text{for } n \geq 1, \quad (2.3)$$

$$\langle 0, k | 0, l \rangle = \delta^{(D)}(k-l).$$

Acting on the space of two-dimensional world-sheet field operators $O(z)$, one defines [32] operators $\hat{\alpha}_n^\mu$ by

$$(\hat{\alpha}_n^\mu O)(z) = \oint_z \frac{dx}{2i\pi} (x-z)^n R(i\partial_x X^\mu(x) O(z)), \quad (2.4)$$

where R denotes R -ordering, and

$$X^\mu(z) = q^\mu - i p^\mu \ln z + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{n} \alpha_n^\mu z^{-n}. \quad (2.5)$$

It is easy to show that

$$[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = n \eta^{\mu\nu} \delta_{n+m,0} \quad (2.6)$$

and

$$n \geq 1 : \quad \hat{\alpha}_n^\mu : e^{ikX(z)} : = 0 \quad (2.7a)$$

$$n=0 : \quad \hat{\alpha}_0^\mu : e^{ikX(z)} : = k^\mu : e^{ikX(z)} : \quad (2.7b)$$

$$n \geq 1 : \quad \hat{\alpha}_{-n}^\mu : e^{ikX(z)} : = : \frac{i}{(n-1)!} \partial_z^n X^\mu(z) e^{ikX(z)} : \quad (2.7c)$$

This establishes a one-to-one correspondence [32] between states in the Fock space and world-sheet operators. Corresponding to any state of the form (2.2), we have the vertex operator

$$\phi(z) = \sum_{\mu_1 \dots \mu_m} C_{\mu_1 \dots \mu_m} \hat{\alpha}_{-n_1}^{\mu_1} \dots \hat{\alpha}_{-n_m}^{\mu_m} : e^{ikX(z)} : = \sum_{\mu_1 \dots \mu_m} C_{\mu_1 \dots \mu_m} \prod_{j=1}^m \frac{i}{(n_j-1)!} \partial_z^{n_j} X^{\mu_j}(z) e^{ikX(z)} : \quad (2.8a)$$

with the relation

$$\lim_{z \rightarrow 0} \phi(z) |0;0\rangle = |\phi; k\rangle. \quad (2.8b)$$

Physical states are defined by the Virasoro gauge conditions

$$(L_0 - \Delta) |\phi; k\rangle = 0, \quad L_n |\phi; k\rangle = 0 \quad \text{for } n \geq 1, \quad (2.9)$$

where

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{n-m}^\mu \alpha_{m\mu} : = \oint_0 \frac{dx}{2i\pi} x^{n+1} : \frac{1}{2} (i \partial_x X^\mu(x))^2 : \quad (2.10)$$

By analogy, one introduces the operators [32]

$$\hat{L}_n \equiv \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \hat{\alpha}_{n-m}^\mu \hat{\alpha}_{m\mu} : = R \oint_z \frac{dx}{2i\pi} (x-z)^{n+1} : \frac{1}{2} (i \partial_x X^\mu(x))^2 : \quad (2.11)$$

Clearly, the operators $\hat{L}_n$ satisfy the same Virasoro algebra as the operators L_n .

Finally, primary conformal fields of conformal weight Δ are defined by the conditions

$$[L_n, \phi(z)] = z^{n+1} \partial_z \phi(z) + \Delta(n+1) z^n \phi(z), \quad n \in \mathbb{Z} \quad (2.12a)$$

or equivalently [32]

$$\hat{L}_{-1} \phi(z) = \partial_z \phi(z), \quad \hat{L}_0 \phi(z) = \Delta \phi(z), \quad \hat{L}_n \phi(z) = 0 \quad \text{for } n \geq 1. \quad (2.12b)$$

The important result [32] can now be stated as follows: any open string state $|\phi; k\rangle$, given in (2.2), is a physical state if and only if the corresponding vertex operator $\phi(z)$, given as in (2.8), is a primary conformal field of conformal weight Δ .

Unitarity of the theory requires that no physical state be of strictly negative norm. The no-ghost theorem gives the necessary conditions to avoid such a situation [34,35]. We must have either $D = 26$ and $\Delta = 1$, or $D \leq 25$ and $\Delta \leq 1$. The pole structure of loop amplitudes however, requires $D = 26$ [36], and thus also $\Delta = 1$. For the critical values of D and Δ , the no-ghost theorem implies more precisely [35] that any physical state $|\phi; k\rangle$ can be written as a superposition of a strictly positive norm state and a null state. More specifically, we then have:

$$|\phi; k\rangle = |\phi_0; k\rangle + L_{-1} |\chi_1; k\rangle + (L_{-2} + \frac{3}{2} L_{-1}^2) |\chi_2; k\rangle \quad (2.13a)$$

with

$$L_0 |\chi_1; k\rangle = 0, \quad L_n |\chi_1; k\rangle = 0 \quad \text{for } n \geq 1, \quad (2.13b)$$

$$(L_0 + 1) |\chi_2; k\rangle = 0, \quad L_n |\chi_2; k\rangle = 0 \quad \text{for } n \geq 1, \quad (2.13c)$$

and $|\phi_0; k\rangle$ a transverse state (i.e. an on-shell DDF state, see Appendix A). The corresponding vertex operator is thus given by

$$\phi(z) = \phi_0(z) + \hat{L}_{-1} \chi_1(z) + (\hat{L}_{-2} + \frac{3}{2} \hat{L}_{-1}^2) \chi_2(z) \quad (2.14)$$

with $\phi_0(z)$, $\chi_1(z)$ and $\chi_2(z)$ the operators associated to the states $|\phi_0; k\rangle$, $|\chi_1; k\rangle$ and $|\chi_2; k\rangle$ respectively. Note that these are primary conformal fields of conformal weight $\Delta = 1$, $\Delta = 0$ and $\Delta = -1$ respectively.

As an example, let us give the physical open string states and vertices at the first three excitation levels:

$$N=0: \quad |\phi; k\rangle = |0; k\rangle, \quad \phi(z) = :e^{ikX(z)}: \quad (2.15)$$

with $k^2 = 2$,

$$N=1: \quad a) |\phi; k\rangle = \epsilon_\mu \alpha_{-1}^\mu |0; k\rangle, \quad \phi(z) = :i\epsilon_\mu \partial_z X^\mu(z) e^{ikX(z)}: \quad (2.16a)$$

with $k^2 = 0$, $\epsilon \cdot k = 0$, $\epsilon^2 = 1$,

$$b) |\phi; k\rangle = L_{-1} |0; k\rangle, \quad \phi(z) = \hat{L}_{-1} :e^{ikX(z)}: \quad (2.16b)$$

with $k^2 = 0$,

$$N=2: \quad a) |\phi; k\rangle = \frac{1}{\sqrt{2}} \lambda_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu |0; k\rangle, \quad \phi(z) = \frac{1}{\sqrt{2}} \lambda_{\mu\nu} :i\partial_z X^\mu(z) i\partial_z X^\nu(z) e^{ikX(z)}: \quad (2.17a)$$

with $k^2 = -2$, $\lambda_{\mu\nu} - \lambda_{\nu\mu} = \lambda_\mu^\mu = k^\mu \lambda_{\mu\nu} = 0$, $\lambda_{\mu\nu} \lambda^{\mu\nu} = 1$,

$$b) |\phi; k\rangle = L_{-1} \epsilon_\mu \alpha_{-1}^\mu |0; k\rangle, \quad \phi(z) = \hat{L}_{-1} :i\epsilon_\mu \partial_z X^\mu(z) e^{ikX(z)}: \quad (2.17b)$$

with $k^2 = -2$, $\epsilon \cdot k = 0$, $\epsilon^2 = 1$,

$$c) |\phi; k\rangle = (L_{-2} + \frac{3}{2} L_{-1}^2) |0; k\rangle, \quad \phi(z) = (\hat{L}_{-2} + \frac{3}{2} \hat{L}_{-1}^2) :e^{ikX(z)}: \quad (2.17c)$$

with $k^2 = -2$.

Clearly, the states (2.15), (2.16a) and (2.17a) have strictly positive norm, whereas all others are null states.

Amplitudes of physical states are obtained as correlation functions of the corresponding vertex operators, integrated over their positions on the boundary of the appropriate Riemann surface. Factors such as $\partial_z^2 X^\mu(z)$ in (2.8a) can be included in the vertex $: \exp(ikX(z)) :$ as $: \exp(ik_\mu X^\mu(z) + i\eta_\mu \partial_z^2 X^\mu(z)) :$; taking the appropriate terms in an expansion in the auxiliary vectors η_μ , the corresponding amplitude is obtained. In general, it is given by the correlation function of the vertices $: \exp(ikX(z)) :$, multiplied by combinations of various derivatives of the correlation function of the scalar field $X^\mu(z)$ on the Riemann surface. For example, any amplitude involving states at the first three excitation levels only can easily be obtained by using the vertex $: \exp(ikX(z) + i\eta \partial_z^2 X(z)) :$ (see Appendix B).

In the present formulation of open string theories, gauge degrees of freedom are introduced through Chan-Paton factors [7], i.e. through matrices λ_α associated to each physical state. Their contribution to amplitudes is given by the trace of their product in the same cyclic order as that of the corresponding states on the boundaries of the world-sheet. Factorisation of tree amplitudes restricts the choice of these matrices [7]; for oriented strings one can only have $n \times n$ complex matrices, and for unoriented strings one can only have $n \times n$ real or $n \times n$ quaternionic matrices. In each case, the Chan-Paton matrices λ_α must

be anti-Hermitian for even mass levels and Hermitian for odd mass levels. These three possibilities lead respectively to gauge symmetries associated with the groups $U(n)$, $SO(n)$ and $USp(n)$.

Through the relation (2.8b), we choose to normalise the vertices $\phi(z)$ for (positive norm) physical states by requiring that the corresponding states $|\phi; k\rangle$ have norm (+1). We also impose that the Chan-Paton factors satisfy $\text{tr } \lambda_a^2 = \pm 1$. For a process with M external states of momenta k_i^μ ($i=1, \dots, M$), the S matrix element is given by

$$S(k_1, \dots, k_M) = \frac{1}{\pi} (2\pi)^{-\frac{(D-1)}{2}} (2|k_i^0|)^{-1/2} A(k_1, \dots, k_M), \quad (2.18)$$

where $A(k_1, \dots, k_M)$ is the corresponding amplitude, obtained in perturbation theory through a summation over different world-sheet topologies.

For quantum consistency of the theory, zero-norm states must decouple from any amplitude, i.e. amplitudes with external zero-norm states must vanish, and such states as well as all other unphysical states cannot contribute to pole and cut singularities of physical amplitudes. At tree level, this is readily satisfied. We then have for the fully normalized amplitude [6,37]

$$A^{(0)}(k_1, \dots, k_M) = (2\pi)^D 2ig^{M-2} \frac{1}{M} \int_{\mathbb{R}} \prod_{i=1}^M d\bar{z}_i \delta(\bar{z}_a - \bar{z}_a^0) \delta(\bar{z}_b - \bar{z}_b^0) \delta(\bar{z}_c - \bar{z}_c^0) \cdot (\bar{z}_a - \bar{z}_b)(\bar{z}_b - \bar{z}_c)(\bar{z}_c - \bar{z}_a) \cdot \langle 0; 0 | \text{tr } R \left(\prod_{i=1}^M \phi_i(\bar{z}_i) \right) | 0; 0 \rangle, \quad (2.19)$$

where g is the open string coupling constant, $\phi_i(z_i)$ are the vertices corresponding to the external open string physical states, "tr" stands for the trace over the Chan-Paton factors, and the factor $\delta(\bar{z}_a - \bar{z}_a^0) \delta(\bar{z}_b - \bar{z}_b^0) \delta(\bar{z}_c - \bar{z}_c^0) (\bar{z}_a - \bar{z}_b)(\bar{z}_b - \bar{z}_c)(\bar{z}_c - \bar{z}_a)$ (with $\bar{z}_a, \bar{z}_b, \bar{z}_c$ any three of the $\bar{z}_i$'s and $\bar{z}_a^0, \bar{z}_b^0, \bar{z}_c^0$ arbitrary real values) is the gauge-fixing factor for the three real conformal Killing vectors of the disk [6]. In the multiperipheral representation, this reduces to:

$$A^{(0)}(k_1, \dots, k_M) = (2\pi)^D 2ig^{M-2} \sum_{\sigma} \text{tr } \lambda_{\sigma(M)} \dots \lambda_{\sigma(1)} \cdot \langle \phi_{\sigma(M)}^- | \phi_{\sigma(M-1)}^{(1)} \frac{1}{L_0 - 1} \dots \frac{1}{L_0 - 1} \phi_{\sigma(2)}^{(1)} | \phi_{\sigma(1)} \rangle, \quad (2.20)$$

where the summation is taken over all cyclically inequivalent permutations σ of the external states, and

$$| \phi_{\sigma(1)} \rangle = \lim_{\bar{z} \rightarrow 0} \phi_{\sigma(1)}(\bar{z}) | 0; 0 \rangle, \quad \langle \phi_{\sigma(M)}^- | = \lim_{\bar{z} \rightarrow \infty} \langle 0; 0 | \phi_{\sigma(M)}(\bar{z}) \bar{z}^2. \quad (2.21)$$

The form (2.20) is the most convenient in establishing [6] that only strictly positive norm physical states couple to tree amplitudes and to pole singularities due to intermediate states. The multiperipheral representation also has a simple physical interpretation. As the in-state $|\phi_\sigma(1)\rangle$ propagates, it absorbs the states $|\phi_\sigma(2)\rangle, |\phi_\sigma(3)\rangle, \dots$ at one of its end points ($z=1$ corresponds to $\sigma=0$ with $z = \exp(\tau + i\sigma)$), until it has transformed into the out-state $|\phi_\sigma(M)\rangle$.

The amplitude $A^{(0)}$ also admits other representations, corresponding to open string states being emitted from both string end points (i.e. with $\phi_i(1)$ or $\phi_i(-1)$). It can be shown [37] that for any such twisted contribution [6], decoupling of (external zero-norm or intermediate unphysical) states does not occur, but that by summing over all twisted contributions corresponding to a given cyclic ordering of the external states on the boundary of the disk, the corresponding multiperipheral contribution is obtained. A similar situation appears for one-loop non-planar and non-orientable diagrams, as is discussed in Appendix A.

At the one-loop level, two different world-sheet topologies appear, namely the cylinder (or orientable) and the Möbius strip (or non-orientable) contributions. The fully normalised amplitude $A^{(1)}(k_1, \dots, k_M)$ is then given by the sum of the corresponding two expressions [37]:

$$A_o^{(1)}(k_1, \dots, k_M) = \frac{-1}{2} i g^M C_o \frac{1}{M} \int_0^1 \frac{dw}{w} \frac{1}{w} f^2(w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_\alpha| \delta(z_\alpha - z_\alpha^0) \cdot \text{Tr } R \left[\prod_{i=1}^M \phi_i(z_i) w^{L_0} \right], \quad (2.22)$$

and

$$A_N^{(1)}(k_1, \dots, k_M) = \frac{-1}{2} i g^M C_N \frac{1}{M} \int_0^1 \frac{dw}{w} \frac{1}{w} f^2(-w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_\alpha| \delta(z_\alpha - z_\alpha^0) \cdot \text{Tr } R \left[\prod_{i=1}^M \phi_i(z_i) w^{L_0} (-1)^N \right], \quad (2.23)$$

where

$$f(w) = \prod_{n=1}^{\infty} (1 - w^n), \quad \Omega(w) = [-1, -w] \cup [w, 1], \quad (2.24)$$

and the coefficients C_o and C_N are given in Table 1. In these expressions, "Tr" now stands for the trace over the zero-modes P^μ , the oscillators α_n^μ and the Chan-Paton factors, and the factor $|z_\alpha| \delta(z_\alpha - z_\alpha^0)$ (with z_α any of the z_i 's and z_α^0 an arbitrary point in $\Omega(w)$) is the gauge-fixing factor for the real conformal Killing vector of the cylinder or of the Möbius strip.

Eq.(2.22) represents the integration over the two boundaries $[-1, -w]$ and $[w, 1]$ of a cylinder defined by $(w \leq |z| \leq 1, \text{Im}z \geq 0)$ and the equivalence $z \sim wz$. Eq.(2.23) represents the integration over the single boundary $\Omega(w)$ of a Möbius strip defined by $(w \leq |z| \leq 1, \text{Im}z \geq 0)$ and the equivalence $z \sim -wz$. The factors w^{L_0} and $w^{L_0(-1)^N}$ in the traces ensure that these amplitudes indeed correspond to such topologies.

The relative sign of C_0 and C_N for $SO(n)$ and $USp(n)$ bosonic strings is opposite [23,24,37] to that of the corresponding Type I superstrings [6]. This is obviously important for the discussion of decoupling of external zero-norm states, and was overlooked in Ref.12.

The integration measures $w^{-1}f^2(\pm w)$ over the modular parameter w of the cylinder or the Möbius strip are unambiguously fixed [38] by the requirement of decoupling of all zero-norm and unphysical intermediate states propagating in the loop, and the correct factorisation into tree-level amplitudes. This is achieved through Feynman's tree theorem [29] and is discussed in Appendix A.

The evaluation of expressions like (2.22) and (2.23) has become standard [6]. We only sketch some of the results. Examples and definitions are given in Appendix B. The integrals in (2.22) and (2.23) take the general form (up to Chan-Paton factors):

$$\delta^{(D)}\left(\sum_{i=1}^M k_i\right) \int_0^1 \frac{dw}{w} \frac{1}{w} \left(\frac{-2\pi}{\ln w}\right)^{\frac{D}{2}} f^2(\pm w) \int_{I_1} \prod_{i=1}^M \frac{d|z_i|}{|z_i|} |z_a| \delta(z_a - z_a^0) \cdot \prod_{i=1}^M |z_i|^{1-\frac{1}{2}k_i^2} \prod_{1 \leq i < j \leq M} \left[\psi\left(\frac{|z_i|}{|z_j|}, w\right) \right]^{k_i k_j} \cdot \left\{ N_i \text{ derivatives in } \frac{\partial}{\partial z_i} \right\} \quad (2.25)$$

Here, I_1 is some region in $[w, 1]^M$, such that $|z_{i+1}| > |z_i|$ ($i=1, \dots, M$) with z_i the position of the vertex $\phi_i(z_i)$ associated to the state of momentum k_i . The total contribution is given by summing over all such R-orderings of the vertices. The function $\psi(|z_i|/|z_j|, w)$ is one of the functions $\psi_P(|z_i|/|z_j|, w)$, $\psi_T(|z_i|/|z_j|, w)$ or $\psi_N(|z_i|/|z_j|, w)$ (see Appendix B), depending on whether, respectively, the vertices ϕ_i and ϕ_j are on the same boundary of the cylinder, on different boundaries of it, or on the single boundary of the Möbius strip. The term in brackets is a combination of products of derivatives of the scalar propagator $\ln \psi(x, w)$, such that there is a total of N_i derivatives in z_i , with $N_i = 1 - \frac{1}{2}k_i^2$. Finally, the partition function $f(w)$ (resp. $f(-w)$) has to be used for the cylinder (resp. Möbius strip).

For orientable contributions, with the change of variables

$$\lambda = \exp\left(\frac{4\pi^2}{\ln w}\right), \quad \gamma_i = \frac{\ln |z_i|}{\ln w}, \quad i=1, \dots, M, \quad (2.26)$$

Eq.(2.25) becomes

$$\delta^{(D)}\left(\sum_{i=1}^M k_i\right) \frac{1}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{D-26}{24} \frac{4\pi^2}{\ln \lambda}\right) \lambda^{\frac{2-D}{24}} f^{2-D}(\lambda) \int_{I_2} \prod_{i=1}^M dv_i \delta(v_i - v_a^0) \cdot \prod_{1 \leq i < j \leq M} \left[\frac{\vartheta(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\vartheta'_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot \left\{ N_i \text{ derivatives in } \frac{-\partial}{\partial v_i} \right\} \quad (2.27)$$

where I_2 is some integration region in $[0, 1]^M$ such that $v_i > v_{i+1}$ ($i=1, \dots, M$), with v_1 associated to the vertex ϕ_1 , and $\vartheta(v_i - v_j | \frac{\ln \lambda}{2i\pi})$ is the θ -function $\theta_1(v|\tau)$ or $\theta_4(v|\tau)$ (see Appendix B) depending on whether the vertices ϕ_i and ϕ_j are on the same boundary or on opposite boundaries of the cylinder. The term in brackets is a combination of products of derivatives in $(\frac{-\partial}{\partial v_i})$ of $\ln(\vartheta(v_i - v_j | \frac{\ln \lambda}{2i\pi}) / \vartheta'_1(0 | \frac{\ln \lambda}{2i\pi}))$, with a total of N_i such derivatives. Note the well-known logarithmic branch cut in λ which appears when $D = 26$ [36].

Summing over all possible R-orderings of the vertices, one then obtains the planar and the non-planar contributions. The former, with all vertices attached to the same boundary of the cylinder, is given as a sum over all cyclically inequivalent permutations of the external states, with each term of the form

$$-i g^M \delta^{(26)}\left(\sum_{i=1}^M k_i\right) C_0 \frac{n \operatorname{tr}(\prod_{i \in A} \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} f^{24}(\lambda) \int_{R_1} \prod_{i=1}^M dv_i \delta(v_i) \cdot \prod_{1 \leq i < j \leq M} \left[\frac{\vartheta_1(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\vartheta'_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot \left\{ N_i \text{ derivatives in } \frac{-\partial}{\partial v_i} \right\} \quad (2.28)$$

Here, the integration region R_1 is defined by $0 \leq v_M \leq v_{M-1} \leq \dots \leq v_1 \leq 1$, corresponding to the cyclic ordering of the vertices on the boundary.

The non-planar contribution, with the vertices split into two sets A and B corresponding to the two boundaries of the cylinder, is given as a sum over all such splits, with cyclic permutations of A and B or within A or within B not counted as different splits, with each term of the form

$$-i g^M \delta^{(26)}\left(\sum_{i=1}^M k_i\right) C_0 \frac{\operatorname{tr}(\prod_{i \in A} \lambda_a^{(i)}) \operatorname{tr}(\prod_{j \in B} \lambda_a^{(j)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} f^{24}(\lambda) \int_{R_2} \prod_{i=1}^M dv_i \delta(v_i) \cdot \prod_{1 \leq i < j \leq M} \left[\frac{\vartheta_{1 \text{ or } 4}(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\vartheta'_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot \left\{ N_i \text{ derivatives in } \frac{-\partial}{\partial v_i} \right\} \quad (2.29)$$

Here, the integration region R_2 is defined by $0 \leq v_{MA}^A \leq v_{MA-1}^A \leq \dots \leq v_1^A \leq 1$ and $0 \leq v_{MB}^B \leq v_{MB-1}^B \leq \dots \leq v_1^B \leq 1$, with v_1^A and v_1^B corresponding to the vertices in the sets A and B defined above, with their respective cyclic orderings.

Similarly for the Möbius strip, by the change of variables

$$\lambda = \exp\left(\frac{\pi^2}{\ln w}\right), \quad v_i = \frac{\ln |z_i|}{2 \ln w}, \quad i=1, \dots, M, \quad (2.30)$$

the non-orientable contribution is given as a sum over all cyclically inequivalent permutations of the external lines, with each term of the form

$$-ig^M \delta\left(\sum_{i=1}^{(26)} k_i\right) C_M \frac{2^{13} \text{tr}(T_i^T \lambda_\alpha^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} f(-\lambda) \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \cdot \prod_{1 \leq i < j \leq M} \left[\frac{\partial_1(v_i - v_j | \frac{\ln(-\lambda)}{2i\pi})}{\partial_1'(0 | \frac{\ln(-\lambda)}{2i\pi})} \right]^{k_i k_j} \cdot \left\{ N_i \text{ derivatives in } \frac{\partial}{\partial v_i} \right\} \quad (2.31)$$

Here, the integration region R_1 is the same as in (2.28).

It must be emphasised that the expressions (2.28), (2.29) and (2.31) are valid for any one-loop amplitude with any external physical open string states. The remarkable thing about the planar and non-orientable contributions is that they only differ by a factor $2^{13} C_N / n C_0$ and the replacement $(\lambda \rightarrow -\lambda)$ in the integrands. This is the very fact which makes finiteness and decoupling of null states possible in open string theories (at least at the one-loop level), and is very suggestive indeed of a much deeper explanation [25,26].

The origin of the factor 2^{13} can be traced back to the trace over the bosonic zero modes - or the transformation relating $f(\pm w)$ to $f(\pm \lambda)$. In the case of open superstrings, the expressions are essentially the same; the factors $\lambda^{-1} f^{-24}(\pm \lambda)$ then do not appear, and the analogous factor 2^5 (which has been established for amplitudes of massless gauge bosons only [9,18,19,20,21]) can presumably be traced back to the trace over the fermionic zero-modes. Hence, the considerations below should also apply to those theories.

III. DIVERGENCES AND FINITENESS

All amplitudes obtained above have divergences of different origin and physical meaning; let us consider them in turn, and discuss their regularisation.

Since $\theta_1(v|\tau)$ vanishes in v , the integrands in (2.28), (2.29) and (2.31) have divergences whenever some vertices approach each other [6,21,39]. This is possible only for vertices attached to the same boundary. Let us assume that these vertices $\phi_k(z)$ are labelled by $i \leq k \leq j$ with $i < j$, and let us scale the successive differences $(v_k - v_{k+1})$ ($k=i, \dots, j$) by a common factor ϵ , so that the singularities arise from the limit $\epsilon \rightarrow 0$. From the general form of amplitudes above, it is easy to show that the singularity is then given by:

$$\sum_{n=0}^{\infty} C_n \int_0^{\infty} \frac{d\epsilon}{\epsilon} \epsilon^{\frac{1}{2} S_{ij} - 1 + n} , \quad (3.1a)$$

where

$$S_{ij} = \left(\sum_{l=i}^j k_l \right)^2 . \quad (3.1b)$$

This expression corresponds to the proper-time representation (see (1.1)) of propagators of particles of invariant momentum s_{ij} and of masses $2(n-1)$, i.e. the open string spectrum. Clearly, it corresponds to the pinching off from the boundary of the cylinder or of the Möbius strip, of a disk to which the vertices $\phi_k(z)$ ($i \leq k \leq j$) are attached. The intermediate open string states in (3.1) propagate through the strip attaching this disk to the remainder of the diagram.

As discussed in the introduction, this makes sense only if $\frac{1}{2} S_{ij} - 1 + n > 0$. In most cases, this can be achieved by analytic continuation in s_{ij} [6,21,39], but there are instances where s_{ij} is fixed by kinematics [21,39]. Hence, the correct contribution from the coincidence limit of vertices is obtained in general as follows:

$$\lim_{\beta \rightarrow 0} \sum_{n=0}^{\infty} C_n \int_0^{\infty} \frac{d\epsilon}{\epsilon} \epsilon^{\beta} \epsilon^{\frac{1}{2} S_{ij} - 1 + n} , \quad (3.2a)$$

with

$$\beta > 1 - \frac{1}{2} S_{ij} . \quad (3.2b)$$

The limit $\beta \rightarrow 0$ is taken only after the complete evaluation of the integrals. The only divergences are then genuine physical intermediate pole singularities.

Kinematics fixes the value of s_{ij} if all vertices but one are coming together [39]. In such a case, in the limit $\beta \rightarrow 0$ of (3.2), there is always a term diverging as $1/\beta$. This is because the vertex which is not approaching the other vertices corresponds to an on-shell intermediate state contributing to (3.2). Such a divergence has to be shifted away through mass renormalisation [39,40].

The integrands in (2.28), (2.29) and (2.31) have divergences in the region $\lambda \rightarrow 0$ [6,21,23,28]. Here, non-planar contributions have a different structure compared to the other two types of amplitudes, since $\theta_4(v|\frac{\ln \lambda}{2i\pi})$ diverges as $\lambda^{-1/8}$ whereas $\theta_1(v|\frac{\ln \lambda}{2i\pi})$ remains finite. From the general analysis in the previous section, a non-planar amplitude has the structure

$$\int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} (2\pi\lambda^{1/8})^{s_{AB}} \sum_{n=0}^{\infty} G_n \lambda^n, \quad (3.3a)$$

where

$$s_{AB} = \left(\sum_{i \in A} k_i \right)^2 = \left(\sum_{i \in B} k_i \right)^2 \quad (3.3b)$$

is the invariant momentum exchanged between the boundaries of the cylinder. This is the proper-time representation of propagators of particles of invariant momentum s_{AB} and of masses $8(n-1)$, i.e. the closed string spectrum. Clearly, it corresponds to the propagation of closed string states in the closed string channel of the amplitude [41].

The representation (3.3) makes sense only if $1/8 s_{AB} - 1 + n > 0$. Hence, the correct contribution from the region $\lambda \rightarrow 0$ is obtained by analytic continuation in s_{AB} [6,21,23,28], or more generally by the same procedure as in (3.2), namely

$$\lim_{\alpha \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \lambda^\alpha \frac{1}{\lambda} (2\pi\lambda^{1/8})^{s_{AB}} \sum_{n=0}^{\infty} G_n \lambda^n, \quad (3.4)$$

where $\alpha > 1 - 1/8 s_{AB}$, and the limit $\alpha \rightarrow 0$ is not taken inside the integral.

Similarly, the structure of planar and non-orientable amplitudes takes the form

$$\int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} \sum_{n=0}^{\infty} F_n \lambda^n. \quad (3.5)$$

Again, this is the proper-time representation of propagators of closed string states at zero-momentum. For this interpretation to make sense, (3.5) has to be regulated by

$$\lim_{\alpha \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \lambda^\alpha \frac{1}{\lambda} \sum_{n=0}^{\infty} F_n \lambda^n \quad (3.6)$$

with $\alpha > 1$ (and the limit $\alpha \rightarrow 0$ not taken in the integral). In analogy with (3.3), the quantity (8α) can be viewed as a fictitious momentum inserted in the closed string channel [23,28].

Finally, the integrands of (2.28), (2.29) and (2.31) diverge exponentially as $\lambda \rightarrow 1$, because of the factors $\Gamma^{-24}(\pm\lambda)$. This region also corresponds to the limit $w \rightarrow 0$, which represents the large distance limit in the open string loop channel. In that representation, the divergence at $w \rightarrow 0$ clearly follows from the open string tachyon, see (2.25) [11,12]. This is due to the proper-time representation of loop amplitudes, inappropriate for tachyonic states [11,12]. As suggested in the introduction, this problem may conveniently be avoided by inserting a factor w^γ in the integration, with $\gamma \geq 1$, so that the infinite but unphysical contribution at $w \rightarrow 0$ is subtracted away.

When performed in a tree channel, the regularisations considered above do not lead to problems; see for example Appendix C for a discussion concerning the limit $\lambda \rightarrow 0$, which corresponds to the closed string channel. For loop channels however, as is the case here for the limit $w \rightarrow 0$, inserting a factor w^γ may introduce cut singularities due to unphysical and/or zero-norm intermediate states, thus breaking the unitarity of the theory. Therefore, when using such a regularisation of the unphysical divergence at $w \rightarrow 0$, one has to build the amplitudes in such a way as to subtract away all contributions from unphysical states propagating in the loop channel.

At the one-loop level, this can be done using Feynman's tree theorem. As is discussed in Appendix C, all additional contributions due to unphysical cuts cancel when the dependence on γ is analytically continued to $\gamma = 0$, so that the expressions (2.22) and (2.23) with a factor w^γ included still lead to proper amplitudes in this limit. Hence, this regularisation procedure of unphysical singularities due to tachyons in the loop channel is consistent for string theories.

Let us emphasize that such a procedure does not break any of the symmetries of the system. Indeed, all amplitudes are still invariant under world-sheet reparametrisations induced by the Virasoro generators, since the vertex operators are primary conformal fields of conformal weight $\Delta = 1$, even for $\gamma \neq 0$. Introducing a factor w^γ does not amount to a shift in the mass

spectrum, since the condition $L_0 = 1$ is still imposed on physical states. The sole role of the factor w^γ is to subtract away in the integrals the divergent but unphysical contribution from $w = 0$.

With the proper regularisation of the different singularities analysed above, let us reconsider the general structure of one-loop amplitudes. For non-planar amplitudes, we have

$$\lim_{\alpha, \gamma \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{4\pi^2 \gamma}{\ln \lambda}\right) \lambda^\alpha \frac{1}{\lambda} (2\pi \lambda^{1/8})^{s_{AB}} \sum_{n=0}^{\infty} G_n \lambda^n, \quad (3.7)$$

where $\alpha > 1 - 1/8$, s_{AB} is the regulator for $\lambda \rightarrow 0$, $\gamma \geq 1$ is the regulator for $\lambda \rightarrow 1$, and the coefficients G_n are given by integration over the vertex insertions v_i , with the understanding that they are obtained through the regularisation discussed in (3.2). Completing the evaluation of (3.7), one obtains

$$\lim_{\gamma \rightarrow 0} (2\pi)^{s_{AB}} \sum_{n=0}^{\infty} G_n \frac{1}{\lambda^{1/8 s_{AB} + n - 1}} K\left[16\pi^2 \gamma \left(\frac{1}{8} s_{AB} + n - 1\right)\right], \quad (3.8)$$

with the Bessel function

$$K(z) = z K_1(z) = 1 + 2 \sum_{k=1}^{\infty} \frac{(z/2)^{2k}}{(k-1)! k!} \left[\ln \frac{z}{2} - \frac{1}{2} \psi(k) - \frac{1}{2} \psi(k+1) \right]. \quad (3.9)$$

As expected, (3.8) has poles at $s_{AB} = 8(n-1)$ ($n=0,1,\dots$) corresponding to closed string states in the tree channel of the amplitude [41]. If the singularity at $\lambda \rightarrow 1$ due to the open string tachyon was absent, the limit $\gamma \rightarrow 0$ could be taken inside the infinite series, and these poles would be the only singularities of the amplitude. However, the limit $\gamma \rightarrow 0$ can be taken only after the series has been resummed. Hence, (3.8) has logarithmic branch cuts if $(1/8 s_{AB} + n - 1) < 0$, an unavoidable consequence of the open string tachyon. Clearly, this is one manifestation of the non-unitarity of tachyonic quantum theories.

Consider now planar and non-orientable amplitudes. We have respectively the general structure

$$\lim_{\alpha_1, \gamma_1 \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{4\pi^2 \gamma_1}{\ln \lambda}\right) \lambda^{\alpha_1} \frac{1}{\lambda} \sum_{n=0}^{\infty} F_n \lambda^n, \quad (3.10a)$$

$$\lim_{\alpha_2, \gamma_2 \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{\pi^2 \gamma_2}{\ln \lambda}\right) \lambda^{\alpha_2} \frac{1}{\lambda} \sum_{n=0}^{\infty} F_n (-\lambda)^n, \quad (3.10b)$$

with $\alpha_{1,2} > 1$ and $\gamma_{1,2} \geq 1$ the corresponding regulators for the regions $\lambda \rightarrow 0$ and $\lambda \rightarrow 1$ respectively, and F_n are obtained by appropriately regulated integrations over the vertex insertions. Once again let us stress that the same coefficients F_n appear in those two amplitudes.

The evaluation of (3.10) leads respectively to

$$\lim_{\alpha_1, \gamma_1 \rightarrow 0} \sum_{n=0}^{\infty} F_n \frac{1}{\alpha_1 + n - 1} K[16\pi^2 \gamma_1 (\alpha_1 + n - 1)] , \quad (3.11a)$$

$$\lim_{\alpha_2, \gamma_2 \rightarrow 0} \sum_{n=0}^{\infty} (-1)^n F_n \frac{1}{\alpha_2 + n - 1} K[4\pi^2 \gamma_2 (\alpha_2 + n - 1)] , \quad (3.11b)$$

These quantities have a pole singularity for $n = 1$, corresponding to the tadpole of the closed string dilaton [22,23]. All other closed string tadpoles give a finite result as $\alpha_{1,2} \rightarrow 0$, with the appearance of a logarithmic branch cut in the contribution of the closed string tachyon due to the open string tachyon. Again, the limits $\gamma_{1,2} \rightarrow 0$ cannot be taken inside the infinite summations.

Combining the results (3.11) with the corresponding normalisation of the amplitudes, the dilaton tadpole singularity can be made to cancel between planar and non-orientable contributions if the gauge group is $SO(n)$ or $USp(n)$, with

$$\lim_{\alpha_1, \alpha_2 \rightarrow 0} \frac{\alpha_1}{\alpha_2} = \frac{C_0}{C_N} 2^{-13} n \quad (3.12)$$

If indeed open string theories can consistently be formulated from an underlying $\mathbb{Z}_2$ - orbifold structure, planar and non-orientable amplitudes are related and the invariant momenta $8\alpha_1$ and $8\alpha_2$ inserted in the closed string channel should be identical. For such a choice, open bosonic string theories are one-loop finite for $SO(2^{13})$ only, a result established so far for M tachyon amplitudes only [23].

However, within the present formulation, any choice for α_1 and α_2 is a priori acceptable. For example, taking $4\alpha_1 = \alpha_2$ [20] the $SO(2^{11})$ open bosonic string is made finite, and generally taking α_1 and α_2 proportional any $SO(n)$ open bosonic string can be made finite [20]. Even more generally, taking α_1 and α_2 to depend on a parameter t such that

$$\lim_{t \rightarrow 0} \alpha_1(t) = 0 , \quad \lim_{t \rightarrow 0} \alpha_2(t) = 0 , \quad (3.13)$$

with a value t_0 such that $\alpha_{1,2}(t_0) > 1$, open bosonic string theories can be made finite for any $SO(n)$ or $USp(n)$ gauge group [14].

At this stage any choice for α_1 and α_2 is acceptable. However, decoupling of null states leads to a unique specification of the gauge group. One then has to make a choice of regularisation which is consistent with this additional requirement.

Most of the discussion above also applies to Type I open superstrings [6,21]. The corresponding amplitudes have singularities in the coincidence limits of vertices and in the limit $\lambda \rightarrow 0$, which may be regulated by the techniques discussed above. The absence of tachyons in those theories, however, leads to simplifications. Since there is no divergence due to an open string tachyon in the loop channel, there is no need for a factor w^γ in the integrands and the associated logarithmic branch cuts in (3.8) and (3.11) are absent. Furthermore, the dilaton tadpole divergence in planar and non-orientable amplitudes now follows from the leading term F_0 in (3.10), so that the condition for finiteness of one-loop amplitudes becomes

$$\lim_{\alpha_1, \alpha_2 \rightarrow 0} \frac{\alpha_1}{\alpha_2} = - \frac{C_0}{C_N} 2^{-5} n . \quad (3.14)$$

However, the sign of C_0/C_N is opposite [6] to that for bosonic strings for $SO(n)$ and $USp(n)$ gauge groups. Hence, the discussion and the conclusions are the same as above. In particular, taking $\alpha_1 = \alpha_2$, Type I superstrings are finite for $SO(2^5)$ [17 - 21], but other choices are also possible at this stage [14,20].

In the case of open superstrings, other regularisations of the limit $\lambda \rightarrow 0$ have also been used, such as Pauli-Villars regularisation [1,6]. In particular, Pauli-Villars masses and coefficients can be chosen such that Type I superstrings are made finite for any $SO(n)$ gauge group [20]. Since such a regularisation inserts factors w^{m^2} in the amplitudes and then takes the limit $m^2 \rightarrow \infty$, what happens of the additional terms required to cancel the unphysical cuts which are then induced remains unclear. Moreover, in the case of open bosonic strings, it can be shown that, because of the closed string tachyon (i.e. the factor $1/\lambda$ in (2.28), (2.29) and (2.31)), there does not exist a choice of Pauli-Villars masses and coefficients such that the amplitude is well defined as $\lambda \rightarrow 0$.

Not only do the regularisations used here avoid these problems, but they also follow from the simple remarks discussed in the introduction and can consistently be used in any string theory, with the advantage of a clear physical interpretation.

IV. DECOUPLING OF NULL STATES

So far, decoupling of intermediate states in the loop channel of amplitudes has played an essential role in our discussion, especially in the determination of the modular integration measure through Feynman's tree theorem. Unitarity however, also requires that zero-norm external states decouple from amplitudes. Having developed a consistent framework for the regularisation of open string amplitudes, the above problem can be approached coherently. Indeed, its analysis usually follows from a delicate cancellation between divergent quantities [1,6,9,10,11].

As recalled in section II, any zero-norm physical open string state (of definite "spin") takes the form

$$L_{-1} |\chi_1; k\rangle, \text{ with } L_0 |\chi_1; k\rangle = 0, L_n |\chi_1; k\rangle = 0, n \geq 1, \quad (4.1)$$

or

$$(L_{-2} + \frac{3}{2} L_{-1}^2) |\chi_2; k\rangle, \text{ with } (L_0 + 1) |\chi_2; k\rangle = 0, L_n |\chi_2; k\rangle = 0, n \geq 1, \quad (4.2)$$

with the corresponding vertex operators

$$\phi(z) = \hat{L}_{-1} \chi_1(z) = \partial_z \chi_1(z), \quad (4.3)$$

$$\phi(z) = (\hat{L}_{-2} + \frac{3}{2} \hat{L}_{-1}^2) \chi_2(z), \quad (4.4)$$

where $\chi_1(z)$ and $\chi_2(z)$ are primary conformal fields of conformal weight $\Delta = 0$ and $\Delta = -1$ respectively.

Amplitudes with null states of the form (4.1) are given by surface terms in the corresponding vertex insertions. By appropriate regularisations of the associated coincidence limits of vertices (see (3.2)), such terms always vanish, and the corresponding null states always decouple.

For states of the form (4.2), the situation is radically different [12]. For a primary conformal field $\chi_2(z)$, we have

$$\hat{L}_{-1}^2 \chi_2(z) = \partial_z^2 \chi_2(z), \quad (4.5)$$

and from (2.11)

$$\hat{L}_{-2} \chi_2(z) = \oint_z \frac{dx}{2i\pi} \frac{1}{(x-z)} R \left(: \frac{1}{2} (i\partial_x X^\mu(x))^2 : \chi_2(z) \right) \quad (4.6)$$

Using relations (2.10) and (2.12a), obtaining the following identity is rather straightforward:

$$\begin{aligned} & \text{Tr } R \left[\left(\hat{L}_{-2} + \frac{3}{2} \hat{L}_{-1}^2 \right) \chi_2(z_M) \phi_{M-1}(z_{M-1}) \dots \phi_1(z_1) w^{L_0} \right] \\ &= \left(\frac{1}{w} f^2(w) \right)^{-1} w \frac{d}{dw} \left[\frac{1}{w} f^2(w) \frac{1}{z_M^2} \text{Tr} \right] \\ &+ \sum_{j=1}^{M-1} \frac{\partial}{\partial z_j} \left\{ \left(-z_j \frac{\partial}{\partial z_j} \ln \Phi \left(\frac{|z_j|}{|z_M|}, w \right) \right) \frac{z_j}{z_M^2} \text{Tr} - \frac{3}{2} \frac{\partial}{\partial z_M} \left(\frac{z_j}{z_M} \text{Tr} \right) \right\} \\ &+ \sum_{j=1}^{M-1} (\Delta_j - 1) \left\{ \frac{\text{Tr}}{z_M^2} \left(1 + z_j \frac{\partial}{\partial z_j} \right) \left(-z_j \frac{\partial}{\partial z_j} \ln \Phi \left(\frac{|z_j|}{|z_M|}, w \right) \right) - \frac{3}{2} \frac{\partial}{\partial z_M} \left(\frac{\text{Tr}}{z_M} \right) \right\} \\ &+ (\Delta_M + 1) \left\{ -\frac{3}{2} \frac{\text{Tr}}{z_M^2} - w \frac{d}{dw} \ln \left[\frac{1}{w} f^2(w) \right] \frac{\text{Tr}}{z_M^2} - \frac{3}{2} \frac{\partial}{\partial z_M} \left(\frac{\text{Tr}}{z_M} \right) \right\} \end{aligned} \quad (4.7)$$

with

$$\text{Tr} \equiv \text{Tr } R \left[\chi_2(z_M) \phi_{M-1}(z_{M-1}) \dots \phi_1(z_1) w^{L_0} \right] \quad (4.8)$$

and $\phi_i(z_i)$ ($i=1, \dots, M-1$) primary conformal fields of conformal weight Δ_i , and $\chi_2(z_M)$ a primary conformal field of conformal weight Δ_M . We have

$$\Phi \left(\frac{|z_i|}{|z_j|}, w \right) = \exp \left(- \frac{\ln^2 \frac{|z_i|}{|z_j|}}{2 \ln w} \right) \psi \left(\frac{|z_i|}{|z_j|}, w \right) \quad (4.9)$$

with $\psi(|z_i|/|z_j|, w)$ being either $\psi_P(|z_i|/|z_j|, w)$ or $\psi_T(|z_i|/|z_j|, w)$ (see Appendix B), depending on whether the vertices labelled by i and j are on the same or on a different boundary of the cylinder.

A similar result holds for non-orientable amplitudes, with the twist operator $(-1)^N$ inserted in the traces. The only differences are that $f(w)$ becomes $f(-w)$ and that $\psi(x, w)$ in (4.9) is now the function $\psi_N(x, w)$. Note the analogy of these expressions with the corresponding results at higher loops for the closed bosonic string in Ref.42.

For physical states, we have $\Delta_i = 1$ ($i=1, \dots, M-1$) and $\Delta_M = -1$. In addition, by appropriate regularisation of the coincidence limits, the surface terms in z_i ($i=1, \dots, M-1$) in (4.7) do not contribute to the amplitudes. Hence, the expressions (2.22) and (2.23) reduce to

$$A_0^{(0)}(k_1, \dots, k_M) = \frac{-1}{2} i g^M C_0 \frac{1}{M} \int_0^1 \frac{dw}{w} w \frac{d}{dw} \left\{ \frac{1}{w} f^2(w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_a| \delta(z_a - z_a^0) \cdot \text{Tr } R[\chi_2(z_M) \phi_{M-1}(z_{M-1}) \dots \phi_1(z_1) w^{L_0}] \right\}, \quad (4.10)$$

and

$$A_N^{(1)}(k_1, \dots, k_M) = \frac{-1}{2} i g^M C_N \frac{1}{M} \int_0^1 \frac{dw}{w} w \frac{d}{dw} \left\{ \frac{1}{w} f^2(-w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_a| \delta(z_a - z_a^0) \cdot \text{Tr } R[\chi_2(z_M) \phi_{M-1}(z_{M-1}) \dots \phi_1(z_1) w^{L_0} (-1)^N] \right\}. \quad (4.11)$$

Amplitudes with null states thus always reduce to surface terms in modular space. This result is also obtained in Ref.12 using a completely different approach. It is expected on general grounds [31,42], and is also known to be valid for the gauge anomaly of Type I superstrings [9]. Note that for this to be the case, the modular integration measure must be exactly the one which follows from Feynman's tree theorem, as explained in Appendix A. Reversing the argument, one can determine [12] this integration measure by requiring that amplitudes with zero-norm states be given by surface terms in modular parameters, without relying on Feynman's tree theorem.

Performing the changes of variables given in (2.26) and (2.30), the amplitude reduces to sums over different cyclic orderings of the vertices, as in (2.28), (2.29) and (2.31), of terms of the form

$$-i g^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) C_0 \frac{\text{tr}(\prod_{i \in A} \lambda_a^{(i)}) \text{tr}(\prod_{i \in B} \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \lambda^\alpha \exp\left(\frac{4\pi^2 \delta}{\ln \lambda}\right) \cdot (-4\pi^2 \lambda \frac{d}{d\lambda}) \cdot \left\{ \frac{1}{\lambda} f^2(\lambda) \int_{R_2} \prod_{i=1}^M dv_i \delta(v_M) \prod_{1 \leq i < j \leq M} \left[\frac{\mathcal{V}_1(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\mathcal{V}'_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}] \right\}, \quad (4.12)$$

$$-i g^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) C_0 \frac{n \text{tr}(\prod_{i \in A} \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\alpha_1} \exp\left(\frac{4\pi^2 \delta_1}{\ln \lambda}\right) (-4\pi^2 \lambda \frac{d}{d\lambda}) \cdot \left\{ \frac{1}{\lambda} f^2(\lambda) \cdot \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \prod_{1 \leq i < j \leq M} \left[\frac{\mathcal{V}_1(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\mathcal{V}'_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}] \right\}, \quad (4.13)$$

$$-i g^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) C_N \frac{2^{13} \text{tr}(\prod_{i \in A} \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\alpha_2} \exp\left(\frac{\pi^2 \delta_2}{\ln \lambda}\right) (-4\pi^2 \lambda \frac{d}{d\lambda}) \cdot \left\{ \frac{1}{\lambda} f^2(-\lambda) \cdot \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \prod_{1 \leq i < j \leq M} \left[\frac{\mathcal{V}_1(v_i - v_j | \frac{\ln(-\lambda)}{2i\pi})}{\mathcal{V}'_1(0 | \frac{\ln(-\lambda)}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}] \right\}, \quad (4.14)$$

respectively for non-planar, planar and non-orientable contributions. The notations are as in section II, and the necessary regularisation factors λ^α and w^γ have already been included, in the notations of section III, with the understanding that the coincidence limits of vertices are also appropriately regulated. We have:

$$N_i = \Delta_i - \frac{1}{2} \frac{p_i^2}{k_i^2}, \quad i=1, \dots, M, \quad (4.15)$$

with $\Delta_i = 1$ ($i=1, \dots, M-1$) and $\Delta_M = -1$. Note that the gauge-fixing of the conformal Killing vector has been performed in terms of z_M , since all surface terms in z_i ($i=1, \dots, M-1$) have been discarded.

The general structure of non-planar amplitudes is thus

$$\lim_{\alpha, \gamma \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \lambda^\alpha \exp\left(\frac{4\pi^2 \gamma}{\ln \lambda}\right) (-4\pi^2 \lambda \frac{d}{d\lambda}) \left[\frac{1}{\lambda} (2\pi \lambda^{1/8})^{S_{AB}} \sum_{n=0}^{\infty} G_n \lambda^n \right], \quad (4.16)$$

with $\alpha > 1 - 1/8 S_{AB}$ and $\gamma \geq 1$. This reduces to

$$\lim_{\gamma \rightarrow 0} (-4\pi^2) (2\pi)^{S_{AB}} \sum_{n=0}^{\infty} G_n K\left[16\pi^2 \gamma \left(\frac{1}{8} S_{AB} + n - 1\right)\right]. \quad (4.17)$$

a contribution entirely due to the open string tachyon. It has logarithmic branch cuts as S_{AB} varies, and is not vanishing. This result however is not an artifact of our regularisation of the open string tachyon contribution, since our approach preserves the symmetries of the amplitudes. Hence, tachyonic theories not only have amplitudes with unphysical analytic structures, but in addition, zero-norm states do not decouple in such theories. Here we have yet another consequence of the breakdown of unitarity in these theories.

In the case of Type I superstrings, the regulator w^γ is not necessary, and the non-planar amplitudes with zero-norm states then vanish identically [1,6]. The contribution from the surface term at $\lambda = 1$ can be shown to vanish, whereas that from $\lambda = 0$ vanishes by analytic continuation in α .

For planar and non-orientable amplitudes, we have respectively

$$\lim_{\alpha_1, \gamma_1 \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\alpha_1} \exp\left(\frac{4\pi^2 \gamma_1}{\ln \lambda}\right) (-4\pi^2 \lambda \frac{d}{d\lambda}) \left[\frac{1}{\lambda} \sum_{n=0}^{\infty} F_n \lambda^n \right], \quad (4.18a)$$

$$\lim_{\alpha_2, \gamma_2 \rightarrow 0} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\alpha_2} \exp\left(\frac{\pi^2 \gamma_2}{\ln \lambda}\right) (-4\pi^2 \lambda \frac{d}{d\lambda}) \left[\frac{1}{\lambda} \sum_{n=0}^{\infty} F_n (-\lambda)^n \right], \quad (4.18b)$$

with $\alpha_{1,2} > 1$, $\gamma_{1,2} \geq 1$, and the same coefficients F_n in both contributions.

These integrals lead respectively to

$$4\pi^2 F_1 + \lim_{\delta_1 \rightarrow 0} (-4\pi^2) \sum_{n=0}^{\infty} F_n K[16\pi^2 \delta_1 (n-1)] , \quad (4.19a)$$

$$-4\pi^2 F_1 + \lim_{\delta_2 \rightarrow 0} (-4\pi^2) \sum_{n=0}^{\infty} (-1)^n F_n K[4\pi^2 \delta_2 (n-1)] . \quad (4.19b)$$

The second terms in these expressions are entirely due to the open string tachyon, as in (4.17). Note their logarithmic branch cut for $n = 0$. On the other hand, the first terms correspond to a genuine anomaly, whose value is independent of any of the regulators and of the existence of tachyons. It follows from a tadpole of a closed string state at excitation level $N = 1$. However, summing these terms for planar and non-orientable amplitudes, their total contribution is given by

$$-i g^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) 2\pi C_0 \left(n - 2^{13} \frac{C_N}{C_0} \right) \sum_{\text{cyclic}} \text{tr} \left(T_i \lambda_a^{(i)} \right) F_1 , \quad (4.20)$$

where $\sum_{\text{cyclic}}$ stands for the summation over all cyclically inequivalent permutations of the M external states. This quantity cancels for the gauge group $SO(2^{13})$. Indeed, the conclusion was first reached in Ref.12, although the relative signs of C_0 and C_N quoted there are not the correct ones for bosonic strings. Moreover, the fact that zero-norm states cannot decouple, even in non-planar amplitudes was also overlooked in Ref.12.

The tadpole leading to (4.20) cannot be that of a physical closed string state, since it appears in the unphysical one-loop amplitude of $(M-1)$ physical states (those of vertices $\phi_i(z_i)$, $i=1, \dots, M-1$ in (4.10) and (4.11)) and an off-shell physical state, of vertex $\chi_2(z)$ (with $\Delta = -1$). Therefore, the anomalous term (4.20) cannot be cancelled through a Fischer-Susskind mechanism [43]; it can only cancel by the choice of the gauge group $SO(2^{13})$.

The same considerations should apply to Type I superstrings although only massless bosonic amplitudes have been analysed in that case. As stated above, none of the terms in (4.17) and (4.19) following from the open string tachyon will appear; only a term analogous to (4.20) will contribute. Since there is no closed string tachyon, it will now be proportional to F_0 , with a factor

$$C_0 \left(n + 2^5 \frac{C_N}{C_0} \right) F_0 . \quad (4.21)$$

The change in sign of the term in C_0/C_N is again compensated by the change in sign of C_0/C_N for $SO(n)$ and $USp(n)$ gauge groups, so that decoupling of all zero-norm states, and not only longitudinal gauge bosons, should hold for $SO(2^5)$ only. Note that F_0 is again a tadpole of an unphysical closed string state, as was explicitly shown in the case of the hexagonal gauge anomaly [11]. F_0 is also related to an unphysical tree level amplitude on the disk [1,6].

V. CONCLUSIONS

In this paper, we have analysed the general structure of one-loop open string amplitudes, concentrating mainly on open bosonic strings [14]. We proposed a unified treatment of all divergences in these amplitudes, based on the simple remark that the proper-time representation of a propagator needs proper analytic continuation in some situations, and, by using Feynman's tree theorem [29], we proved the consistency of this approach in the context of string theory. More specifically, we included a proper treatment of divergences due to tachyons in loop channels, a feature which seems not to have been addressed before. This approach to the regularisation of string amplitudes should be applicable to any string theory. It also has a straightforward physical interpretation, and preserves the symmetries of the theory.

Bearing these considerations in mind, we showed that open string tachyons lead to unphysical logarithmic cuts in one-loop amplitudes with any number of any open string physical states. Herein lies one manifestation of the non-unitarity of tachyonic theories. We also discussed how, by appropriate regularisations of planar and non-orientable amplitudes, open string theories can be made finite for any $SO(n)$ or $USp(n)$ gauge group [14,20], although the group $SO(2^{D/2})$ certainly seems to distinguish itself from the others [17,18,19,21,23]. Let us remark here that any of those open string theories could also be made finite by the Fischler-Susskind mechanism [43].

We then addressed the question of decoupling of all zero-norm open string states in bosonic strings, a problem also considered in Ref.12. We showed that the corresponding amplitudes, when not vanishing, are given by two types of terms [14]. The first type, when summed over planar and non-orientable diagrams, follows from the tadpole of an unphysical closed string state and vanishes only for the gauge group $SO(2^{D/2})$; this result is independent of any of the regularisation parameters introduced in the discussion and of the existence

of tachyons. The second type of terms follows directly from the open string tachyon. These terms do not vanish, even for non-planar amplitudes. Our regularisation procedure being consistent with local world-sheet reparametrisation invariance, these terms are not an artifact of the regularisation but are rather yet another consequence of the non-unitarity of tachyonic theories. Moreover, trying to compensate for these terms by some counterterm would certainly spoil the correct factorisation of one-loop amplitudes. Indeed, the expressions (2.22) and (2.23) were uniquely determined from such a requirement. The cancellation of terms of the first type was indeed given in Ref.12, but that work overlooked the second type of terms by lacking a careful discussion of regularisation. To conclude then, decoupling of null states is not realised in tachyonic open string theories. An analogous situation is to be expected in the case of tachyonic closed string theories, where the same kind of regularisation of tachyon divergences should be applicable.

Similar considerations should likewise apply to Type I superstrings where decoupling of all zero-norm states, including fermionic ones, should require the gauge group $SO(2^5)$, as is the case for decoupling of longitudinal gauge bosons [1,6,9,10,11].

Open string theories with gauge group $SO(2^{D/2})$ are thus singled out from the consideration of their quantum consistency, i.e. finiteness and unitarity. This certainly points to something deeper than merely a numerical coincidence [21,25]. Indeed, it has been suggested [26] that these theories can be obtained through a $\mathbb{Z}_2$ - orbifold construction in the world-sheet. Hence, these theories are placed within the same framework as closed string theories, thus opening the way for the construction of four-dimensional open string theories [8,14].

Finiteness and decoupling are therefore expected to be direct consequences of modular invariance, as is the case for closed string theories. Decoupling of all zero-norm states, including those of the closed string sector [44], a problem not addressed in this paper, should follow from such considerations.

Appendix A

Let us consider a null vector $k_0^\mu = (k_0^0, \vec{k}_0)$, normalised by the condition

$$k_0 k_0^* = -1, \quad \text{with} \quad k_0^{*\mu} = (k_0^0, -\vec{k}_0), \quad (\text{A.1})$$

and a set of $(D-2)$ transverse vectors $t^{(i)\mu}$ ($i=1, \dots, D-2$) such that

$$k_0 t^{(i)} = 0 = k_0^* t^{(i)}, \quad t^{(i)} t^{(j)} = \delta^{ij}, \quad i, j = 1, \dots, D-2. \quad (\text{A.2})$$

One then defines the DDF operators [45] by

$$A_m^i(k_0) = \oint_0 \frac{dz}{2i\pi} : i t^{(i)} \partial_z X(z) e^{-im \frac{k_0 X(z)}{P^+}} : , \quad m \in \mathbb{Z} \quad (\text{A.3})$$

where $P^+ \equiv -k_0 P$ (this is a slight generalisation of the usual definition). The DDF operators are well defined on the subspace of $\mathcal{F}(k_0)$ of the full Fock space, of all states of momentum p^μ such that $p^+ = -k_0 p \neq 0$. They satisfy the properties [6,45]:

$$A_m^{i+} = A_{-m}^i, \quad A_0^i = t^{(i)} p \equiv p^i, \quad [p^\mu, A_n^i] = -n k_0^\mu A_n^i \frac{1}{p^+}, \quad (\text{A.4a})$$

$$[A_n^i, A_m^j] = n \delta^{ij} \delta_{n+m,0}, \quad (\text{A.4b})$$

$$[L_n, A_m^i] = 0, \quad [N, A_n^i] = -n A_n^i, \quad (\text{A.4c})$$

$$A_n^i |0; p\rangle = 0, \quad n \geq 1. \quad (\text{A.4d})$$

The subspace of $\mathcal{F}(k_0)$ of DDF states [45] is generated by the states

$$\prod_{n=1}^{\infty} \prod_{i=1}^{D-2} (A_{-n}^i)^{\beta_{i,n}} |0; p - k_0 \frac{N'_0}{p^+}\rangle \quad (\text{A.5a})$$

where

$$N'_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} n \beta_{i,n}, \quad p^+ = -k_0 p \neq 0, \quad \beta_{i,n} \in \mathbb{N}. \quad (\text{A.5b})$$

From the properties above, DDF states have strictly positive norm and satisfy the off-shell conditions $L_n = 0$ ($n \geq 1$). Transverse states are on-shell DDF states (i.e. they satisfy $L_0 = 1$).

A complete basis for $\mathcal{F}(k_0)$ is given by the states [6,35]

$$|\{\lambda, \mu, \beta\}; p\rangle = \prod_{n=1}^{\infty} (L_{-n})^{\lambda_n} \prod_{n=1}^{\infty} (K_{-n})^{\mu_n} \prod_{n=1}^{\infty} \prod_{i=1}^{D-2} (A_{-n}^i)^{\beta_{i,n}} |0; p - \frac{N_0'}{p^+} k_0\rangle, \quad (\text{A.6a})$$

where

$$K_n = \frac{-k_0 \alpha_n}{p^+}, \quad \lambda_n, \mu_n, \beta_{i,n} \in \mathbb{N} \quad (\text{A.6b})$$

In (A.6a), the operators L_{-n} and K_{-n} have to be multiplied in some specific order [6,35].

Let us now introduce the operator [38]

$$E(k_0) = \mathcal{L}_0(k_0) - N \quad (\text{A.7a})$$

where

$$\mathcal{L}_0(k_0) = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} A_{-n}^i(k_0) A_n^i(k_0), \quad N = \sum_{n=1}^{\infty} \sum_{\mu=0}^{D-1} \alpha_{-n}^{\mu} \alpha_{n\mu}. \quad (\text{A.7b})$$

From the properties above, it is evident that the operator

$$\mathcal{L}(k_0) = \oint \frac{dz}{2i\pi z} z^{E(k_0)} \quad (\text{A.8})$$

is a projection operator, which projects any state in $\mathcal{F}(k_0)$ onto a DDF state. Moreover, since L_0 commutes with $E(k_0)$, $\mathcal{L}(k_0)$ is the on-shell transverse projector [38], i.e. it projects on-shell states (which satisfy $L_0 = 1$ but not necessarily $L_n = 0, n \geq 1$) onto transverse states.

For $D = 26$, we have the remarkable identity [38]

$$E = (D_0 - 1)(L_0 - 1) + \sum_{n=1}^{\infty} (D_{-n} L_n + L_{-n} D_n) \quad (\text{A.9})$$

where

$$D_n \equiv \oint \frac{dz}{2i\pi z} z^{n-1} : \frac{p^+}{-ik_0 \partial_z X(z)} : , \quad (\text{A.10})$$

and the relations

$$D_0 - 1 = - \sum_{n=1}^{\infty} (D_{-n} K_n + K_{-n} D_n) , \quad (\text{A.11a})$$

$$[X_n, E] = -n X_n , \quad (\text{A.11b})$$

$$X_n \alpha = 0 = \alpha X_{-n} , \quad n \geq 1 , \quad (\text{A.11c})$$

valid for $X_n = L_n, K_n, D_n$.

Let us now turn to the determination of one-loop amplitudes, by first considering the vacuum amplitude. From (1.2), the total vacuum amplitude of a mass distribution

$$\rho(m^2) = \sum_{k=0}^{\infty} d_k \delta(m^2 - 2(k-1)) \quad (\text{A.12})$$

is

$$- \frac{1}{2} i \delta^{(D)}(0) \int_0^1 \frac{dw}{w} \left(\frac{-2\pi}{\ln w} \right)^{D/2} \sum_{k=0}^{\infty} d_k w^{k-1} \quad (\text{A.13})$$

The problems related to the tachyon at $m^2 = -2$ are dealt with in Appendix C. The degeneracies d_k are given as follows. For $U(n)$ oriented bosonic strings, we have

$$d_k = n^2 d_0(k) \quad (\text{A.14})$$

and for $SO(n)$ and $USp(n)$ unoriented bosonic strings

$$d_{2k} = \frac{1}{2} n(n+l) d_0(2k) , \quad d_{2k+1} = \frac{1}{2} n(n-l) d_0(2k+1) , \quad (\text{A.15})$$

where $l = +1$ (resp. $l = -1$) corresponds to $SO(n)$ (resp. $USp(n)$), and $d_0(k)$ is defined by

$$\sum_{k=0}^{\infty} d_0(k) q^k = \frac{1}{q} \zeta(q) . \quad (\text{A.16})$$

(For the sake of the discussion, we keep D arbitrary, although the above spectra make sense for string theory only when $D = 26$).

With those choices, the total vacuum amplitude (A.13) reduces to

$$-\frac{1}{2} i \delta^{(D)}(0) n^2 C_0 \int_0^1 \frac{dw}{w} \left(\frac{-2\pi}{\ln w} \right)^{D/2} \frac{1}{w} f^2(w) + \frac{1}{2} i \delta^{(D)}(0) n C_N \int_0^1 \frac{dw}{w} \left(\frac{-2\pi}{\ln w} \right)^{D/2} \frac{1}{w} f^2(-w) \quad (\text{A.17a})$$

$$= -\frac{1}{2} i C_0 \int_0^1 \frac{dw}{w} \frac{1}{w} f^2(w) \text{Tr } w^{L_0} + \frac{1}{2} i C_N \int_0^1 \frac{dw}{w} \frac{1}{w} f^2(-w) \text{Tr } w^{L_0} (-1)^N, \quad (\text{A.17b})$$

with the coefficients C_0 and C_N given in Table 1, and the same meaning for "Tr" as in (2.22) and (2.23). Incidentally, this is the simplest way to fix those coefficients and the normalisation of amplitudes. Eq.(A.17) is recognised as the sum of a planar and of a Möbius strip contribution. If the factors $f^2(\pm w)$ were absent, (A.17) would further reduce to

$$-\frac{1}{2} i C_0 \text{Tr} \left[\frac{1}{L_0 - 1} \right] + \frac{1}{2} i C_N \text{Tr} \left[\frac{1}{L_0 - 1} (-1)^N \right] \quad (\text{A.18})$$

Hence, the net effect of the factors $f^2(\pm w)$ is to compensate for the propagation of the unphysical states in the vacuum loop.

Within the operator formalism, this simple remark can be generalised to arbitrary amplitudes [38] with the help of Feynman's tree theorem [29]. Essentially, this theorem is a trick to ensure that only positive norm physical states contribute to the singularities of one-loop amplitudes, i.e. that (one-loop) unitarity is preserved in theories with unphysical states. This theorem states that the correct expression for a given one-loop diagram is obtained by cutting its internal lines (i.e. by putting these lines on-shell and allowing only positive norm physical states to contribute to the cut) and then summing over all possible ways of cutting the diagram [29,38].

The use of Feynman's tree theorem in string theory is straightforward enough. Any internal line leads to the propagator $(L_0 - 1)^{-1} = \int_0^1 \frac{dx}{x} x^{L_0 - 1}$; hence, putting it on-shell simply amounts to replacing this factor by $\oint \frac{dx}{2i\pi x} x^{L_0 - 1}$, whereas projecting onto transverse states is then readily achieved by the operator $\mathcal{Q}$ introduced above [38].

Let us first apply this to the planar vacuum amplitude. The cut diagram is simply

$$\oint \frac{dw}{2i\pi w} \text{Tr } w^{L_0 - 1} \mathcal{Q} = \oint \frac{dw}{2i\pi w} \oint \frac{dy}{2i\pi y} \text{Tr } w^{L_0 - 1} y^E. \quad (\text{A.19})$$

Using the algebraic properties of E given in (A.9), (A.10), (A.11), one easily establishes the equation [37]

$$y \frac{\partial}{\partial y} \left[f^2\left(\frac{w}{y}\right) \text{Tr } w^{L_0-1} y^E \right] = 0 \quad (\text{A.20})$$

which implies

$$\text{Tr } w^{L_0-1} \partial = \frac{1}{w} f^2(w) \text{Tr } w^{L_0} \quad (\text{A.21})$$

Hence, by Feynman's tree theorem, the correct planar vacuum amplitude is (up to its absolute normalisation)

$$\int_0^1 \frac{dw}{w} \frac{1}{w} f^2(w) \text{Tr } w^{L_0} \quad (\text{A.22})$$

as obtained before. Similarly, the analysis for the Möbius strip contribution readily leads to the second term in (A.17b).

Let us now consider scattering amplitudes, and firstly, a planar contribution. We expect it to have the form, up to normalisation (see (2.22)):

$$\int_0^1 \frac{dw}{w} \frac{1}{w} F(w) \int_w^1 \prod_{i=1}^M dz_i |z_a| \delta(z_a - z_a^*) \text{Tr} [\phi_M(z_M) \dots \phi_1(z_1) w^{L_0}] \quad (\text{A.23})$$

where the primary conformal fields $\phi_i(z_i)$ are ordered such that $z_i < z_{i+1}$ ($i=1, \dots, M-1$), and the modular integration measure $F(w)$ is to be determined such that only positive norm physical states contribute to the singularities of the amplitude. It is quite remarkable indeed that all these complications only lead to such a simple modification of the amplitude. Using the properties (2.12a) with $\Delta = 1$, (A.23) may also be expressed as

$$\int_0^1 \prod_{i=1}^M \frac{dx_i}{x_i} F\left(\prod_{i=1}^M x_i\right) \text{Tr } P(x) \quad , \quad (\text{A.24})$$

where

$$P(x) = x_M^{L_0-1} \phi_M(1) x_{M-1}^{L_0-1} \phi_{M-1}(1) \dots x_1^{L_0-1} \phi_1(1) \quad . \quad (\text{A.25})$$

If the factor $F(w)$ could be ignored, (A.24) would reduce to a naïve guess following from the multiperipheral representation of tree level amplitudes, namely

$$\text{Tr} \left[\frac{1}{L_0-1} \phi_M(1) \frac{1}{L_0-1} \phi_{M-1}(1) \dots \frac{1}{L_0-1} \phi_1(1) \right] \quad . \quad (\text{A.26})$$

To determine $F(w)$, let us for example examine the diagram with the M^{th} line cut:

$$\oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} \text{Tr } P(x) \mathcal{L} = \oint \frac{dx_M}{2i\pi x_M} \oint \frac{dy}{2i\pi y} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} \text{Tr } P(x) y^E \quad (\text{A.27})$$

Following Ref.38, where the analysis is performed for tachyon states (and planar diagrams) only, one establishes the identity

$$\begin{aligned} \text{Tr } P(x) y^E &= f^2\left(\frac{w}{y}\right) f^2(w) \text{Tr } P(x) + f^2\left(\frac{w}{y}\right) x_M \frac{\partial}{\partial x_M} \int_1^y \frac{dz}{z} f^2\left(\frac{w}{z}\right) \text{Tr } P(x) (D_0 - 1) y^E \\ &+ f^2\left(\frac{w}{y}\right) \sum_{i=M}^2 D_i \left\{ \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \int_1^y \frac{dz}{z} f^2\left(\frac{w}{z}\right) \frac{1}{1 - \left(\frac{w}{z}\right)^n} (x_M \dots x_i)^n (1 - x_{i-1}^n) \text{Tr } P(x) z^E D_{-n} \right\}, \end{aligned} \quad (\text{A.28})$$

where

$$w = \prod_{i=1}^M x_i, \quad D_i = x_M \frac{\partial}{\partial x_M} - x_{i-1} \frac{\partial}{\partial x_{i-1}}, \quad i=2, \dots, M \quad (\text{A.29})$$

Integrating (A.28), one obtains for (A.27)

$$\oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} \text{Tr } P(x) \mathcal{L} = \oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} f^2(w) \text{Tr } P(x) \quad (\text{A.30})$$

When more than one internal line is cut, because of the decoupling of unphysical intermediate state in the multiperipheral representation of tree level amplitudes, it is enough to insert the on-shell transverse projector on only one of the cuts. Thus, the identity (A.28) is still relevant, and one obtains the result (A.30) for an arbitrary number of cuts. Feynman's tree theorem then implies that for planar amplitudes [38], the measure $F(w)$ is simply given by

$$F(w) = f^2(w). \quad (\text{A.31})$$

The analogous analysis for non-planar and non-orientable contributions is complicated by the fact that twisted representations of tree level amplitudes have pole singularities corresponding to unphysical states also [37]. Up to normalisation, a non-planar amplitude is expected to have the form

$$\int_0^1 \frac{dw}{w} \frac{1}{w} F(w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_a| \delta(z_a - z_a^0) \text{Tr} [\phi_M(z_M) \dots \phi_1(z_1) w^{L_0}], \quad (\text{A.32})$$

where the primary conformal fields $\phi_i(z_i)$ are ordered such that $|z_i| < |z_{i+1}|$ ($i=1, \dots, M-1$), but with some z_i 's belonging to $[w, 1]$, and the others to $[-1, -w]$. As in (A.24), this can be expressed as

$$\int_0^1 \prod_{i=1}^M \frac{dx_i}{x_i} F\left(\prod_{i=1}^M x_i\right) \text{Tr } P_T(x) , \quad (\text{A.33})$$

with now

$$P_T(x) = x_M^{L_0-1} \phi_M(\eta_M) \dots x_1^{L_0-1} \phi_1(\eta_1) , \quad (\text{A.34})$$

where $\eta_i = +1$ (resp. $\eta_i = -1$) if the vertex $\phi_i(z_i)$ belongs to the boundary $[w, 1]$ (resp. $[-1, -w]$).

In the absence of the factor $F(w)$, (A.33) would again follow from the naive guess inferred from the corresponding twisted contribution to the tree level amplitude. Considering for example the diagram with the M^{th} line cut, one derives the relation [37]:

$$\oint_{2i\pi x_M} \frac{dx_M}{x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} \text{Tr } P_T(x) = \oint_{2i\pi x_M} \frac{dx_M}{x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} f^2(w) \text{Tr } P_T(x) - \sum_{i=M}^2 \oint_{2i\pi y} \frac{dy}{y} \oint_{2i\pi x_M} \frac{dx_M}{x_M} \int_0^1 \prod_{j=1}^{M-1} \frac{dx_j}{x_j} f^2\left(\frac{w}{y}\right) \int_1^y \frac{dz}{z} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f^2\left(\frac{w}{z}\right) \frac{(x_M \dots x_i)^n (\eta_i^n - \eta_{i-1}^n)}{1 - \left(\frac{w}{z}\right)^n} \text{Tr } P_T(x) z^E D_{-n} , \quad (\text{A.35})$$

with the notations of (A.29). Were it not for the second term in (A.35), one would once again conclude that the measure $F(w)$ appropriate for non-planar contributions be given by (A.31). However, we know [37] that by summing all twisted contributions corresponding to a given cyclic order of the external states in tree level amplitudes, only physical states contribute to its pole structure, although unphysical states also contribute to individual twisted amplitudes. Hence, by summing (A.35) appropriately, such that in effect one is integrating over the vertex positions on the two boundaries independently but in a way which preserves their respective cyclic order, all additional terms in (A.35) should compensate and only transverse states should contribute to cut singularities of non-planar amplitudes for $F(w) = f^2(w)$ [37].

Incidentally, there is another way to look at this problem. One-loop open string amplitudes also correspond to closed string tree amplitudes; the modular integration measure should be such that one obtains the correct singularity structure in the closed string channel. This can only be achieved with the choice (A.31) for oriented amplitudes [46].

The discussion for non-oriented amplitudes is essentially the same. The only modification is that the twist operator $(-1)^N$ is inserted in the trace of (A.32), which leads to (A.33) with $P_T(x)$ replaced by $P_N(x) = P_T(x)(-1)^N$. As a result, the relation (A.35) is still obtained, with $P_T(x)$ replaced by $P_N(x)$ and w replaced by $(-w)$ [37]. Hence, with the same considerations as above concerning pole singularities of twisted or closed string channel contributions, the correct modular integration measure for non-oriented amplitudes is [37]

$$F_N(w) = f^2(-w). \quad (A.36)$$

The above analysis for planar amplitudes has also been performed including the reparametrisation Faddeev-Popov ghosts [47], and has been extended to fermionic strings [38,48].

Appendix B

As an example of one-loop amplitudes discussed in section II, let us give the expressions relevant for amplitudes with any open string states at the first three excitation levels. It is enough then to consider the vertices $V(z, k, \eta) = : \exp(ikX(z) + i\eta\partial_z X(z)) :$, and perform the appropriate expansions in the auxiliary vector η_μ .

Given the variables z_i ($i=1, \dots, M$) in $\Omega(w)$ such that $|z_{i+1}| > |z_i|$ ($i=1, \dots, M-1$), one obtains for orientable contributions

$$\begin{aligned} & \text{Tr} \left[V(z_M, k_M, \eta_M) \dots V(z_1, k_1, \eta_1) w^{L_0} \right] \\ &= \delta^{(D)} \left(\sum_{i=1}^M k_i \right) \left(\frac{-2\pi}{\ln w} \right)^{D/2} f^2(w) \prod_{i=1}^M z_i^{-1/2} k_i^2 \prod_{i=1}^M \exp \left\{ -\frac{1}{2} \frac{k_i \eta_i}{z_i} + \frac{\eta_i^2}{z_i^2} w \frac{d}{dw} \ln \left[\left(\frac{-2\pi}{\ln w} \right)^{D/2} f^2(w) \right] \right\} \\ & \cdot \prod_{1 \leq i < j \leq M} \left[\psi \left(\frac{|z_i|}{|z_j|}, w \right) \right]^{k_i k_j} \cdot \prod_{1 \leq i < j \leq M} \exp \left\{ \left[\frac{k_i \eta_j}{z_j} - \frac{k_j \eta_i}{z_i} \right] \left(-x \frac{\partial}{\partial x} \ln \psi(x, w) \right) \right\}_{x=\frac{|z_i|}{|z_j|}} \\ & \quad + \frac{\eta_i \eta_j}{z_i z_j} x \frac{\partial}{\partial x} \left(-x \frac{\partial}{\partial x} \ln \psi(x, w) \right) \Big|_{x=\frac{|z_i|}{|z_j|}} \Bigg\}, \quad (B.1) \end{aligned}$$

where $\psi(|z_i|/|z_j|, w)$ is one of the functions ψ_P or ψ_T given below, depending on whether respectively the corresponding vertices $V(z_i, k_i, \eta_i)$ and $V(z_j, k_j, \eta_j)$ are attached to the same or to different boundaries of the cylinder.

For non-orientable contributions, the same result is obtained with the only differences that the partition function $f(w)$ is replaced by $f(-w)$ and that the function $\psi(x, w)$ is the function $\psi_N(x, w)$ below. We have:

$$\psi_p(x, w) = \frac{1-x}{\sqrt{x}} \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \prod_{n=1}^{\infty} \frac{(1-w^n x)(1-w^n x^{-1})}{(1-w^n)^2} \quad (\text{B.2a})$$

$$\psi_T(x, w) = \frac{1+x}{\sqrt{x}} \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \prod_{n=1}^{\infty} \frac{(1+w^n x)(1+w^n x^{-1})}{(1-w^n)^2} \quad (\text{B.2b})$$

$$\psi_N(x, w) = \frac{1-x}{\sqrt{x}} \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \prod_{n=1}^{\infty} \frac{(1-(-w)^n x)(1-(-w)^n x^{-1})}{(1-(-w)^n)^2} \quad (\text{B.2c})$$

The same results were essentially obtained in Ref.49, but the authors seem to have overlooked the fact that the terms involving the polarisation vectors η_1^μ are logarithmic derivatives of $f(\pm w)$ and $\psi(x, w)$. In particular, the terms in η_1^2 were discarded in Ref.49, since they do not contribute to amplitudes of massless gauge bosons. Actually, they do not contribute to amplitudes with the massive tensor states given in (2.17a) either, but they play an essential role in the decoupling of the null states given in (2.17c). Indeed, it is the very fact that those terms are total derivatives in the modular parameter w , which implies that amplitudes with these null states are given as surface terms in modular space, as shown in section IV for arbitrary zero-norm states.

The functions $\psi(x, w)$ above can be related to θ -functions [6]. Through the change of variables

$$\lambda = \exp\left(\frac{4\pi^2}{\ln w}\right), \quad \nu = \frac{\ln x}{\ln w}, \quad (\text{B.3})$$

we have

$$\psi_p(x, w) = \frac{-4\pi^2}{\ln \lambda} \frac{\vartheta_1\left(\frac{\ln x}{\ln w} \middle| \frac{\ln \lambda}{2i\pi}\right)}{\vartheta_1'(0 \mid \frac{\ln \lambda}{2i\pi})} = \frac{-4\pi^2}{\ln \lambda} \frac{\sin \pi \nu}{\pi} \prod_{n=1}^{\infty} \frac{(1-2\lambda^n \cos 2\pi \nu + \lambda^{2n})}{(1-\lambda^n)^2}, \quad (\text{B.4a})$$

$$\psi_T(x, w) = \frac{-4\pi^2}{\ln \lambda} \frac{\vartheta_4\left(\frac{\ln x}{\ln w} \middle| \frac{\ln \lambda}{2i\pi}\right)}{\vartheta_4'(0 \mid \frac{\ln \lambda}{2i\pi})} = \frac{-4\pi^2}{\ln \lambda} \frac{1}{2\pi \lambda^{1/8}} \prod_{n=1}^{\infty} \frac{(1-2\lambda^{n-1/2} \cos 2\pi \nu + \lambda^{2n-1})}{(1-\lambda^n)^2}, \quad (\text{B.4b})$$

and

$$w^{1/24} f(w) = \left(\frac{-\ln \lambda}{2\pi}\right)^{1/2} \lambda^{1/24} f(\lambda). \quad (\text{B.5})$$

Through the change of variables

$$\lambda = \exp\left(\frac{\pi^2}{\ln w}\right), \quad \gamma = \frac{\ln x}{2 \ln w}, \quad (B.6)$$

we have

$$\psi_N(x, w) = \frac{-2\pi^2}{\ln \lambda} \frac{\vartheta_1\left(\frac{\ln x}{2 \ln w} \middle| \frac{\ln \lambda}{2i\pi} - \frac{1}{2}\right)}{\vartheta_1'(0 \middle| \frac{\ln \lambda}{2i\pi} - \frac{1}{2})} = \frac{-2\pi^2}{\ln \lambda} \frac{\sin \pi \gamma}{\pi} \prod_{n=1}^{\infty} \frac{(1 - 2(-\lambda)^n \cos 2\pi \gamma + \lambda^{2n})}{(1 - (-\lambda)^n)^2}, \quad (B.7)$$

and

$$w^{1/24} f(-w) = \left(\frac{-\ln \lambda}{\pi}\right)^{1/2} \lambda^{1/24} f(-\lambda). \quad (B.8)$$

Appendix C

In this appendix, we discuss how the unphysical singularities due to the regularisation of the open string tachyon can be subtracted from amplitudes through Feynman's tree theorem. Only planar amplitudes are considered; a similar analysis applies to the other amplitudes.

Inserting a factor w^γ amounts to shifting the mass eigenvalues in the propagators in (A.26), as $(L_0 - 1 + \gamma)^{-1}$, without changing the conformal properties of the vertex operators. The naive guess for the amplitude would thus be:

$$\int_0^1 \prod_{i=1}^M \frac{dx_i}{x_i} \left(\prod_{i=1}^M x_i \right)^\gamma \text{Tr } P(x) \quad (C.1)$$

However, we require that only transverse states (i.e. on-shell DDF states) contribute to the cut singularities of amplitudes when the regulator w^γ is removed. This can most easily be formulated through Feynman's tree theorem. For example, the contribution of the diagram with the M^{th} line cut is then

$$\oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} R(w) \text{Tr } P(x)^\omega, \quad (C.2)$$

where we allow for a more general function $R(w)$ than simply w^γ , in order to also consider the effect of the regularisation discussed in section III of the limit $\lambda \rightarrow 0$ or $w \rightarrow 1$. From the identity (A.28), one obtains, in analogy with (A.30)

$$\begin{aligned} \oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} R(w) \text{Tr } P(x)^\omega &= \oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} R(w) f^2(w) \text{Tr } P(x) \\ &- \oint \frac{dy}{2i\pi y} \oint \frac{dx_M}{2i\pi x_M} \int_0^1 \prod_{i=1}^{M-1} \frac{dx_i}{x_i} \left(w \frac{d}{dw} R(w) \right) f^{-2}\left(\frac{w}{y}\right) \int_1^y \frac{dz}{z} f^2\left(\frac{w}{z}\right) \text{Tr } P(x) (\mathcal{D}_{0-1})^\omega \mathbb{E}. \end{aligned} \quad (C.3)$$

The second term in this result compensates the unphysical cuts of the first term when $\gamma \neq 0$.

For the regularisations discussed in section III, we would now like to argue that this additional term vanishes when the regulators are removed. Let us first consider the regularisation of the open string tachyon, namely $R(w) = w^\gamma$. Since $w \frac{d}{dw} R(w) = \gamma w^\gamma$, the additional term then vanishes as $\gamma \rightarrow 0$, unless the quantity that γ multiplies is not finite for $\gamma = 0$. However, this cannot happen; it is understood that the limit $\gamma \rightarrow 0$ is not taken in the integrals, so that effectively all unphysical divergent contributions at $w \rightarrow 0$, due to the inappropriate proper-time representation of the amplitudes, are subtracted away. Thus, the quantity that γ multiplies should be finite as $\gamma \rightarrow 0$, and the additional term in (C.3) cancels in this limit.

Let us now consider the complete regularisation factor $R(w)$ discussed in section III, namely

$$R(w) = w^\gamma \exp\left(\frac{4\pi^2 \alpha}{\ln w}\right) \quad (C.4)$$

with $\gamma \geq 1$ and $\alpha > 1$. We have

$$w \frac{d}{dw} R(w) = \left(\gamma - \frac{4\pi^2 \alpha}{(\ln w)^2}\right) R(w) \quad (C.5)$$

As $\gamma \rightarrow 0$, the discussion is the same as above, since the dominant contributions then come from $w \rightarrow 0$ and only the first term in (C.5) is then important. As $\alpha \rightarrow 0$, it is the second term in (C.5) which dominates, with $w \rightarrow 1$, and the corresponding term in (C.3) then also vanishes.

This thus shows that the regularisations advocated in section III can consistently be used for one-loop amplitudes as given in (2.22) and (2.23), with the understanding that additional terms are necessary to cancel unphysical cuts but that those terms do not contribute in the limit where the regulators are taken away.

As a side remark, let us note that the so-called point-splitting or Gaussian regularisation [1,6] used in Type I superstrings also leads to additional terms in (A.30); they follow from the third kind of term in (A.28) evaluated at $x_{i-1} = e^{-\eta_i} - 1$. Again, those terms do not contribute in the limit where the regulators η_i are removed.

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	c_0	c_N
$U(n)$	1	0
$SO(n)$	$1/2$	$1/2$
$USp(n)$	$1/2$	$-1/2$

Table 1.