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CONSISTENT EQUATIONS FOR INTERACTING MASSLESS FIELDS OF ALL SPINS IN THE FIRST ORDER IN CURVATURES
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Consistent Equations for Interacting Massless Fields of All Spins
In the First Order in Curvatures

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ABSTRACT

A new form of equations of motion is suggested for d=4 massless fields of all spins interacting with gravity: equations of all massless fields, including the gravitational field itself, are described in terms of a free differential algebra constructed from 1-forms and 0-forms belonging both to an adjoint representation of the superalgebra of higher-spin and auxiliary fields proposed previously by E.S. Fradkin and the author. In this construction, 1-forms describe gauge massless and auxiliary fields, while 0-forms describe lower-spin fields and Weyl tensors corresponding to gauge 1-forms. The equations of motion are constructed explicitly in the first order in the Weyl 0-forms (and in all orders in 1-forms) that exceeds significantly the results of refs. /1,2/ on the cubic higher-spin-gravitational interaction.

The equations obtained are shown to remain consistent when all quantities take on their values in an arbitrary associative algebra. This enables us to describe simultaneously a class of extended-type theories with Yang-Mills gauge groups $U(n) \times U(n)$ corresponding to massless spin-1 fields ($n$ is arbitrary). Various consistent truncations of these extended theories are also discussed including those with Yang-Mills gauge groups $SO(n) \times SO(n)$. 
§1. Introduction

It has been shown recently /1,2/ that despite a wide-spread belief a consistent gravitational interaction of massless higher-spin fields exists at least in the first nontrivial (cubic) order. In particular, it has been shown in refs./1,2/ that the difficulties of introducing a gravitational interaction of massless higher-spin fields \((s\geq 2)\), which were discovered in refs./3-5/, originate from the fact that the higher-spin-gravitational interaction turns out to be non-analytical in the cosmological constant and, therefore, it becomes meaningless in the framework of the expansion over the flat background used in refs./3-5/.

The basic aim of the present paper is to develop an approach to massless higher-spin fields which is more efficient than that of refs./1,2/. The approach suggested enables us to construct various higher-spin interactions which lie beyond the cubic order of refs./1,2/, and we hope that this is an important step towards a closed description of all interactions of massless fields (including the gravitational field itself). However, at the first stage, the price for this is that we deal with motion equations instead of an action as in refs./1,2/. In fact, we demonstrate that equations of motion for massless fields of all spins \(s\geq 0\) admit a quite natural formulation in terms of some free differential algebra\(^\text{2}\) (henceforth, FDA). Although this FDA is not presently known to us in its closed form, it can be investigated in the framework of an expansion in powers of 0-forms generalizing the gravitational Weyl tensor. In the present paper, we construct the FDA, describing equations of motion of massless fields, in zero and first orders in the "Weyl 0-forms". Let us emphasize that the expansion procedure used here is more powerful than that of refs./1,2/ where the expansion was carried out in both 1-forms and curvatures. Therefore, the results of the present paper exceed considerably those of refs./1,2/ at least on the level of motion equations, and we hope that ultimately the approach suggested will enable one to formulate a complete consistent dynamics

\(^{2}\) Free differential algebras were introduced by Sullivan in ref./6/ where some their important properties were investigated. An extension of these results to superalgebras was given by van Nieuwenhuizen /7/. For physical applications of FDA's to supergravity see, e.g., ref./8/ and references therein. All facts on FDA's needed in the present paper, are collected in §3.
of all massless fields not only on the level of motion equations but on the action level too.

The same as in refs./1,2/, our approach is based on the results of refs./9-12/ where a new form was developed for description of free massless fields of all spins $\mathfrak{gs}/3/2$ in both flat /9/ and anti-de Sitter (AdS) /10/ spaces, ordinary (N=1) /11/ and extended (N>1) /12/ non-abelian higher-spin superalgebras were constructed, and their simple operator realization was found /12/. In addition, we use the results of refs./18,19/ where new systems of auxiliary fields were discovered /18/ which obey differential equations of motion, and an appropriate generalization of the higher-spin superalgebra of ref./11/ was found /19/ that led naturally to both massless fields of ref./10/ and auxiliary fields of ref./18/.

Let us mention that some interesting results on non-gravitational interactions of massless higher-spin fields in the flat space were obtained during last years in refs./20-23/. Specifically an important qualitative conclusion was made in refs. /20,21/ that higher-spin interactions contain higher derivatives, while in refs. /21/ it was shown that consistent interactions of massless higher-spin fields require for introducing infinite systems of fields of infinitely increasing spins.

These general properties of interacting systems of massless higher-spin fields manifest themselves quite naturally in terms of superalgebras of higher-spin and auxiliary fields constructed in refs./11,12,19/ (see also refs./1,2/). Moreover, we believe that higher-spin interactions of refs./20-23/ can be derived from the cubic interactions constructed in refs./1,2/ and in the present paper by means of some flat contraction such that all those interactions are switched off (including the gravitational interaction itself) that become meaningless in the flat limit. Because such a procedure can consistently be applied only to cubic interactions (as emphasized in refs./21,2/, cubic interactions are independent when analysed in the lowest order),

* For fermionic massless fields in the flat space, the results analogous to those of ref./9/ were independently obtained by Aragone and Doser /13/. For the standard approach, in which massless fields are described by symmetric tensor(-spinors), see, e.g., refs./14-17,3-5/ (for a more complete list of references see refs./9,10/). It is worth mentioning that the approach of refs./9,13,10/ is completely equivalent to the standard one. In fact, the former generalizes the tetradic formulation of gravity, while the latter generalizes its metric formulation (the latter parallelism was emphasized by De Wit and Freedman in ref./5/).
it seems natural to expect that no extension of the results of refs. 20-23/ exists beyond the cubic order when working in the flat space (except for trivial interactions which do not lead to any deformation of the initial (abelian) higher-spin symmetries as is for example the case for the interactions considered in ref. 24/).

The general properties of interacting systems of massless higher-spin fields mentioned above resemble in many respects those known for string field theories (see for example refs. 25-28/ and references therein). In particular, in the both cases infinite systems of fields of infinitely increasing spins are present. However, the qualitative difference is that only some lower-spin fields have vanishing mass in string field theories, while all higher-spin fields are massive. Another essential point is that, presently, string field theories are formulated in the flat d-dimensional space, that makes general coordinate invariance implicit enough in string theories incorporating gravitation. On the other hand, the theories of massless higher-spin fields under consideration are formulated in explicitly general coordinate fashion in refs. 1, 2, 11/ and in the present paper. Moreover, as emphasized above, the theories of massless higher-spin fields admit no expansion over the flat background at all because interactions of massless fields turn out to be non-analytical in the cosmological constant.

Since any spin s+1 massless field possesses its own gauge symmetry, physical gauge symmetries of theories of massless higher-spin fields strongly exceed gauge symmetries of physical fields emerging in string field theories. This enables one to speculate that theories of massless higher-spin fields are in fact more fundamental than string theories. Perhaps, string theories may be viewed as some spontaneously broken phases of theories of massless higher-spin fields. Indeed, there are several reasons for gauge symmetries of massless higher-spin fields to be broken in a physical phase. For example, this breakdown may lead to a physical theory with vanishing cosmological constant (which is to be re-defined in a broken phase). Since, as usual, originally massless higher-spin fields are expected to acquire (large) masses after appropriate spontaneous breakdown (for more detailed discussion of this point see §8), corresponding spontaneously broken theories become very similar to string field theo-
ries. Whether theories of massless higher-spin fields are related to string theories or not, they deserve great attention as forming a new nontrivial class of gauge theories. We hope that the results of the present paper will provide a deeper insight into intrinsic structure of these theories.

The rest part of the paper is organized as follows.

In §2, we re-formulate equations of motion of free massless and auxiliary fields suggested in refs. /10,18/ in an equivalent form which, however, enables us to treat these equations as some FDA. It is shown that motion equations of lower-spin massless fields, which could not be described within the approach of refs. /9,10/, admit a quite natural description in terms of this FDA too. In addition, it is argued here that a full FDA, describing full non-linear equations, is some deformation of the FDA formed by the gauge 1-forms and Weyl 0-forms belonging to the adjoint representation of the non-abelian superalgebra of higher-spin and auxiliary fields, shsa(1), proposed in ref. /19/.

In §3, some general properties of FDA's are discussed which are currently throughout the paper.

In §4, an operator realization of shsa(1) is described that enables us to apply the Berezin's theory of symbols of operators in our analysis of the FDA under investigation.

In §5, a deformation of the original FDA, based on shsa(1), is described in the first nontrivial order in the Weyl 0-forms, i.e. consistent motion equations of massless and auxiliary fields are found in this order.

In §6, some general properties are discussed of the deformation of §5. It is emphasized that the whole analysis remains valid if all quantities take on their values in an arbitrary associative algebra. Specifically, this enables us to generalize trivially the results obtained to a class of extended theories with the Yang-Mills gauge

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Recently, massless higher-spin fields and some their cubic interactions were described in refs. /29-31/ in string-like terms. However, no progress was achieved in these works towards introducing the gravitational interaction of massless higher-spin fields. As usual, this is mainly because of using the flat background. In fact, this seems to be the main obstacle preventing from applying the ordinary string-like formalism for description of dynamics of massless higher-spin fields (see also ref. /35/).
groups $U(n) \times U(n)$ corresponding to massless spin-1 fields. Hermiticity conditions and a number of automorphisms are found which respect the deformation constructed. Consistent truncations of equations of massless and auxiliary fields are described, including those with the Yang-Mills gauge group $SO(n) \times SO(n)$. Finally, the gauge symmetries are listed here that remain undeformed within the deformed EDA of §5.

In §7, an action is presented which is consistent in the cubic order within the expansion procedure analogous to that of refs. /1,2/. This action describes a dynamics of the gauge fields (1-forms) related to $sha(1)$ including the case when they take on their values in an arbitrary associative algebra. However, the approximation used in this section is weaker than that of other sections. This manifests itself in the fact that the action constructed here is incomplete as it describes only some part of relevant dynamical fields (the same is true for the actions of refs. /1,2/).

In §8, we summarize the main results of the paper once again and discuss briefly some peculiarities of spontaneous breakdown of higher-spin gauge symmetries.

In Appendix, an explicit proof is given for the fact that the deformation of §5 correctly reproduces free equations of massless and auxiliary fields on the linearized level as constructed in §2.

§2. Free Equations of Massless and Auxiliary Fields

In this section, we re-formulate equations of motion of free massless and auxiliary fields suggested in refs. /9,10,18/ in somewhat different but equivalent form. This will lead us quite naturally to a general problem setting for full (non-linear) equations investigated in the subsequent sections.

The key point consists of the fact that, as shown in ref. /10/, free equations of spin $s \geq 3/2$ massless fields in AdS space can be reduced to the form\[\text{we use conventions introduced in ref. /10/}. \text{Spinorial indices } \alpha, \beta, \gamma, \ldots, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \ldots \text{take on values } 1,2. \text{A number of indices is indicated in parentheses (except for the case of a single index). Upper (lower) spinorial indices denoted by one and the same letter are assumed to be fully symmetrized. When this symmetrization is carried out, the maximal possible number of upper and lower indices denoted by one and the same letter should be contracted. Indices of components of differential forms } \mu, \nu, \rho, \sigma = 0-3 \text{ are from the middle of Greek alphabet.} \]
\[ R^r_{\gamma \nu \alpha \mu, d(n), \beta(m)} = \delta(n) \delta_\rho_\nu^\delta_\rho_\mu^\gamma_\beta^\alpha_\delta_\delta C_{d(n)} \gamma(m) + \delta(n) \delta_\rho_\nu_\delta_\rho_\mu^\gamma_\beta^\alpha_\delta_\delta C_{\beta(m)} \delta(2) \] (2.1)

where \( R^r \) are linearized higher-spin curvatures constructed in ref.10/.

\[ R^r_{\gamma \nu \alpha \mu, d(n), \beta(m)} = D^b_\gamma \omega_\nu, d(n), \beta(m) + n \hbar_\nu_\delta_\nu_\omega_\mu_\alpha d(n) \delta_\beta(m) + m \hbar_\nu_\delta_\nu_\omega_\mu_\alpha(\omega_\mu, d(n), \beta(m)) \delta_\beta(m) - (\gamma \leftrightarrow \mu), \]

\[ \hbar_\nu_\delta_\nu_\omega_\mu_\alpha \] is the vierbein of the background AdS space and \( D^b_\gamma \) is the background Lorentz covariant derivative with respect to background Lorentz connection \( \hbar_\nu_\delta_\nu_\omega_\mu_\alpha(2) \) and \( \hbar_\nu_\delta_\nu_\omega_\mu_\alpha(2) \).

\[ D^b_\gamma \omega_\nu, d(n), \beta(m) = \partial^b_\gamma \omega_\nu, d(n), \beta(m) + n \hbar_\nu_\delta_\nu_\omega_\mu_\alpha d(n) \delta_\beta(m) + m \hbar_\nu_\delta_\nu_\omega_\mu_\alpha(\omega_\mu, d(n), \beta(m)) \delta_\beta(m) - (\gamma \leftrightarrow \mu), \] (2.3)

In addition, the following notations are currently used through the text:

\[ \delta(n) = \begin{cases} 1 & \text{at } n = 0 \\ 0 & \text{at } n \neq 0 \end{cases}, \quad \Theta(n) = \begin{cases} 1 & \text{at } n = 0 \\ 0 & \text{at } n < 0 \end{cases}, \quad \zeta(n) = \Theta(n) - \Theta(-n). \] (2.4)

The background fields \( \hbar, \omega, \hbar_\nu_\delta_\nu_\omega_\mu_\alpha \) are supposed to be chosen in such a way that the corresponding AdS curvatures (in fact, sp(4) Yang-Mills curvatures), which have the form

\[ \iota_\gamma \omega_\mu, d(2) = \partial_\gamma \omega_\mu, d(2) + \omega_\nu, d(2) \omega_\mu, d(2) + \hbar_\nu_\delta_\nu_\omega_\mu_\alpha \delta_\gamma_\beta_\mu_\delta_\alpha_\delta \delta - (\gamma \leftrightarrow \mu), \] (2.5)

\[ \iota_\gamma \omega_\mu, \beta(2) = \partial_\gamma \omega_\mu, \beta(2) + \omega_\nu, \beta(2) \omega_\mu, \beta(2) + \hbar_\nu_\delta_\nu_\omega_\mu_\alpha \delta_\gamma_\beta_\mu_\delta_\alpha_\delta \delta - (\gamma \leftrightarrow \mu), \] (2.6)

\[ \gamma_\gamma \omega_\mu, d\beta = D^b_\gamma \hbar_\mu, d\beta - D^b_\mu \hbar_\nu, d\beta, \] (2.7)

are all equal to zero. In the present paper, we fix the parameter \( \lambda \) characterizing the inverse AdS radius (for more detail see refs.10,11/) to be equal to one.

The fields \( \omega_\nu \) are assumed to obey the following hermiticity conditions:

\[ (\gamma_\gamma \omega_\nu, d(n), \beta(m))^\dagger = \omega_\nu, \beta(2) \omega_\mu, d(n) \omega_\mu, \beta(m). \]

The quantities \( C_{d(n+2)} \) and \( \overline{C_{\beta(m+2)}} \) on the r.h.s. of eq.(2.1) generalize the gravitational Weyl tensor to massless fields of arbitrary spins \( s \geq 3/2 \). In fact, it was shown in ref.10/ that, for any fixed \( n+m=2(s-1)+1 \), the part of equations (2.1), which does not contain \( C \) and \( \overline{C} \), is equivalent to the free motion equations.
for spin-s physical massless fields, $\omega(n,m)$ with $|n-m| \leq 1$, supplemented by some constraints expressing all other fields, $\omega(n,m)$ with $|n-m| > 1$, in terms of the physical fields and their derivatives (whenever possible, we use a shorthand notation $\omega(n,m)$ instead of $(\omega)_{\nu}^{\alpha}(n), \dot{\omega}(m)$). In practice, it is convenient to treat the "higher-spin Weyl tensors", $C$ and $\bar{C}$, as independent dynamical variables which, however, can also be expressed in terms of (derivatives of) physical fields by means of eq.(2.1). Although one can easily express $C$ and $\bar{C}$ in terms of physical fields on the linearized level by solving eq.(2.1) (for explicit expressions see ref./1/), a non-linear generalization of these expressions would be complicated enough, and it is convenient to handle with $C$ and $\bar{C}$ (as well as with 1-forms $\omega(n,m)$ with $|n-m| > 1$) without referring to an explicit form of their relation to the physical fields, assuming, however, that this relation does exist and is governed by some non-linear generalization of eq.(2.1).

It follows from the Frobenius consistency conditions for eq.(2.1) that $C$ and $\bar{C}$ obey the following restrictions

$$\varepsilon^{\gamma\mu\rho\delta} \partial_{\nu} C_{\alpha(n)\gamma}^{\beta\delta} = 0, \tag{2.8}$$

$$\varepsilon^{\gamma\mu\rho\delta} \partial_{\nu} \bar{C}_{\alpha(n)\gamma}^{\beta\delta} = 0. \tag{2.9}$$

Equivalently, one can say that eqs.(2.8), (2.9) follow from the Bianchi identities for the curvatures $R^\gamma_r^c$ (2.2) (explicit form of the Bianchi identities is given in ref./10/ and can readily be derived from eq.(2.2) itself; also recall that background curvatures (2.5)-(2.7) are assumed to vanish in this analysis).

It is not difficult to make sure that eqs.(2.8), (2.9) are equivalent to the following:

$$\partial_{\rho} C_{\alpha(n+2)}^{\gamma\delta} - i \partial_{\rho} \dot{\gamma} \partial_{\delta} C_{\alpha(n+2)\gamma}^{\beta} = 0, \tag{2.10}$$

$$\partial_{\rho} \bar{C}_{\beta(m+2)}^{\gamma\delta} - i \partial_{\rho} \dot{\gamma} \partial_{\delta} \bar{C}_{\beta(m+2)\gamma}^{\beta} = 0. \tag{2.11}$$

where $C_{\alpha(n+3)}^{\gamma\delta}$ and $\bar{C}_{\beta(m+3)}^{\gamma\delta}$ are some new arbitrary quantities. In their turn, the consistency conditions for eqs.(2.10), (2.11) give some differential restrictions on $C(n+3,1)$ and $\bar{C}(1,m+3)$, and so on. Ultimately, one finds that the following infinite chains of differential relations should hold:
\[ D^\mu C_{\alpha(n),\beta(m)} - i \hbar \nabla^\mu C_{\alpha(n),\beta(m)} + \imath nm \lambda_\alpha \lambda_\beta C_{\alpha(n-1),\beta(m)} = 0, \tag{2.12} \]
\[ D^\mu \tilde{C}_{\alpha(n),\beta(m)} - i \hbar \nabla^\mu \tilde{C}_{\alpha(n),\beta(m)} + \imath nm \lambda_\alpha \lambda_\beta \tilde{C}_{\alpha(n-1),\beta(m)} = 0. \tag{2.13} \]

Eqs. (2.12), (2.13) turn out to be consistent both among themselves and with eq. (2.1).

By their very construction, the quantities \( C(n,m) \) and \( \tilde{C}(n,m) \) emerge only with \( n-m=2s \geq 2 \) and with \( m-n=2s \geq 2 \), respectively. Although \( C \) and \( \tilde{C} \) seem to behave quite similarly on the linearized level in eqs. (2.12), (2.13), we do not combine them into a unique set because, as will become clear later, they distinguish between the non-linear level. Note, however, that \( C(n,m) \) and \( \tilde{C}(m,n) \) are related to each other by hermitian conjugation.

The full linearized system of equations (2.1), (2.12), (2.13) splits into independent subsystems corresponding to all spins \( s \geq 3/2 \) (any spin emerges once and only once). Each spin \( s \geq 3/2 \) is described here by the fields \( \omega(n,m) \) with \( n+m=2(s-1) \), \( C(n,m) \) with \( n-m=2s \) and \( \tilde{C}(n,m) \) with \( m-n=2s \). Physical spin-\( s \) fields are identified with the 1-forms \( \omega(n,m) \) at \( \text{Im} \Phi \leq 1 \) and \( n+m=2(s-1) \). Equations (2.1) contain dynamical equations for physical fields and express all other 1-forms \( \omega(n,m) \) as well as 0-forms \( C(2s,0) \) and \( \tilde{C}(0,2s) \) in terms of derivatives of physical fields. It is also clear from the construction above that eqs. (2.12), (2.13) express consistently all Weyl 0-forms \( C(n,m) \) and \( \tilde{C}(n,m) \) with \( n-m \geq 0 \) in terms of \( C(2s,0) \) and \( \tilde{C}(0,2s) \), respectively, thus expressing all Weyl 0-forms in terms of derivatives of physical fields too. As a result, we see that eqs. (2.1), (2.12), (2.13) are equivalent to physical higher-spin motion equations supplemented by some relations expressing auxiliary quantities \( \omega(n,m) \) with \( \text{Im} \Phi \geq 1 \), \( C(n,m) \) and \( \tilde{C}(n,m) \) in terms of derivatives of the physical higher-spin fields \( \omega(n,m) \) with \( \text{Im} \Phi \leq 1 \).

Eqs. (2.1), (2.12), (2.13) have been derived above for spin \( s \geq 3/2 \) massless fields. It is a remarkable property of these equations that they describe a dynamics of lower-spin (\( s \leq 1 \)) massless fields equally well. In order to demonstrate this, let us consider the following consequences of eq. (2.12) at \( n>0 \), \( m=0 \) and eq. (2.13) at \( n=0 \), \( m>0 \):

\[ \eta \lambda_\alpha \lambda_\beta \partial^\mu C_{\alpha(n-1),\beta(0)} = 0, \tag{2.14} \]
\[ \begin{align*}
\frac{m}{\hbar^2} \partial_\delta \hbar^2 \partial_\delta C_{a(0),\dot{a}(m-1)\dot{\delta}} &= 0 \quad \text{(2.15)}
\end{align*} \]
where \( \hbar^2 \partial_\delta \hbar^2 \partial_\delta = 2 \delta_\alpha^\gamma \delta_\beta^\delta \).

For all spins \( s \geq 3/2 \) (\( n,m \geq 2 \)), eqs.\((2.14),(2.15)\) are consequences of eq.\((2.1)\), but for spins \( s=1 \) or \( 1/2 \) (\( n,m=1 \) or \( 2 \)) eqs.\((2.14),(2.15)\) are independent from eq.\((2.1)\) (for \( s=1 \), it only follows from eq.\((2.1)\) that the difference vanishes between the l.h.s. of eq.\((2.14)\) and the l.h.s. of eq.\((2.15)\) that expresses the spin-1 Bianchi identities. As for spins 0, and \( 1/2 \), eq.\((2.1)\) is not related at all to their linearized dynamics).

It is easily seen that eq.\((2.1)\) at \( n=m=0 \), eq.\((2.14)\) at \( n=2, m=0 \) and eq.\((2.15)\) at \( n=0, m=2 \) are equivalent to the spin-1 motion equations,
\[ D_\mu^L R_\nu^\mu,\mu^a(0),\dot{a}(0) = 0, \quad \in^\gamma \gamma \rho^\beta \partial_\mu^L R_\rho^\mu,\rho^a(0),\dot{a}(0) = 0. \quad \text{(2.17)} \]
Similarly, eq.\((2.14)\) at \( n=1, m=0 \) and eq.\((2.15)\) at \( n=0, m=1 \) coincide with the Pauli equation for massless spin-1/2 field described by the conjugated 0-forms \( C(1,0) \) and \( \overline{C}(0,1) \). It is important that for the both cases, \( s=1 \) and \( s=1/2 \), all other equations among eqs.\((2.12),(2.13)\), which do not reduce to eqs.\((2.14),(2.15)\), merely express corresponding 0-forms \( C(n,m) \) and \( \overline{C}(n,m) \) with \( n=m \geq 0 \) in terms of derivatives of \( C(2s,0) \) and \( \overline{C}(0,2s) \), containing simultaneously all consistency conditions for these expressions.

For the spin-0 case, dynamical equations coincide with eqs.\((2.12),(2.13)\) taken at \( n=m=0 \) and \( n=m=1 \). For example, we have for 0-forms \( C \)
\[ \begin{align*}
D_\mu^L C - i \hbar^2 \partial_\delta C \gamma^\delta &= 0, \\
D_\mu^L C_{a,\dot{a} \dot{\delta}} - i \hbar^2 \partial_\delta C_{a,\dot{a}, \dot{\delta}} + i \hbar^2 \partial_\delta C &= 0. \quad \text{(2.18)} \quad \text{(2.19)}
\end{align*} \]
Excluding the auxiliary 0-forms \( C(1,1) \) and \( C(2,2) \) from eqs.\((2.18),(2.19)\), one derives the Klein-Gordon equation for \( C(0,0) \),
\[ g^{\mu \nu} D_\mu D_\nu C - 8 C = 0. \quad \text{(2.20)} \]
Here the metric tensor is defined by the relation
\[ g^{\mu \nu} = \frac{1}{2} \ h^{\mu}_{\phantom{\mu} \alpha \beta} h^{\nu \alpha \beta}, \]  

(2.21)

while \( D_{\gamma} \) is the full background covariant derivative with zero torsion Christoffel symbols defined via the metric postulate, \( D_{\gamma} g^{\mu \nu} = 0 \).

In tensorial terms, eqs. (2.18), (2.19) read

\[ D_{\rho} C = i c_{\rho}, \quad D_{\rho} C_{\beta} = i c_{\rho \beta} + 2 i \ g_{\rho \beta} C = 0 \]  

(2.22)

where \( C_{\rho \beta} \) is the traceless symmetric tensor corresponding to \( C(2,2) \), and it is now obvious that eq. (2.20) follows from eq. (2.22). Let us note that the mass-like term on the l.h.s. of eq. (2.20) is in fact proportional to \( \lambda^2 \) where \( \lambda \) is the inverse radius of the background AdS space, which is set equal to one in this paper but goes to zero in the flat limit. Indeed, it is well-known (see, for example, refs. /36, 16, 17/) that motion equations of massless fields in AdS space contain mass-like terms with the mass parameters proportional to \( \lambda \).

Note also that quite similarly one can derive Klein-Gordon equations from eqs. (2.12), (2.13) for all other 0-forms \( \tilde{C}(0,m) \) and \( C(n,0) \), but for spins \( s > 0 \) these equations are consequences of the Pauli-like equations (2.14), (2.15).

Thus, we have shown that eqs. (2.1), (2.12), (2.13) describe free massless fields of all spins \( s \geq 0 \) in AdS space. Any spin \( s \) is described in this construction by means of a chain of fields formed by 1-forms \( \Lambda(n,m) \) with \( n+m=2(s-1) \) (when \( s \geq 1 \)), 0-forms \( C(n,m) \) with \( n-m=2s \) and 0-forms \( \tilde{C}(n,m) \) with \( m-n=2s \). Physical spin-\( s \) fields coincide with \( \Lambda(n,m) \) at \( |n-m| \leq 1 \) when \( s \geq 1 \) or with \( C(2s,0) \) and \( \tilde{C}(0,2s) \) when \( s \leq 1 \). All other fields, belonging to the chains above, can be expressed algebraically in terms of the physical fields and their derivatives by means of eqs. (2.1), (2.12), (2.13) (in fact, this is the case on the mass shell and up to a pure gauge part in the sector of 1-forms \( \Lambda(n,m) \) - for more detail see ref. /10/). It is important that the motion equations for the physical massless fields are the only nontrivial differential equations contained in eqs. (2.1), (2.12), (2.13), while all other relations in eqs. (2.1), (2.12), (2.13) either express some supplementary fields in terms of physical ones or become identities when the latter expressions and the physical motion equations are taken into account.
Now let us demonstrate that free motion equations for auxiliary fields of ref./18/ can be dealt in a quite similar (or, better, supplementary) fashion. These systems of auxiliary fields were described in ref./18/ by the sets of 1-forms \( \alpha_{\gamma \mu \nu} \delta \beta(m) \) with arbitrary (fixed) integer parameters \( t=n-m \). They are auxiliary in the sense that possess no dynamical degrees of freedom, i.e. a number of initial functions of three spatial coordinates, providing consistency of the corresponding Cauchy problem, turns out to be equal to zero after complete gauge fixing /18/.

Linearized curvatures for these fields are of the form /18/

\[
A^\ell_{\gamma \mu \nu, \delta} \alpha(n) \delta \beta(m) = \mathcal{D}^\ell_{\gamma \mu \nu, \delta} \alpha(n) \delta \beta(m) \delta + i n m H_{\gamma \mu \nu} \delta \beta(n-1) \delta \beta(m-1) - (n \leftrightarrow \mu).
\]

(2.23)

It was shown in ref./18/ that the equations

\[
A^\ell_{\gamma \mu \nu, \delta} \alpha(n) \delta \beta(m) = \mathcal{S}(m) \Theta(n-2) H_{\gamma \mu \nu} \delta \beta(n) \delta \mathcal{D} \alpha(n-2) + \mathcal{S}(n) \Theta(m-2) H_{\gamma \mu \nu} \delta \beta \mathcal{E} \beta(m-2)
\]

(2.24)

describe the sets of fields which are auxiliary in the above sense. Here \( \mathcal{D}(n-2,0) \) and \( \mathcal{E}(0,m-2) \) are "auxiliary Weyl 0-forms" which can be viewed as some new independent variables analogous to the higher-spin Weyl 0-forms introduced previously.

Quite similarly to the case of massless fields, one can make sure that eq.(2.24) leads to the following chains of consistency conditions:

\[
\mathcal{D}^\ell_{\gamma \mu \nu, \delta} \mathcal{D}(n) \delta \beta(m) + n H_{\rho \delta \beta} \mathcal{D}(n-2) \delta \beta(m) \delta + m H_{\rho \gamma \delta} \mathcal{D}(n) \delta \beta \delta \beta(m-2) = 0
\]

(2.25)

\[
\mathcal{D}^\ell_{\gamma \mu \nu, \delta} \mathcal{E}(n) \delta \beta(m) + n H_{\rho \delta \beta} \mathcal{E}(n-2) \delta \beta(m) \delta + m H_{\rho \gamma \delta} \mathcal{E}(n) \delta \beta \delta \beta(m-2) = 0
\]

(2.26)

where 0-forms \( \mathcal{D}(n,0) \) and \( \mathcal{E}(0,m) \) are those which emerge in eq.(2.24). Note that eqs.(2.25),(2.26) contain no additional dynamical conditions and merely express all 0-forms \( \mathcal{D}(n,m) \) and \( \mathcal{E}(n,m) \) in terms of \( \mathcal{D}(k,0) \) and \( \mathcal{E}(0,1) \), containing simultaneously all consistency conditions for eq.(2.24) and eqs.(2.25),(2.26) themselves (in other words, there are no counterparts within eqs. (2.25),(2.26) of nontrivial dynamical equations for lower-spin massless fields con-
tained in eqs. (2.12), (2.13)).

Thus, eqs. (2.24)-(2.26) describe auxiliary fields. The fields \( a(n,m) \) in eqs. (2.23), (2.24) are assumed to be complex in the sense that their conjugates do not belong to the same set of fields \( a(n,m) \) but form some new set (this point is explained in detail in ref. /19/; see also below). Therefore, from the point of view of the analysis above, the 0-forms \( D \) and \( E \) should also be considered as independent complex fields. Nevertheless, as will become clear soon later, it is natural to restrict the auxiliary Weyl 0-forms by the following hermiticity conditions:

\[
(\mathcal{D}a(n), \beta(m))^\dagger = a(n,m) \mathcal{D}^\dagger \beta(m), a(n)
\]

\[
(Ea(n), \beta(m))^\dagger = \beta(n,m) E^\dagger \beta(m), a(n)
\]

(2.27)

(2.28)

where \( a(n,m) \) and \( \beta(n,m) \) are some numerical coefficients whose explicit form will be given later. In this case, eqs. (2.25)-(2.28) impose some additional restrictions on the fields \( a(n,m) \) in eq. (2.24). However, since original equations (2.24)-(2.26) describe some auxiliary fields with zero numbers of degrees of freedom, the same equations supplemented by the additional hermiticity conditions (2.27), (2.28) describe some auxiliary fields too.

Now we observe that the structure of l.h.s. of eqs. (2.12), (2.13) for \( C \) and \( \tilde{C} \) is analogous to the structure of the "auxiliary curvatures" (2.23), while the structure of l.h.s. of eqs. (2.25), (2.26) for auxiliary Weyl tensors, \( \mathcal{D} \) and \( E \), is close to that of the higher-spin curvatures (2.2). In fact, this important observation enables us to suppose that both 1-forms \( (\mathcal{W}, a) \) and Weyl 0-forms \( (C, \tilde{C}, \mathcal{D}, E) \) belong to the adjoint representation of one and the same Lie superalgebra incorporating both massless fields of ref. /10/ and auxiliary fields of ref. /18/.

Such non-abelian superalgebra of higher-spin and auxiliary fields was already constructed by E.S. Fradkin and the author in ref. /19/. This infinite-dimensional superalgebra, denoted in ref. /19/ as shsa(1), gives rise to the set of gauge fields

\[
(\mathcal{W})^A_B a(n), \beta(m)
\]

with the indices \( A, B \) taking on values 0 or 1 \( (n,m=0,1,\ldots,\infty) \).

The curvatures of shsa(1) read /19/

\[
R_{\gamma \mu}^{AB} a(n), \beta(m) = (\partial_{\gamma} (\omega)_{\mu}^{AB} a(n), \beta(m) + \ldots
\]

(2.29)
\[ + \frac{15}{2} \sum_{\ell \geq 0} \frac{S(n+p-q) S(m-k-\ell) S(1A+C+F\ell) S(1B+D+G\ell) x}{n! \ell! (p+3)! (k+t)!} \times \]
\[ \times (\omega)_{\delta}^{A} d(p) \mu^{x} \beta(k) \hat{\beta}(\ell) \mu^{y} \beta(\ell) \hat{\beta}(\ell) \hat{\beta}(\ell) - (x \rightarrow y) \]
\[ \text{where} \]
\[ |n|_2 = \begin{cases} 0 & \text{at } n = 2k \\ 1 & \text{at } n = 2k + 1 \end{cases} \]

The fields \( \omega^{AA}(n, m) \) with \( A=0 \) and \( 1 \) are identified with massless fields, while the fields \( \omega^{AB}(n, m) \) with \( A+B=1 \) are auxiliary. This identification is due to the fact that, after linearization, the curvatures (2.29) lead \( /9/ \) to the linearized higher-spin curvatures (2.2) when \( A=B \) and to the linearized auxiliary curvatures (2.23) when \( A+B=1 \).

The doubling of fields of both types in the framework of shsa(1) is necessary in order to ensure appropriate hermiticity conditions in the fermion sector \( /9/ \) (in the pure bosonic case this doubling is unnecessary since the even sector of shsa(1) decomposes into the direct sum of two subalgebras each generating all bosonic fields of both types once and only once \( /9/ \)). The hermiticity conditions, corresponding to shsa(1), are of the form

\[ (\omega)_{\delta}^{AB} d(n) \beta(m) \hat{\beta}(m) = (-1)^{An+Bm} (\omega)_{\delta}^{BA} d(n) \beta(m) \hat{\beta}(m) . \]  

(2.31)

All Weyl 0-forms introduced above will be assumed to belong to the adjoint representation of shsa(1), i.e. they will be described by the quantities \( C^{AB} d(n) \beta(m) \) restricted by the analogous hermiticity conditions

\[ (C^{AB} d(m) \beta(m))^\dagger = (-1)^{An+Bm} C^{BA} d(n) \beta(m) \hat{\beta}(m) . \]  

(2.32)

Taking into account that all fields are doubled within shsa(1), one can now rewrite all linearized equations for massless and auxiliary fields, found above, in the following quite uniform way

\[ R^{\delta}_{\mu} \omega^{AB} d(n) \beta(m) + \]
\[ + \eta \delta(n) S(1A+B|2) C^{1A+1B} d(0) \beta(m) \hat{\beta}(2) \mu \gamma \gamma \mu \delta \delta \]  

(2.33)
\[ + \bar{\eta}_{1} \delta(m) S(1A + B + 1) C^{A^1 A^2} d_{\gamma}^{(2)} \hat{\alpha}(0) \hat{\beta}(0) \hat{y} \delta \hat{h} \mu \delta \]
\[ - \eta_{1} \delta(m) m(n+1) S(1A + B + 1) C^{B_B} d_{\gamma}^{(2)} \hat{\alpha}(0) \hat{\beta}(m) \hat{y} \delta \hat{h} \mu \delta \]
\[ - \bar{\eta}_{1} \delta(m) n(n+1) S(1A + B + 1) C^{A^1 A^2} d_{\gamma}^{(2)} \hat{\alpha}(0) \hat{\beta}(m) \hat{y} \delta \hat{h} \mu \delta \]
\[ D^{\delta}_{\gamma} C^{AB} d_{\gamma}^{(n)} \hat{\alpha}(m) = 0 \]  

(2.33)

(2.34)

where \(R^{\delta}_{\gamma} \ldots\) are linearized curvatures of shsa(1), while \(D^{\delta}_{\gamma} \ldots\) is the linearized covariant derivative in the adjoint representation of shsa(1) (an arbitrary complex parameter \(\eta_{1} \neq 0\) and the concrete values of the coefficients on the l.h.s. of eq. (2.33) are introduced for future convenience). Note that eq. (2.32) leads to the hermiticity conditions (2.27), (2.28) for the 0-forms \(C^{A^1 A^2} \) which describe (up to numerical factors) the auxiliary Weyl 0-forms \(D\) and \(E\) introduced previously.

(On the other hand, if a massless Weyl 0-form \(C(n, m)\), introduced previously, is identified with one of the fields \(C^{A^1 A^2} (n, m)\), then corresponding 0-form \(C(m, n)\) is described by the conjugate field \(C^{A^1 A^2} (m, n)\). This explains why we have not combined the massless 0-forms \(C\) and \(\bar{C}\) into one and the same set when analysing equations of massless fields at the beginning of this section.

The structure of linearized motion equations (2.33), (2.34) puts on an idea that full equations for interacting fields corresponding to shsa(1) can be reduced to the form

\[ R + f(\omega, C) = 0 , \quad D^{\delta}_{\gamma} C + g(\omega, C) = 0 \]  

(2.35)

where \(R\) are full (non-linearized) curvatures of shsa(1), \(D\) is covariant derivative in the adjoint representation of shsa(1), and \(f\) and \(g\) are constructed from 1-forms \(\omega\) and 0-forms \(C\) corresponding to the adjoint representation of shsa(1). It is assumed here that \(f\) and \(g\) are respectively of at least first and second orders in 0-forms \(C\). Let us emphasize that the language of exterior algebra of differential forms will only be used from now on (\(\omega = dx^\nu \omega^\nu \), \(dx^\nu dx^\mu = -dx^\mu dx^\nu \)), although the wedge multiplication sign \(\wedge\) is omitted everywhere because it is only the exterior product of differential forms which is implied through the rest text.

Now the problem of constructing equations for massless and auxiliary fields corresponding to shsa(1) reduces to determination of such "functions" \(f\) and \(g\) in eq. (2.35) that possess the following two properties: (i) Eqs. (2.35) are consistent
(corresponding Frobenius conditions are satisfied) and (ii) $f$ should reproduce
eq. (2.33) on the linearized level.

It is quite natural to look for a solution of this task by expanding $f$ and $g$
in powers of $O$-forms $\mathcal{C}$. The main result of the paper consists of constructing $f$
and $g$ in the first nontrivial orders. Namely, we find those parts of $f$ and $g$ which
are, respectively, of the first and second orders in $O$-forms $\mathcal{C}$. As for contribution
of 1-forms ($\omega$), it is found completely in this order in $\mathcal{C}$ thus producing a
lot of terms beyond the cubic order of the expansion procedure of refs. [1,2].

The structure of eq. (2.35) is typical for FDA's [6-8]. Therefore, before proceeding
to the detailed analysis of the dynamical system under consideration, we
comment briefly some general properties of FDA's, which are needed in this analysis.

§ 3. Free Differential Algebras

Consider an arbitrary set of differential forms $\mathcal{W}^A$ involving $p$-forms with
arbitrary $p \geq 0$. Let "curvatures" $R^A$ be defined by the relations

$$R^A = d\mathcal{W}^A + F^A(\mathcal{W})$$  \hfill (3.1)

where $F^A(\mathcal{W})$ are some functions of $\mathcal{W}^A$ (only exterior product of differential
forms is used in construction of $F^A$). Following to refs. [6-8], we say that eq.
(3.1) defines some free differential algebra (FDA) if $dR$ is proportional to $R$ or,
equivalently, if

$$F^B \frac{\partial F^A}{\partial \mathcal{W}^B} \equiv 0$$ \hfill (3.2)

(derivatives over $\mathcal{W}^B$ are everywhere assumed to be left with respect to both possi-
bile Grassmann grading of $\mathcal{W}^B$ and its grading in the exterior algebra of differential
forms). Eq. (3.2) leads to the following Bianchi identities for the curvatures $R^A$ (3.1):

$$dR^A = R^B \frac{\partial F^A}{\partial \mathcal{W}^B}$$ \hfill (3.3)

which ensure Frobenius consistency conditions to be satisfied for equations

$$R^A = 0 .$$  \hfill (3.4)
In fact, we seek for equations of motion for the dynamical system under consideration in the form (3.1), (3.4) with \( W^A = (\Omega, C) \), where \( \Omega \) and \( C \) are respectively 1- and 0-forms belonging both to the adjoint representation of \( \text{shsa}(1) \). Let us note that the requirement is often imposed in the literature when defining FDA's that the set of forms \( W^A \) does not contain 0-forms. This simplifies the situation considerably because only polynomial functions \( F^A \) are allowed on the r.h.s. of eq.(3.1) in this case (obviously, only 0-forms can give a non-polynomial contribution in the framework of exterior algebra). On the other hand, we are interested in the case when 0-forms are present in the set \( W^A \), and the whole analysis of this section is applicable to this most general case.

FDA's generalize directly ordinary Lie algebras and superalgebras which in their turn correspond to the specific case when all differential forms \( W^A \) are 1-forms (Lie superalgebras correspond to the case when 1-forms \( W^A \) possess some additional Grassmann grading which is in accordance with the structure of eq.(3.1)). As can be readily seen, eq.(3.2) is equivalent in this case to the Jacobi identities for an underlying Lie (super)algebra.

Let us define infinitesimal gauge transformations as follows:

\[
\delta W^A = d\varepsilon^A - \varepsilon^B \frac{\delta F^A}{\delta W^B}
\]

(3.5)

where \( \varepsilon^A \) is a \((\text{deg}^A -1)\)-form if \( W^A \) is a \(\text{deg}^A\)-form (0-forms do not give rise to any gauge parameters). By using eqs.(3.1), (3.2), one can readily see that

\[
\delta R^A = - R^c \frac{\delta}{\delta W^c} \left( \varepsilon^B \frac{\delta F^A}{\delta W^B} \right)
\]

(3.6)

and, therefore, gauge transformations (3.5) describe some symmetry of equations (3.4).

For Lie (super)algebras, transformations (3.5) coincide with ordinary (Yang-Mills) infinitesimal gauge transformations. However, in the general case, FDA's and gauge transformations (3.5) corresponding to them are not related directly to any algebras. As a result, the composition properties of the transformations (3.5) resemble those of "open algebras" which are familiar for supergravity theories without auxiliary fields. Indeed, it follows from eq.(3.5) that
\[ \left[ \delta_2, \delta_1 \right] W^A = \delta_{1,2} W^A + \Delta_{1,2} W^A \]  

(3.7)

where \( \delta_{1,2} \) is the transformation (3.5) with the parameter \( \varepsilon_{1,2} \) of the form

\[ \varepsilon_{1,2}^A = \varepsilon_1^B \varepsilon_2^C (-1)^{\delta_{1}(B) \delta_{2}(B)} S^2 F^A \]  

(3.8)

and

\[ \Delta_{1,2} W^A = (-1)^{\delta_{2}(A) \delta_{1}(A)} \varepsilon_1^B \varepsilon_2^C R^D \frac{S^3 F^A}{S W^2 S W^2 S W^2}. \]  

(3.9)

It follows from eq. (3.9) that the transformations (3.5) close if \( F^A(W) \) are

more than quadratic in all differential forms \( W^A \) (in addition, because no gauge parameters are related to 0-forms, \( F^A(W) \) may contain any terms which are no more than linear in all p-forms with \( p > 0 \), while depending arbitrarily on 0-forms).

In particular, this is the case for Lie (super)algebras.

Although, strictly speaking, FDA's are not algebras at all, many useful operations relevant to usual algebras admit natural generalizations to FDA's as well.

For example, we say that some invertible linear mapping \( \mathcal{T} \) of the exterior (\( \otimes \) Grassmann) algebra, in which \( W^A \) take on their values, provides an automorphism of the FDA (3.1) if

\[ \mathcal{T}(F^A(W)) = F^A(\mathcal{T}(W)) \]  

(3.10)

(quite similarly one can define conjugations, involutions etc.).

When analysing motion equations for massless and auxiliary fields, we confine ourselves to the specific case of FDA's formed by 1-forms \( \omega^a \) and 0-forms \( \zeta^b \) belonging to one and the same linear space (i.e. the indices \( a \) and \( b \) take on equal numbers of values). Let us consider a further subclass of the FDA's above, such that

\[ \omega R^a = d \omega^a + F^a(\omega, C), \]  

(3.11)

\[ c R^a = d C^a - C^b S F^a(\omega, C) \]  

\( = - C^b \frac{S \omega^a}{S \omega^a} \omega R^a. \)  

(3.12)

In this case, consistency conditions (3.2) for the curvatures \( \omega R^a \) read

\[ \frac{S F^a(\omega, C)}{S \omega^a} - C^d \frac{S F^d(\omega, C)}{S \omega^a} \frac{S F^a(\omega, C)}{S C^g} = 0. \]  

(3.13)
One can readily see that eq. (3.13) ensures simultaneously the consistency of eq. (3.12). In fact, this is the most important property of the FDA's (3.11), (3.12).

When $F^{\alpha}(\omega, \mathfrak{c})$ do not depend on $\mathfrak{c}$, eq. (3.11) describes some Lie (super)algebra, while eq. (3.12) describes 0-forms in its adjoint representation. In the general case, any FDA (3.11), (3.12) can be treated as some deformation of the FDA (3.11), (3.12) with $\hat{F}^{\alpha}(\omega, \mathfrak{c}) = F^{\alpha}(\omega, \mathfrak{c})$ (i.e. as a deformation of FDA related to some Lie superalgebra as explained above). Indeed, let us redefine 0-forms $C^\alpha$ by introducing a deformation parameter $\eta \neq 0$ as follows:

$$C^\beta = \eta \ C^i \delta^i_j \tag{3.14}$$

Inserting eq. (3.14) into eqs. (3.11), (3.12) and normalizing correctly the "curvatures" for 0-forms $C^i$, one finds that eqs. (3.11), (3.12) take the form

$$\omega R^\alpha = d\omega^\alpha + F^{\alpha}(\omega, \eta \mathfrak{c}) - \sum_{i=1}^{n} F_{\alpha}^{\gamma}(\omega, \eta \mathfrak{c}) \frac{\partial F^{\gamma}(\omega, \eta \mathfrak{c})}{\partial \omega^i} \tag{3.15}$$

$$C R^\alpha = dC^\alpha - C^\beta \frac{\partial F^{\alpha}(\omega, \eta \mathfrak{c})}{\partial \omega^\beta} \tag{3.16}$$

(the primes for $C$ are omitted). Thus, one can investigate a full FDA (3.15), (3.16) by expanding in powers of $\eta$ when a zero-order FDA is known corresponding to $\eta = 0$. In the case of massless and auxiliary fields, we identify this zero-order FDA with the Lie superalgebra shs(1) supplemented by 0-forms in its adjoint representation.

Let us emphasize that whence a zero-order FDA is governed by some Lie superalgebra (and 0-forms in its adjoint representation) in accordance with eqs. (3.15), (3.16) at $\eta = 0$, it is natural to expect that full consistent "curvatures" $\omega R$ and $C R$ can be reduced to the form (3.15), (3.16) in all orders too (if they exist). Indeed, this is supported by the observation that if eqs. (3.15), (3.16) are shown to be consistent up to the $n^{th}$ order, and if one finds also $\omega R$ in the $(n+1)^{th}$ order, such that the consistency condition of the type (3.13) holds in this order, then the $(n+1)^{th}$-order deformation of $C R$, defined via eq. (3.16), turns out automatically to be consistent due to eq. (3.13). It is also important here that one needs only to know $\omega R$ and $C R$ up to the $n^{th}$ order when determining $\omega R$ in the $(n+1)^{th}$ order, i.e. $\omega R$ in the $(n+1)^{th}$ order can be found out independently on $C R$ in the $(n+1)^{th}$ order.
Another important point is that one can pass from one FDA (3.15),(3.16) to another by introducing new variables,

\[ \omega'\alpha = f^\alpha(\omega, \gamma \xi) \quad C'\alpha = C \frac{\delta f^\alpha(\omega, \gamma \xi)}{\delta \omega} \]  

at the condition \( \text{det} \left| \frac{\delta f^\alpha(\omega, \omega)}{\delta \omega} \right| \neq 0 \). The corresponding function \( F' \) in eqs. (3.15),(3.16) for \( \omega' \) and \( C' \) is of the form

\[ F'(\omega', C') = F'_{(\omega, C)} \frac{\delta f^\alpha(\omega, C)}{\delta \omega} - C \frac{\delta F_{(\omega', C')}}{\delta C'} \frac{\delta f^\alpha(\omega, C)}{\delta C} \]  

(\( \omega \) and \( C \) on the r.h.s. of eq. (3.18) should be expressed in terms of \( \omega' \) and \( C' \) with the aid of eq. (3.17)). Note that the fact is used in the derivation of this result that \( f^\alpha(\omega, C) \) is linear in \( \omega \) because \( \omega^\alpha \) are 1-forms.

Obviously, equations of the form (3.4) for the variables \( \omega \) and \( C \) are equivalent to those for \( \omega' \) and \( C' \) (3.17) at least in the framework of the perturbation expansion in powers of \( \eta \). Therefore, in practice, one has to factorize all deformations (3.15),(3.16) over trivial deformations originating from the transition to new variables \( \omega', \xi' \) in the initial (undeformed) FDA related to some Lie superalgebra and its adjoint representation. The deformation constructed in the subsequent sections is nontrivial in the sense that it cannot be obtained from the undeformed FDA by means of any admissible field redefinition (3.17).

Now we would like to discuss a method which enables one to construct some extensions of FDA's. Given FDA containing differential forms \( W^A \), we consider some d-dimensional manifold \( M \) such that \( d > \text{deg}(W^A) \) for all \( A \), and introduce an "action"

\[ S = \int_M R^A \lambda_A \]  

where \( \lambda_A \) are new differential forms such that \( \text{deg}(\lambda_A) = d - \text{deg}(W^A) - 1 \).

Require the action \( S \) to be invariant under the transformations (3.5). Then it follows from eq. (3.6) that the transformation law for \( \lambda_A \) should have the form

\[ \delta \lambda_A = \frac{\delta}{\delta W^A} (\epsilon^B \frac{\delta f^D}{\delta W^B}) \lambda^D \]  

(3.20)

On the other hand, it follows from the Bianchi identities (3.3) that \( S \) is invariant under additional transformations.
\[ \delta W^A = 0, \quad \delta \lambda_A = d \tilde{\mathcal{F}}_A - (-1)^{\text{deg}(\tilde{\mathcal{F}}_A)} \frac{S F^B}{\delta W^A} \tilde{\mathcal{F}}_B \]  

(3.21)

where \( \tilde{\mathcal{F}}_A \) are new gauge parameters \( \text{deg}(\tilde{\mathcal{F}}_A) = d - \text{deg}(W^A) - 2 \) when \( \text{deg}(\lambda_A) > 2 \) and no gauge parameters correspond to 0-forms among \( \lambda_A \).

As is easily seen, the action \( S \) leads to the following motion equations

\[ R^A_W = 0, \quad \tilde{R}^A_A = 0 \]  

(3.22)

where \( R^A_W \) are curvatures (3.1) of the original FDA, while the "curvatures" \( \tilde{R}^A_A \) are of the form

\[ \tilde{R}^A_A = d \lambda_A - (-1)^{\text{deg}(W^A)} \frac{S F^B}{\delta W^A} \lambda_B. \]  

(3.23)

By using eq. (3.2), one can easily see that the curvatures \( R^A_W \) (3.1) and \( \tilde{R}^A_A \) (3.23) describe some new FDA, while the transformations (3.5), (3.20), (3.21) are nothing but the gauge transformations (3.5) for the full extended FDA (3.1), (3.23).

Since it is assumed that equations of motion for massless and auxiliary fields are of the form (3.4) for some FDA, one can attempt to describe a dynamics of these fields by means of the action (3.19). In accordance with the general rules above, 2-forms \( \lambda \alpha \) and 3-forms \( \lambda \sigma \) should be introduced in the case under consideration \( (d=4) \) with the indices \( \alpha \) corresponding to the adjoint representation of shsa(1).

An apparent defect of the action (3.19) consists of the fact that it changes its sign when \( \lambda_A \rightarrow -\lambda_A \), and, therefore, this action cannot lead to positive-definite energy. However, one should take into account that the hamiltonian vanishes in any general coordinate invariant theory when constraints are satisfied and, therefore, this question needs for a more detailed investigation. If one will succeed in finding some mechanism providing non-zero vacuum values for the forms \( \lambda_A \) (for example, this will be the case if \( \hat{\lambda}_A \) will contain a part proportional to the gravitational vierbein \( h^\alpha \)), which belongs to the full set of 1-forms \( \omega^A \) related to shsa(1), then, perhaps, this problem can be avoided. However there is an apparent difficulty for such a mechanism due to the fact that equations \( \tilde{R}^A_A = 0 \) are linear in \( \lambda_A \).

Another potential difficulty of using the action \( S \) (3.19) is that the differenti-
al forms $\lambda_A$ may lead to some new degrees of freedom and the indefinite metric problem (ghosts). Although this question was not investigated by us carefully enough, there are some indications that the fields $\lambda_A$ are most likely of auxiliary type in the case of the FDA of massless fields, i.e. that the true number of independent degrees of freedom related to $\lambda_A$ vanishes after imposing a complete set of gauge conditions. This is possible because the system of fields $W^A$ under consideration is infinite-dimensional (with respect to the index $A$) and the kernels may distinguish between the differential operators in the linearized equations $R^A_w = 0$ and $\tilde{R}^A_A = 0$. It should be noted however that simultaneously such a structure of the quadratic part of the action (3.19) may lead to some troubles for quantization (when constructing propagators etc.):

Although the action (3.19) for massless fields deserves a more detailed investigation, it seems that this is too naive to be used as a physical action. As pointed out in §1, it is most likely that some broader systems of fields are needed in order to construct a physical action (i.e. to give an off-mass-shell formulation) which contain additional sets of auxiliary fields that become trivial on the mass shell.

4. Operator Realization of Superalgebra of Higher-Spins and Auxiliary Fields

In this section, we focus on the operator realization of the superalgebra $shs_a(1)$ proposed in ref./19/, because this enables us to use an efficient language of symbols of operators by Berezin /37,38/ when constructing nontrivial motion equations of massless and auxiliary fields.

It was shown in ref./12/ that higher-spin superalgebra $shs_0(1)$, constructed in ref./11/, admits the "operator realization" in which elements of $shs_0(1)$ are identified with polynomials constructed from the operators $\hat{\gamma}_a$ and $\hat{\lambda}_\beta$ that form associative Heisenberg algebra,

$$[\hat{\gamma}_a, \hat{\gamma}_\beta] = 2i \epsilon_{a\beta}, \quad [\hat{\lambda}_a, \hat{\lambda}_\beta] = 2i \epsilon_{a\beta}, \quad [\hat{\gamma}_a, \hat{\lambda}_\beta] = 0$$

(4.1)
\( (\varepsilon_{a\beta} = -\varepsilon_{\beta a}, \varepsilon_{12} = 1) \) As already mentioned in ref./19/, the superalgebra shsa(1) can be realized as the algebra formed by arbitrary polynomials constructed from the operators \( \hat{Q}_a, \hat{Z}_\beta \) \(^{(4.1)}\) and the operators \( \hat{Q} \) and \( \hat{K} \) obeying the relations
\[
\hat{Q} \hat{Q}_a = \hat{Q}_a \hat{Q}, \quad \hat{Q} \hat{Z}_\beta = \hat{Z}_\beta \hat{Q}, \quad \hat{Q}^2 = 1; \quad \hat{K} \hat{Q}_a = \hat{Q}_a \hat{K}, \quad \hat{K} \hat{Z}_\beta = -\hat{Z}_\beta \hat{K}, \quad [\hat{K}, \hat{Q}] = 0, \quad \hat{K}^2 = 1. \quad (4.3)
\]
The following hermiticity conditions are imposed on the operators \( \hat{Q}, \hat{Z}, \hat{\varphi} \) and \( \hat{R} \):
\[
(\hat{Q}_a)^\dagger = \hat{Q}_a, \quad (\hat{Z}_\beta)^\dagger = \hat{Q}_\beta, \quad \hat{\varphi}^+ = \hat{R}, \quad \hat{R}^\dagger = \hat{\varphi}. \quad (4.4)
\]
Note that the operators \( \hat{Q} \) and \( \hat{K} \) can be viewed as counterparts of the Klein operator for the numbers of undotted and dotted spinorial indices, respectively.

Eqs.\( (4.1)-(4.3) \) define associative algebra \( B \) of all polynomials constructed from the operators \( \hat{Q}, \hat{Z}, \hat{\varphi} \) and \( \hat{R} \). Complex Lie superalgebra shsa(1; \( \mathbb{C} \)) is related to \( B \) in the standard fashion after fixing the automorphism of boson-fermion parity.

f, as follows:
\[
f(\hat{Q}_a) = -\hat{Q}_a, \quad f(\hat{Z}_\beta) = -\hat{Z}_\beta, \quad f(\hat{\varphi}) = \hat{\varphi}, \quad f(\hat{R}) = \hat{R}. \quad (4.5)
\]

Real superalgebra shsa(1) coincides with the real form of shsa(1; \( \mathbb{C} \)) extracted by the conditions \( (4.4) \) (note that all necessary general facts on the relation between Lie superalgebras and associative algebras, extracting real forms out complex superalgebras, and some others are discussed in detail in ref./12/).

The whole set of gauge 1-forms, corresponding to shsa(1), is described by the generating function
\[
\omega(\hat{Q}, \hat{Z}, \hat{\varphi}, \hat{R}) = \sum_{a, \beta = 0, 1} (\hat{Q})^A (\hat{R})^B \omega^{A B}(\hat{Q}, \hat{Z}), \quad (4.6)
\]
\[
\omega^{A B}(\hat{Q}, \hat{Z}) = \sum_{n, m = 0}^{\infty} \frac{1}{2^n! n!} \omega^{A B m}(\hat{Q})^d \cdots \hat{Q}_a \hat{Z}_\beta \cdots \hat{Z}_\beta. \quad (4.7)
\]
The summation over \( A \) and \( B \) in eq.\( (4.6) \) is carried out from 0 to 1 since \( \hat{Q}^2 = \hat{K}^2 = 1 \).

The multispinors \( \omega^{A B m}(\hat{Q})^d \cdots \hat{Q}_a \hat{Z}_\beta \cdots \hat{Z}_\beta \) are assumed to be symmetric over spinorial indices since we use the Weyl ordering for the operators \( \hat{Q} \) and \( \hat{Z} \). Note that expansions of the type \( (4.6),(4.7) \) will be assumed to hold for all quantities which take
on their values in Lie superalgebra shsa(1) (i.e. for Weyl O-forms, curvature 2-forms etc.).

In accordance with the normal relation between spin and statistics, components of all differential forms carrying (odd) even numbers of spinorial indices are assumed to be (anti)commuting. For example,

\[
\omega^{AB}_{d(n), \delta(m)} \omega^{CD}_{y(k), \xi(e)} = -(1)^{(n+m+\lambda+k+\varepsilon)} \omega^{y(k), \xi(e)} \omega^{AB}_{d(n), \delta(m)}
\]  

(4.8)

(the additional minus sign on the r.h.s. of eq. (4.8) is due to the fact that \(\omega\) are 1-forms, and it is absent for Weyl O-forms). As for hermiticity conditions for \(\omega\), which follow from eq. (4.4), these are in accordance with eq. (2.31) (see also §6 where a class of superalgebras is discussed including shsa(1) as a particular case).

In operator terms, the curvature 2-forms for shsa(1) read

\[
\hat{\mathbf{R}}_{\hat{\mathbf{q}}, \hat{\mathbf{r}}, \hat{\mathbf{q}}, \hat{\mathbf{r}}} = d \hat{\mathbf{\omega}}(\hat{\mathbf{q}}, \hat{\mathbf{r}}, \hat{\mathbf{q}}, \hat{\mathbf{r}}) + \hat{\mathbf{\omega}}(\hat{\mathbf{q}}, \hat{\mathbf{r}}, \hat{\mathbf{q}}, \hat{\mathbf{r}}) \hat{\mathbf{\omega}}(\hat{\mathbf{q}}, \hat{\mathbf{r}}, \hat{\mathbf{q}}, \hat{\mathbf{r}}).
\]

(4.9)

The fact that they correspond to some Lie superalgebra follows from the Grassmann properties of \(\omega\) (4.8).

It is the Berezin's theory of symbols of operators /37, 38/ (see also ref. /12/), which will be used in practice for description of shsa(1) and of the corresponding FDA. Namely, instead of the operators \(\hat{q}_{\alpha}\) and \(\hat{\zeta}_{\beta}\), one introduces their symbols, \(q_{\alpha}\) and \(\zeta_{\beta}\), obeying the commutation relations

\[
[q_{\alpha}, q_{\beta}] = 0, \quad [\zeta_{\alpha}, \zeta_{\beta}] = 0, \quad [q_{\alpha}, \zeta_{\beta}] = 0.
\]

(4.10)

However, the symbols \(q\) and \(r\) are supposed to satisfy commutation relations of the type (4.2) with the "Klein operators" \(\hat{q}\) and \(\hat{r},

\[
q_{\alpha} \hat{q} = -\hat{q} q_{\alpha}, \quad \zeta_{\beta} \hat{q} = \hat{q} \zeta_{\beta}, \quad \hat{q}^2 = 1;
\]

\[
\zeta_{\alpha} \hat{r} = -\hat{r} \zeta_{\alpha}, \quad q_{\alpha} \hat{r} = \hat{r} q_{\alpha}, \quad \hat{r}^2 = 1, \quad [\hat{q}, \hat{r}] = 0.
\]

(4.11)  

(4.12)

Instead of operators \(\omega\) (4.6), (4.7), we shall use their symbols \(\omega(q, r, q, r)\) with respect to \(\hat{q}\) and \(\hat{r},

\[
\omega(q, r, q, r) = \sum_{A_{\alpha}, B_{\beta} = 0, 1} (\hat{q})^A (\hat{r})^B \omega^{AB}(q, r)
\]

\[
(4.13)
\]

\[
\omega^{AB}(q, r) = \sum_{n, m = 0}^{\infty} \frac{1}{n! m!} \omega^{AB}_{d(n), \delta(m)} q_{\alpha} \ldots q_{\alpha} r_{\beta} \ldots r_{\beta}
\]

(4.14)
(and similarly for Weyl 0-forms, curvature 2-forms etc.).

The associative product \( \hat{\phi} \hat{\chi} \) of operators \( \hat{\phi} \) and \( \hat{\chi} \), constructed from \( \hat{q} \) and \( \hat{r} \), induces an associative product \( f \star g \) of their (Weyl) symbols (\( f \star g \) is the Weyl symbol of \( \hat{f} \hat{g} \)). This product law can be described for example by the following formula /37,38/ (see also ref./12/):

\[
(f \star g)(Z_0) = \left[ \exp -i(\alpha_{0} + \alpha_{2} + \alpha_{1} + \beta_{0} + \beta_{2} + \beta_{1}) \right] \times
\]

\[
x f(Z_1) g(Z_2) |_{Z_1 = Z_2 = 0}.
\]

Here \( Z_i \) \( (i=0,1,2) \) denote three independent sets of variables \( (q_{\alpha}, \tau_{\beta}) \), while \( a_{ij} \) and \( b_{ij} \) are differential operators defined by the relations

\[
a_{ij} = P_{i\alpha} P_{j\beta}, \quad b_{ij} = t_{i\beta} t_{j\alpha} \tag{4.16}
\]

where, by definition,

\[
P_{\alpha \alpha} = q_{0\alpha}, \quad t_{\alpha \beta} = \tau_{0\beta} \tag{4.17}
\]

and

\[
P_{j\alpha} = i \frac{\partial}{\partial q_{j\alpha}}, \quad t_{i\beta} = i \frac{\partial}{\partial \tau_{i\beta}} \tag{4.18}
\]

at \( \beta > 0 \). By their very construction, the operators \( a_{ij} \) and \( b_{ij} \) are skewsymmetric,

\[
a_{ij} = -a_{ji}, \quad b_{ij} = -b_{ji}. \tag{4.19}
\]

Note that although the indices \( i \) and \( j \) take on only three values in this section, \( i,j = 0-2 \), in the subsequent sections the indices in eqs. (4.16)-(4.19) will be allowed to take on some other numbers of values as well.

With the aid of eqs. (4.11), (4.12), (4.15) one can readily see that the components of the type (4.13) for the curvatures (4.9), \( R^A_B(Z) \), take the form

\[
R^A_B(Z_0) = d \omega^A_B(Z_0) + \sum_{C,F,G} S(1A + C + F|2) \xi(IA + D + G|2) \times
\]

\[
x \exp \left(-i \sum_{m,n} (\alpha_{nm} + \beta_{nm}) \omega^C_D(Z_1^F \omega^F_G(Z_2)) \right) |_{Z_1 = Z_2 = 0}
\]

where, by definition,

\[
\sum_{m,n} (-1)^N q_{\alpha} (-1)^M \tau_{\beta} = (-1)^N q_{\alpha} \xi_{i\alpha} (-1)^L \tau_{i\beta} \tag{4.21}
\]

Analogously, the covariant derivative in the adjoint representation of \( \text{shsas}(1) \) reads
\[
[\mathcal{D}_0(C)]^{AB}(Z_0) = dC^{AB}(Z_0) + \sum_{C,F, G=0}^{27} S(1A+C+Fl_2) S(1B+D+Gl_2) x \exp(-i \sum_{m,n=0-2}^{\infty} \alpha_{nm} + \beta_{nm}) \left[ \omega^{CD}(Z_1) C^{FG}(Z_2) - C^{CD}(Z_1) \omega^{FG}(Z_2) \right]_{Z_2=Z_0}^{(4.22)}
\]

where \(C^{AB}(Z_0)\) are components of the type \((4.13)\) for the Weyl 0-forms. The curvatures of \(\text{sha}(1)\) and the covariant derivative in its adjoint representation are denoted as \(R_0\) and \(\mathcal{D}_0\) because they correspond to the zero order (undeformed) part of the FDA of massless and auxiliary fields. It is a simple exercise to show that eq.\((4.20)\) leads to eq.\((2.29)\) for components \((4.14)\).

§ 5. First Order Deformation of Free Differential Algebra of Massless and Auxiliary Fields

Assuming that the "curvatures" and "covariant derivatives" corresponding to a full FDA of massless and auxiliary fields are expanded as follows:

\[
R = \sum_{n=0}^{\infty} (\eta)^n R_n(\omega, C), \quad \mathcal{D}(C) = \sum_{n=0}^{\infty} (\eta)^n \mathcal{D}_n(\omega, C),
\]

where the deformation parameter \(\eta\) is introduced by means of the substitution \((3.14)\), while \(R_0\) and \(\mathcal{D}_0\) are defined by eqs.\((4.20)\) and \((4.22)\) respectively, we look for a first order deformation \(R_1(\omega, C)\) in the form

\[
R_1^{FG}(Z_0) = \sum_{A, B, N, M=0-1} \exp(-i \sum_{m,n=0-3}^{\infty} \alpha_{nm} + \beta_{nm}) x \left[ S(1A + X + N + F + 1l_2) S(1B + L + M + G + 1l_2) x \right.
\]
\[
\times \left[ f_I(P_0, P_1, P_2, P_3) \omega^{AB}(Z_1) \omega^{KL}(Z_2) C^{NM}(Z_3) \right] \right)
\]
\[
\left. + f_{II}(P_0, P_1, P_2, P_3) \omega^{AB}(Z_1) \omega^{KL}(Z_2) C^{NM}(Z_3) \right)
\]
\[
\left. + f_{III}(P_0, P_1, P_2, P_3) \omega^{AB}(Z_1) \omega^{KL}(Z_2) C^{NM}(Z_3) \right]
\]
\[
\times [S(1A + X + N + F + 1l_2) S(1B + L + M + G + 1l_2) x \left[ g_I(t_0, t_1, t_2, t_3) \omega^{AB}(Z_1) \omega^{KL}(Z_2) C^{NM}(Z_3) \right]
\]
where the differential operators $a_{ij}, b_{ij}, p_i$ and $t_i$ are defined by eqs. (4.16) - (4.18) as before. As for the deformation $\mathcal{D}_2(C)$, it is assumed to be related to $R_2$ (5.2) by the general relation (3.12).

Let us stress that all possible orderings of $\omega$ and $C$ are written in eq. (5.2), but no product factors will be commuted in the subsequent analysis (only the associativity of the multiplication will be used). In fact, this implies that we are solving the problem not only for $\text{shsa}(1)$ itself but also for the much more general case when all quantities at hand take on their values in an arbitrary associative algebra $A$ (for more detail see next section).

Note also that the deformation (5.2) is not of the most general form (for example, $f_K$ depend only on $p$ while $g_K$ depend only on $t$). Nevertheless, eq. (5.2) is in accordance with the structure of the linearized deformation (2.33), and the practical analysis led us to the conclusion that only deformations of this kind are relevant to the problem under investigation.

The functions $f_K$ and $g_K$ in eq. (5.2), are supposed to depend only on the Lorentz covariant combinations, $a_{ij}$ and $b_{ij}$ (4.16), of their arguments, $p_i$ and $t_i$. Moreover, it is assumed that $f_K$ and $g_K$ can be expanded in power series of $a_{ij}$ and $b_{ij}$. This very important condition guarantees that if $\omega(Z)$ and $C(Z)$ are polynomial functions of their arguments, the r.h.s. of eq. (5.2) is polynomial in $Z_0$ too (if $f$ and $g$ are singular at the point $a_{ij} = b_{ij} = 0$, then the r.h.s. of eq. (5.2) becomes meaningless for the expansions of the type (4.14)).

It is not very difficult to make sure that, in the first order in the deformation parameter $\eta$, the consistency conditions (3.2) impose the following restrictions on the functions $f_K$:

$$
\begin{align*}
&f_I(p_0, p_1, p_2, p_3, p_4) - f_I(p_0, p_2, p_2 + p_3, p_4) + \\
&+ f_I(p_0, p_1, p_2, p_3 + p_4) - f_I(p_0 + p_2, p_2, p_3, p_4) = 0,
\end{align*}
$$

(5.3)
\(- f_I(p_0, p_1, p_2, p_3 + p_4) + f_{II}(p_0, p_1, p_2, p_3, p_4) - f_{III}(p_0, p_1, p_2, p_3, p_4)\) 
\(+ f_{IV}(p_0 + p_4, p_1, p_2, p_3, p_4)\exp(2i \sum_{j=0}^{3} \alpha_j) - f_{V}(p_0 + p_3, p_2, p_3, p_4) = 0\) 
\(f_{II}(p_0, p_1, p_2 + p_3, p_4) - f_{II}(p_0, p_1, p_2, p_3 + p_4) + f_{II}(p_0, p_1, p_2, p_3, p_4)\) 
\(- f_{III}(p_0 + p_3, p_2, p_3, p_4) + f_{III}(p_0 + p_4, p_2, p_3, p_4)\exp(2i \sum_{j=0}^{3} \alpha_j) = 0\) 
\(- f_{IV}(p_0, p_2 + p_3, p_3, p_4) + f_{IV}(p_0, p_2, p_3, p_4) - f_{IV}(p_0, p_2, p_3, p_4 + p_4)\) 
\(+ f_{IV}(p_0 + p_4, p_2, p_3, p_4)\exp(2i \sum_{j=0}^{3} \alpha_j) = 0\) 

(the l.h.s. of eqs. (5.3)-(5.6) coincide with the coefficients against the terms in eq. (3.2), which are, respectively, of the form \(\omega^3 C, \omega^2 C^2, \omega C, C^2, C^3\). As for the analogous restrictions on the functions \(f_k\), they can be obtained from eqs. (5.3)-(5.6) by means of the replacement \(f_k \rightarrow g_k, p_k \rightarrow t_1, \alpha \rightarrow c_1\). 

Eqs. (5.3)-(5.6) admit many solutions. However, almost all of them originate from the trivial field redefinitions (3.17) applied to the undeformed FPA based on \(R_0\) and \(D_0(C)\). Indeed, by making redefinitions of the form 
\[\omega^{FE}(Z_0) = \omega^{FE}(Z_0) + \sum_{\alpha, \beta, \gamma, \delta} S(I\alpha + N + P + Z_1) S(IB + M + E) x \exp\left(-\sum_{m,n} (\alpha m + \beta n)\right) \left[ \frac{\partial}{\partial \alpha \beta \gamma \delta} \left( C^{NM}(Z_1) \right) \right] \bigg|_{Z_1 = Z_2 = 0}\]

one can see that these produce curvatures \(R_1\) with the functions \(f_k\) of the form 
\[f_I(p_0, p_1, p_2, p_3) = - \frac{\partial}{\partial \alpha} \left( p_0, p_2, p_3 + p_4 \right) + \frac{\partial}{\partial \beta} \left( p_0, p_1, p_2 + p_3 \right)\]
\[f_{II}(p_0, p_1, p_2, p_3) = - \frac{\partial}{\partial \alpha} \left( p_0, p_2, p_3 + p_4 \right) + \frac{\partial}{\partial \beta} \left( p_0, p_1, p_2 + p_3 \right)\]
\[f_{III}(p_0, p_2, p_3, p_4) = \frac{\partial}{\partial \alpha} \left( p_0, p_2, p_3 + p_4 \right) + \frac{\partial}{\partial \beta} \left( p_0, p_1, p_2 + p_3 \right)\]

It is not too difficult to verify that eqs. (5.8)-(5.10) provide for solutions of eqs. (5.3)-(5.6). However, as already mentioned in §3, all such solutions are dynami-
in the sector of gauge fields (1-forms).

ocally trivial; they do not lead to nontrivial equations for massless fields. As a
result, a typically cohomological problem arises of constructing those solutions of
eq (5.3)-(5.6) (cocycles) which do not reduce to trivial solutions (5.8)-(5.10)
(co-boundaries). Let us also mention that there exist functions $\mathcal{P}_{\mu \nu}$ trivial twice*,
leading to vanishing functions $f_{\mu \nu} (5.8)-(5.10)$. These originate from various gau-
ge transformations of the undeformed FDA with the gauge parameters depending linear-
ly on the Weyl 0-forms $C$.

A nontrivial solution of eqs. (5.3)-(5.6), leading to appropriate higher-spin
equations, is

$$f_{\mu} (p_i) = \alpha_{i2} \int d^4 \xi \delta (d_2) \theta (d_2) \theta (d_3 - d_3) \delta (2 - d_3) \times$$

$$\times e^{\exp \left(-i \frac{3}{2} d \alpha_{nm} \right)} \quad (5.11)$$

$$f_{\mu \nu} (p_i) = f_{\mu \nu} (p_i) = 0 \quad (5.12)$$

where $\alpha_{nm}$ are defined in eq. (4.16). Formally, an integration in eq. (5.11) is car-
rried out over $\mathbb{R}^4$. However, a non-vanishing contribution is only from some compact
domain (triangle in the plane of $\alpha_{i2}$ and $d_2$). It is important that $f_{\mu \nu} (5.11)$,
(5.12) are therefore analytical functions of their arguments. In Appendix, we show
explicitly that this solution reproduces correctly the linearized deformation (2.33)
after appropriate field redefinitions (i.e. they belong to the same cohomology
class).

Since the construction of the deformation (5.11),(5.12) is one of the central
results of the paper, let us comment briefly how it was found out. The main trick
was to look for such singular functions $\mathcal{P}_{\mu \nu}$ that nevertheless lead to analytical
functions $f_{\mu \nu} (5.8)-(5.10)$. If one succeeds in finding such functions $\mathcal{P}_{\mu \nu}$, then
this would most probably lead to some nontrivial deformation because $\mathcal{P}_{\mu \nu}$ do not be-

* Unfortunately, we have no space to re-formulate completely all these points in
the cohomological terms (for general problem setting see refs. [6,7]). Perhaps, this
may be justified by the fact that the fine cohomological language does not ensure by
itself constructing nontrivial cohomology classes of interest. If desires, the rea-
der can easily proceed in this direction by identifying the operation $\delta$ possessing
the property $\delta^2 = 0$ with the linear operators acting on the r.h.s. of eqs. (5.3)-(5.6)
(5.8)-(5.10) and their further generalizations to arbitrary numbers of arguments.
long to the allowed class (let us emphasize once again that only the functions \( \varphi \) and \( f \) which can be expanded in power series are meaningful when operating with the expansions of the type (4.13), (4.14) in which the coefficients \( (\omega)^{A B a} (\mathbf{B})^{(m)} \) are identified with physical fields). Indeed, the resulting (analytical) functions \( f \) are expected to obey eqs. (5.3)-(5.6) because, formally, all functions of the form (5.8)-(5.10) obey these equations. Nevertheless, an explicit verification of this fact is needed because one must prove that eqs. (5.3)-(5.6) hold operating with regular functions only (that is without referring to the original decomposition (5.8)-(5.10) containing singular functions \( \varphi_K \)).

In practice, we sought the singular functions \( \varphi_K \) in the form \( \varphi_K \sim K_k \exp \left( \sum \mu_{ij}^k d_{ij} \right) \) where \( \mu_{ij}^k \) are some constants, while the factors of \( K_k \) are singular in the point \( d_{ij} = 0 \). A structure of exponentials was guessed easily enough and the main problem was to find out singular factors of \( K_k \). The appropriate functions \( K_k \) turned out to be of the form \( d_{ij} \left( \xi, \eta, \epsilon_0 \right)^{\sum \omega_{nm} d_{nm}} \) where \( \xi \) and \( \eta \) are some fixed coefficients. In fact, their double pole structure can be seen from the final result (5.11) after completing the integration over the variables \( d \xi \). Let us however stress once again that the function (5.11) is analytical because the integration in eq.(5.11) is carried out over a compact domain.

Leaving aside a more detailed discussion of this procedure which is in fact tedious enough, we verify explicitly that eqs. (5.11), (5.12) provide for some solution of eqs. (5.3)-(5.6). As already mentioned, such verification is necessary in any case.

Eqs. (5.5), (5.6) hold trivially due to eq. (5.12). Insertion of eqs. (5.11), (5.12) into eq. (5.4) leads to the condition

\[
\alpha_{12} \int d \phi \delta(\phi_0) \Theta(\phi_1) \Theta(\phi_2-\phi_1) \delta(\phi_3-2) \delta(\phi_3-\phi_4) \times
\exp(-i \frac{\phi}{\gamma, m=0} d_n d_{nm}) =
\]

\[
= \alpha_{12} \int d \phi \delta(\phi_0) \delta(\phi_0-\phi_1) \delta(\phi_3-2) \Theta(\phi_2) \Theta(\phi_2-\phi_1) \times
\Theta(2-\phi_2) \exp(-i \frac{\phi}{\gamma, m=0} d_n d_{nm} + 2i \frac{\phi}{\gamma, n=0} d_{nm})
\]

(5.13)
which is trivially satisfied after transition to the new integration variable \( \alpha'_4 = \alpha_4 + 2 \) on the r.h.s. of eq.(5.13). Finally, the l.h.s. of eq.(5.3) is of the form

\[
\chi = \int d^5 \alpha \exp(-i \sum_{n,m=0}^{N-1} \alpha_n \alpha_{nm}) S(\alpha_0) S(\alpha_4 - 2) \times
\]

\[
\times \left\{ (\alpha_{13} + \alpha_{23}) \Theta(\alpha_2 - \alpha_0) \Theta(\alpha_3 - \alpha_2) \Theta(\alpha_4 - \alpha_3)
- (\alpha_{12} + \alpha_{13}) \Theta(\alpha_1 - \alpha_0) \Theta(\alpha_2 - \alpha_1) \Theta(\alpha_3 - \alpha_2) \Theta(\alpha_4 - \alpha_3)
+ \alpha_{12} \Theta(\alpha_1 - \alpha_0) \Theta(\alpha_2 - \alpha_1) \Theta(\alpha_3 - \alpha_2) \Theta(\alpha_4 - \alpha_3)
- \alpha_{23} \Theta(\alpha_2 - \alpha_0) \Theta(\alpha_3 - \alpha_2) \Theta(\alpha_4 - \alpha_3) \right\}
\]

(5.14)

In order to show that \( \chi = 0 \), we have to use the fact that spinorial indices take on only two values (this is the only but very important point when the two-componentness of spinorial indices is actually taken into account). Namely, the following identity is to be used:

\[
ad_{ij} d_{ke} + ad_{ik} d_{ej} + ad_{ie} d_{jk} = 0
\]

(5.15)

which is the consequence of the definition of \( ad_{ij} \) (4.16) and of the simple fact that the full antisymmetrization over any three two-component spinorial indices gives zero. In its turn, it follows from eq.(5.15) that the following relation is true:

\[
\int d^N \alpha \exp(-i \sum_{n,m=0}^{N-1} \alpha_n \alpha_{nm}) \times
\]

\[
\times \left[ ad_{ij} \frac{\partial}{\partial \alpha_k} + ad_{ik} \frac{\partial}{\partial \alpha_j} + ad_{ie} \frac{\partial}{\partial \alpha_k} \right] \rho(\alpha) = 0
\]

(5.16)

when the measure \( \rho(\alpha) \) has a compact support, as is readily seen by the integration by parts. Now, by rewriting \( \chi \) (5.14) in the form

\[
\chi = -\int d^5 \alpha \exp(-i \sum_{n,m=0}^{N-1} \alpha_n \alpha_{nm}) \left[ a_{33} \frac{\partial}{\partial \alpha_1} + a_{31} \frac{\partial}{\partial \alpha_2} + a_{12} \frac{\partial}{\partial \alpha_3} \right] \times
\]

\[
\times \Theta(\alpha_1 - \alpha_0) \Theta(\alpha_2 - \alpha_1) \Theta(\alpha_3 - \alpha_2) \Theta(\alpha_4 - \alpha_3) S(\alpha_4 - 2),
\]

(5.17)

one proves that \( \chi = 0 \), that is eq.(5.3) holds as well. (Note that eq.(5.15) is also to be used when constructing the deformation (5.11), (5.12) via singular functions \( \gamma_{1,II} \)).
Thus, it is shown that eqs. (5.11), (5.12) describe some (consistent) deformation $R_{FG}^\mathcal{I}$ (5.2). Since various deformations (5.2), which differ by trivial deformations (5.8)-(5.10), are equivalent from the point of view of description of dynamics of massless fields (at least within the perturbation expansion over Weyl 0-forms), one can equally well choose some other representative of the same equivalence class. Specifically, one can readily verify that the deformation (5.11), (5.12) is equivalent to the following
\begin{equation}
\tilde{f}_I = \tilde{f}_{II} = 0,
\end{equation}
\begin{equation}
\tilde{f}_{I} \equiv \alpha_{23} \int d^3x \, \delta(d_0) \delta(d_1) \delta(d_2-d_3) \theta(2-d_2) \exp(-i \frac{3}{2} \alpha n \alpha nm).
\end{equation}
The difference $\tilde{f}_I - f_I$ is described by eqs. (5.8)-(5.10) with the functions $\varphi_\mathcal{I}$ of the form
\begin{equation}
\varphi_\mathcal{I}(p_0, p_1, p_2) = - \varphi_\mathcal{I}(p_0, p_1, p_2) = \alpha_{12} \int d^3x \, \delta(d_0) \delta(d_1) \delta(d_2-d_3) \theta(2-d_2) \exp(-i \frac{3}{2} \alpha n \alpha nm)
\end{equation}
(to prove this, one has to use eq. (5.16) once again).

Choosing $\eta_1(f+\tilde{f})$ as a representative of the nontrivial cohomology class and taking into account that the whole setting is completely symmetric with respect to the replacement $f_I \leftrightarrow \tilde{f}_I$, $\varphi_\mathcal{I} \leftrightarrow \varphi_\mathcal{I}$, $\rho_n \leftrightarrow \tau_n$, $\alpha nm \leftrightarrow \beta nm$, we arrive at the following final result
\begin{equation}
R_{FG}^\mathcal{I}(Z_0) = \sum_{A, B, K, L, M, N = 0} \int d^4x \, \delta(d_0) \delta(d_2-d_3) \theta(2-d_2) \exp(-i \frac{3}{2} \alpha n \alpha nm) \times \frac{1}{2} [\alpha_{12} \delta(d_2) \theta(2-d_3) (\omega)^AB(z_1^*)^{K+N+1} L + M) \omega^{K+1} L (Z_2^*)^{N+1} M) C^{NY}(Z_3^{10}) + \alpha_{23} \delta(d_4) \theta(2-d_3) C^{AB}(Z_1^*)^{K+N+1} L + M) \omega^{K+1} L (Z_2^*)^{N+1} M) \omega^{NM}(Z_3^{10})] + \gamma_2 \delta(d_4) \theta(2-d_3) (\omega)^AB(z_1^*)^{K+N+1} L + M) \omega^{K+1} L (Z_2^*)^{N+1} M) \omega^{NM}(Z_3^{10})]
\end{equation}
\[ x \left[ b_{12} \Theta(d_{1})(2-d_{3})(\omega^{AB}(Z_1^{x+N+L+M+1}) \omega^L(Z_2^{N+2M+1}) C^{NM}(Z_3^{O+1}) \right. \]
\[ + b_{23} \Theta(d_{2})(2-d_{3}) C^{AB}(Z_1^{x+N+L+M+1}) \omega^L(Z_2^{N+2M+1}) \omega^N(Z_3^{O+1}) \right] \bigg|_{Z_1=Z_2=Z_3=0} \]

where the deformation parameters \( \eta_1 \) and \( \eta_2 \) are introduced which remain arbitrary (the complex deformation constructed has two independent deformation parameters which however turn out to be conjugate in the real case - see §6). As for the deformation \( D_4(C) \) it is assumed to be related to \( R_4 \) (5.21) by the general relation 3.12. As mentioned in §3, \( D_4(C) \) constructed in such a way is automatically consistent as a consequence of the consistency of \( R_4 \). It reads
\[ D_4(C)_{FG}(Z_0) = \sum_{A,B} \int d^{3} \theta(d_{1}) \theta(d_{2}) \theta(d_{3}) \theta(d_{4}) x \]
\[ \times \exp(-i \frac{1}{2} \sum_{n,m=0}^{\infty} \alpha_{nm} + \beta_{nm}) \left( \eta_1 S(|A+K+N+F+1/2)|B+L+M+G|/2) \right) \]
\[ x \exp(-i \frac{1}{2} \sum_{n,m=0}^{\infty} \alpha_{nm}) \left[ b_{12} \Theta(d_{1})(2-d_{3})(\omega^{AB}(Z_1^{x+N+L+M+1}) \omega^L(Z_2^{N+2M+1}) C^{NM}(Z_3^{O+1}) \right. \]
\[ - C^{AB}(Z_1^{x+N+L+M+1}) \omega^L(Z_2^{N+2M+1}) C^{NM}(Z_3^{O+1}) \right] \]
\[ x \exp(-i \frac{1}{2} \sum_{n,m=0}^{\infty} \alpha_{nm} \beta_{nm}) x \]
\[ x \left[ b_{12} \Theta(d_{1})(2-d_{3})(\omega^{AB}(Z_1^{x+N+L+M+1}) \omega^L(Z_2^{N+2M+1}) C^{NM}(Z_3^{O+1}) \right. \]
\[ - C^{AB}(Z_1^{x+N+L+M+1}) \omega^L(Z_2^{N+2M+1}) C^{NM}(Z_3^{O+1}) \right] \bigg|_{Z_1=Z_2=Z_3=0} \]

Note that there exist more compact forms for \( D_4(C) \) which are not of the adjoint structure (3.12) and differ from \( D_4(C)(5.22) \) by some terms generated by trivial transformations \( C^T=h(C) \) where \( h(C) \) is some quadratic "function" of \( C \) which is not however related to the transformation of \( L \) by eq. (3.17).

In Appendix, it is shown that one can add such trivial deformation (5.8)-(5.10)
to the deformation (5.21) that, when linearized, the resulting deformation will coincide with the linearized deformation (2.33). In particular, it is therefore shown in Appendix that the deformation (5.21) is actually nontrivial. Although we have not a closed proof of the fact that the deformations (5.21) exhaust all nontrivial deformations of the initial (undeformed) FDA, the practical analysis gave us a belief that this is really the case.

In conclusion of this section, we rewrite the deformation (5.21) in terms of the fields of the type (4.13) depending on the operators \( \hat{Q} \) and \( \hat{R} \),

\[
R_2(Z_0, \hat{Q}, \hat{R}) = \int d^4 x \, S(d_0) \Theta(d_2-d_3) \Theta(d_3-d_2) \text{exp} \left( -i \sum_{m,n=0}^\infty (a_{nm} + b_{nm}) \right) \times \\
\left[ \eta_1 \text{exp} \left( -i \sum_{m,n=0}^\infty a_n c_{nm} \right) \right] \begin{bmatrix} \alpha_{12} \Theta(d_2) \Theta(d_3-d_2) \omega(Z_1, \hat{Q}, \hat{R}) \omega(Z_2, \hat{Q}, \hat{R}) \omega(Z_3, \hat{Q}, \hat{R}) \\
+ \alpha_{23} \Theta(d_2) \Theta(d_3-d_2) C(Z_1, \hat{Q}, \hat{R}) \omega(Z_2, \hat{Q}, \hat{R}) \omega(Z_3, \hat{Q}, \hat{R}) \end{bmatrix}_{Z_1=Z_2=Z_3=0} \\
\eta_2 \text{exp} \left( -i \sum_{m,n=0}^\infty b_n c_{nm} \right) \begin{bmatrix} \beta_{12} \Theta(d_2) \Theta(d_3-d_2) \omega(Z_1, \hat{Q}, \hat{R}) \omega(Z_2, \hat{Q}, \hat{R}) \omega(Z_3, \hat{Q}, \hat{R}) \\
+ \beta_{23} \Theta(d_2) \Theta(d_3-d_2) C(Z_1, \hat{Q}, \hat{R}) \omega(Z_2, \hat{Q}, \hat{R}) \omega(Z_3, \hat{Q}, \hat{R}) \end{bmatrix}_{Z_1=Z_2=Z_3=0} \\
\end{array}
\]

(5.23)

§ 6. Some Properties of the Free Differential Algebra of Massless and Auxiliary Fields

As already mentioned, the analysis of §5 remains valid when all quantities (\( \omega \), C, R...) take on their values in an arbitrary complex associative algebra A. This enables one to consider a broad class of extended-type theories of massless and auxiliary fields. A simplest possibility is to identify A with the associative algebra of n×n complex matrices, Mat\(_n\)(C). As follows from the Wedderburn's structural theorem /39/, any finite-dimensional semi-simple associative complex algebra A reduces to the direct sum of matrix algebras, \( A = \text{Mat}_{n_1}(C) \oplus \ldots \oplus \text{Mat}_{n_k}(C) \).

Since any associative algebra A of the form \( A = A_1 \oplus A_2 \) leads to two independent mutually non-interacting systems of fields (FDA's), only simple algebras \( A = \text{Mat}_n(C) \) are of interest (non-semi-simple algebras are disregarded because, as usual, they are ex-
pected to cause troubles on the lagrangian level due to the lack of a non-degene-
rate invariant bilinear form needed for construction of a lagrangian - see §7). Let
us note that extended higher-spin superalgebras of ref./12/ were based on the Clif-
ford algebras $A = \mathbb{C}_n(C)$. However, these are naturally included in the above class
due to isomorphisms $C_{2M}(C) \sim \text{Mat}_{2^n}(C)$ and $C_{2M+1}(C) \sim \text{Mat}_{2^n}(C) \oplus
\oplus \text{Mat}_{2^n}(C)$ ($C_n(C)$ are algebras of $\gamma$-matrices and it is well-known that vari-
ous products of $\gamma$-matrices form a complete set of matrices of an appropriate dimen-
sionality). Thus, in the subsequent analysis, we identify $A$ with $\text{Mat}_n(C)$ at some
fixed $n$. In fact, the only way for a nontrivial generalization of this class of as-
sociative algebras is to consider infinite-dimensional algebras. These will lead to
theories with infinite numbers of fields of any spin.

For physical applications, some reality (hermiticity) conditions should be im-
posed on the fields $\Omega$ and $C$, which respect the structure of deformed curvatures $R =
R_0 - \Omega R_1 \ldots$. We define an involution $\mu$, which extracts out an appropriate real
form of FDA under investigation, by identifying it with the operation $\dagger$ (4.4) when
acting on the operators $\hat{g}, \hat{f}, \hat{G}$ and $\hat{H}$ and with the ordinary hermitian conjugation
when acting on elements of $A = \text{Mat}_n(C)$. The former requirement is grounded in ref./19/
while the latter merely follows from the requirement that the gauge group related to
massless spin-1 (Yang-Mills) fields be compact (as discussed below, the resulting
Yang-Mills group is $U(n) \times U(n)$).

The hermiticity conditions, corresponding to the involution $\mu$ defined in such
a way, read

$$ (\Omega^{AB}_{ij}(\gamma, \tau))^\dagger = - (\Omega^{BA}_{ij}((-1)^B \gamma, (-1)^A \gamma) , \tag{6.1} $$

$$ (C^{AB}_{ij}(\gamma, \tau))^\dagger = - C^{BA}_{ij}((-1)^B \gamma, (-1)^A \gamma) \tag{6.2} $$

where the indices $i, j = 1 \ldots n$ describe matrices belonging to $A = \text{Mat}_n(C)$ (for more de-
tails on the correspondence between involutions and hermiticity conditions on physi-
cal fields see ref./12/). Note that the operation $\dagger$ in eqs. (6.1), (6.2) is assumed to
invert an order of all product factors except for exterior differentials $dX^\nu_1 \ldots dX^\nu_k$

, i.e. $\dagger$ is some involution of the algebra of components of differential
forms but not of the exterior algebra (this leads to the additional minus sign in eq. (6.1)). It is worth mentioning that after imposing conditions (6.1), (6.2) the quantities like \( \omega \), \( C \) ... cannot be regarded as taking values in any real associative algebra and one can speak only about corresponding Lie superalgebras and/or FDA's.

The hermiticity conditions (6.1), (6.2) are consistent with both undeformed FDA and its deformation (5.21), (5.22) if the deformation parameters \( \eta_1 \) and \( \eta_2 \) are complex conjugate,

\[
\eta_2 = \bar{\eta}_2.
\]

(6.3)

In other words, it follows from eqs. (6.1)-(6.3) that the full curvatures \( R \) and covariant derivatives \( \partial \) (including \( R_\partial \) and \( \partial \) ) obey the hermiticity conditions of the type (6.1), (6.2) too. In order to verify this explicitly, one has to re-enumerate arguments \( Z_i \) in \( R^+_\partial \) and reduce it to \( R_1 \) (5.21) by an appropriate re-definition of the integration variables \( n \).

Now, let us analyse which automorphisms of shsa(1) found in ref. 19 admit generalizations to the FDA under consideration in the first nontrivial order.

The following trivial automorphisms of shsa(1):

\[
f(\omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{g}, \mathfrak{q}, \mathfrak{z})) = \omega^{A\bar{B}}_{\bar{i}j}(-\mathfrak{g}, -\mathfrak{q}, -\mathfrak{z}) \quad f(C^A_{\bar{i}j}(\mathfrak{g}, \mathfrak{q}, \mathfrak{z})) = C^{A\bar{B}}_{\bar{i}j}(-\mathfrak{g}, -\mathfrak{q}, -\mathfrak{z})
\]

(6.4)

\[
p(\omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{g}, \mathfrak{q}, \mathfrak{z})) = \omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{q}, \mathfrak{g}, \mathfrak{z}) \quad p(C^A_{\bar{i}j}(\mathfrak{g}, \mathfrak{q}, \mathfrak{z})) = C^{A\bar{B}}_{\bar{i}j}(\mathfrak{q}, \mathfrak{g}, \mathfrak{z})
\]

(6.5)

serve as automorphisms of its deformation as well (for arbitrary associative algebra \( A \) described by the indices \( i, j \)). Obviously, all automorphisms of \( A \) can trivially be extended to automorphisms of the deformation constructed too. Let us note that \( f \) (6.4) is the automorphism of boson-fermion parity while the automorphism \( p \) (6.5) which will be used for construction of \( P \)-reversal automorphisms, acts only when \( \eta_1 = \eta_2 \).

The original superalgebra shsa(1;i;\( \mathcal{Q} \)) admitted also automorphisms \( \hat{T}_Q \) and \( \hat{T}_R \) corresponding to sign changes of the operators \( \hat{Q} \) and \( \hat{R} \), respectively (i.e.,

\[
\hat{T}_Q(\omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{z})) = (-1)^A \omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{z}) \quad \hat{T}_R(\omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{z})) = (-1)^B \omega^{A\bar{B}}_{\bar{i}j}(\mathfrak{z})
\]

However, these automorphisms do not act in the FDA under consideration as is most obvious from eq. (5.23) the r.h.s. of which contains explicitly odd powers of \( \hat{Q} \) and \( \hat{R} \).

It is remarkable that nevertheless the automorphisms \( k = \hat{T}_Q \circ \hat{T}_R \) admits the fol-
lowing extension to the deformed FDA:

\[ k(\omega^{AB}_{ij}(Z)) = (-1)^{A+B} \omega^{AB}_{ij}(Z), \quad k(C^{AB}_{ij}(Z)) = (-1)^{A+B} C^{AB}_{ij}(Z) \quad (6.6) \]

Indeed, now the simultaneous sign change of \( \hat{q} \) and \( \hat{r} \) on the r.h.s. of eq.(5.23) is compensated by the additional sign factor in the transformation law of \( C \quad (6.6) \). Let us note that the possibility of introducing this additional sign factor is due to the fact that the original (undeformed) FDA possesses the automorphism \( \mathcal{F}(\omega) = \omega, \mathcal{F}(C) = -C \).

Finally, a most interesting and important automorphism, \( t \), originates from the antiautomorphism \( \eta \) of \( \text{shs}(1) \), which is defined by the relations

\[ \eta(\hat{q}_a) = i \hat{q}_a, \quad \eta(\hat{z}_\beta) = i \hat{z}_\beta, \quad \eta(\hat{\phi}) = \hat{\phi}, \quad \eta(\hat{r}) = \hat{r} \quad (6.7) \]

(for more details about antiautomorphisms and their relation to automorphisms of Lie superalgebras see, for example, ref./12/). When all fields take on their values in some associative algebra \( A \), one has also to define how the antiautomorphism \( \eta \) acts on \( A \). For \( A = \text{Mat}_n(\mathbb{C}) \) we identify \( \eta \) with the transposition, \( \eta(B_{ij}) = B_{ji} \) for all \( b \in A \). Acting along the lines of ref./12/ for \( \text{shs}(N,4) \), one can make sure that the automorphism \( t \) of the undeformed FDA, which is induced by the antiautomorphism \( \eta \), acts as follows

\[ t(\omega^{AB}_{nm}(q, r)) = -\frac{1}{2} \left[ (1+i) \omega^{AB}_{nm} (i(-1)^A q, i(-1)^B r) + (1-i) \omega^{AB}_{nm} (-i(-1)^A q, -i(-1)^B r) \right] \quad (6.8) \]

\[ t(C^{AB}_{nm}(q, r)) = -\frac{1}{2} \left[ (1+i) C^{AB}_{nm} (i(-1)^A q, i(-1)^B r) + (1-i) C^{AB}_{nm} (-i(-1)^A q, -i(-1)^B r) \right] \quad (6.9) \]

It is remarkable that \( t \quad (6.8), (6.9) \) serves as automorphism of the deformed FDA \( (5.21), (5.22) \) too. The proof for this fact is analogous to that for the consistency of the hermiticity conditions \( (6.1), (6.2) \). Note that \( t \) transforms the terms proportional to \( \alpha_{12} \) and \( \beta_{12} \) on the r.h.s. of eqs.\( (5.21), (5.22) \) into those proportional
to $a_{23}$ and $b_{23}$ and vice versa. When proving that $t$ (6.8), (6.9) provides for some automorphism of the FDA under consideration, one should also take into account some sign factors caused by the Grassmann properties (4.8) (an order of product factors is to be inverted in this proof) and the fact that when $\omega$ possesses definite Grassmann parity $\chi = 0$ or 1, eq. (6.8) takes the form

$$ t(\omega_{nm}^{AB}(q, \tau)) = i\chi \omega_{mn}^{AB}(-1)^A i\gamma_5 (-1)^B i\gamma_5. $$

(6.10)

All automorphisms $f$, $p$, $k$, $t$ (6.4)–(6.8) are involutive and commute with the involution $\mu$ leading to the hermiticity conditions (6.1), (6.2), that is $f$, $p$, $k$ and $t$ are automorphisms of both complex FDA and its real form extracted by the conditions (6.1), (6.2). Acting in the same fashion as in ref. /19/ for shsa(1), one can now construct a class of P-reversal automorphisms corresponding to the reflection of the $x_2$-axis,

$$ P(\omega) = p_0 f^a_0 k^b_0 t^c(\omega), \quad P(C) = p_0 f^a_0 k^b_0 t^c(C) $$

(6.11)

where $a$, $b$ and $c$ are arbitrary parameters taking on values 0 or 1 ($\omega_5 = (\omega_0, \omega_4, -\omega_2, \omega_3)$).

In ref. /19/, the automorphisms $f$, $k$ and $t$ were used for construction of such subalgebras of shsa(1) that form themselves consistent superalgebras of higher spins and auxiliary fields. Quite similarly, these automorphisms can now be used for extracting FDA's which form nontrivial deformations of the subalgebras above. In other words, using the automorphisms $f$, $k$ and $t$, one can obtain consistent truncations of the deformation constructed by means of imposing conditions of the form

$$ \hat{T}_\Omega(\omega) = \omega, \quad \hat{T}_\Omega(C) = C $$

(6.12)

where $\hat{T}_\Omega$ is an arbitrary set of automorphisms constructed from $f$, $k$ and $t$. The fact that conditions (6.12) do indeed lead to consistent truncations becomes extremely obvious by noting that automorphisms $f$, $k$ and $t$ are involutive (and mutually commuting) and, therefore, they give rise to certain superselection rules ($KZ_2$ - grading), while eqs. (6.12) reduce to extracting even subspaces with respect to some combinations of these superselection rules.
It is an important property of the automorphisms \( f, k \) and \( t \) that any conditions (6.12) based on these automorphisms do not spoil the chains of fields which were identified in §2 with independent massless and auxiliary fields (let us recall that according to the results of §2, a massless spin \( s \) is described by the chain of fields \( \Omega^{AA}(n,m) \) with \( n+m=2(s-1) \) (and some fixed \( A=0 \) or \( 1 \)), \( C^{A_1A_2}(n,m) \) with \( n-m=2s \), and \( C^{A_1A_2}(n,m) \) with \( n-m=2n \), whereas an auxiliary system of fields with some fixed parameter \( U \geq 0 \) includes \( C^{A}(n,m) \) with \( n+m=2 \) at \( u \geq 2 \) (\( A=0 \) or \( 1 \) serves as some fixed parameter too), \( \Omega^{A_1A_2}(n,m) \) with \( n-m=u \) and \( \Omega^{A_1A_2}(n,m) \) with \( m-n=u \). In other words, any conditions (6.12) constructed from \( f, k \) and \( t \) either annihilate some chain completely or leave it unmodified. This means that the linearized dynamics described in §2 remains valid for subsystems which can be extracted with the aid of eqs. (6.12) based on the automorphisms \( f, k \) and \( t \). It is worth mentioning that this is not the case for arbitrary automorphisms of the FDA under consideration. Specifically, the property above does not take place for the automorphism \( p \) (6.5) and automorphisms \( f_q \) and \( f_r \) discussed in ref. /12/ (in fact, this is the reason for disregarding the latter automorphisms in the present paper).

The simplest examples of the conditions (6.12) are

\[ f(\Omega) = \Omega, \quad f(C) = C \]  

(6.13)

and

\[ k(\Omega) = \Omega, \quad k(C) = C. \]  

(6.14)

The condition (6.13) extracts out the even (bosonic) sector while the condition (6.14) extracts out the sector of all massless fields, \( \Omega^{AA} \) and \( C^{AB} \) with \( A+B=1 \). Thus, in the framework of the deformation constructed, all auxiliary fields can be consistently set equal to zero.

However, one cannot now avoid the doubling of massless fields by considering e.g. the subset of fields \( \Omega^{AB} \) at \( A=B=0 \). Indeed, this was possible in the framework of shsa(1) because the afore-mentioned automorphisms \( T_q \) and \( T_R \) acted separately in shsa(1) and one could impose the conditions \( T_q(\Omega) = T_R(\Omega) = \Omega \). As already pointed out, only the combination \( T_q \circ T_R \) admits an extension to the con-
structured FDA and, therefore, the conditions above become inconsistent in the framework of this FDA. It seems that the doubling of bosonic massless fields is inevitable at least in presence of fermions. The reason is that Weyl 0-forms corresponding to massless fields coincide with $C^{AB}$ at $A+B=1$. These transform as the auxiliary gauge fields, $\Omega^{AB}$ with $A+B=1$, of shsa(1). As emphasized in ref./19/, the doubling of such fields is necessary in presence of fermions because no consistent hermiticity conditions exist without it. On the other hand, the even (bosonic) subalgebra of shsa(1) was shown in ref./19/ to decompose into the direct sum of two subalgebras each generating any integer spin once and only once. These are subalgebras projected out by the projectors $\Pi^{\pm} = \frac{1}{\sqrt{2}} (1 \pm \hat{R} \hat{G})$ commuting with all bosonic generators of shsa(1). As is easily seen, the analogous projections exists for the bosonic sector of the FDA constructed and, therefore, one can consider bosonic theories containing each spin once and only once (or each spin belonging to some simple associative algebra).

Let us stress that the results above imply that the higher-spin superalgebras of refs./11,12/ do not admit themselves a consistent highest-order interaction of the gauge fields corresponding to them (in presence of fermions). In other words, their extension to the superalgebra of higher-spin and auxiliary fields of ref./19/ is essential for introducing an interaction. Note that this could not be seen in the framework of the cubic approximation as discussed in refs./1,2/, and the results above, which in fact rule out the purely higher-spin superalgebras of refs./11,12/, simultaneously provide for a nontrivial manifestation of the fact that we are working here far beyond the cubic approximation of refs./1,2/.

Most interesting truncations correspond to restrictions (6.12) involving the automorphism $t$ (6.8),(6.9). Let us consider the subsystems extracted by the conditions

$$t \circ \kappa^a_\alpha_f^b(\omega) = \omega, \quad t \circ \kappa^a_\alpha_f^b(C) = C$$

which generalize naturally the systems of fields corresponding to superalgebras shsa(1) of ref./19/ to the case when all fields take on their values in the associative algebra $A$. A trivial generalization of the analysis of ref./19/ shows that
the systems extracted by the conditions (6.15) contain the following fields (only the structure of 1-forms $\omega$ is explicitly described below, but it is implicitly assumed that the 1-forms $\omega$ are supplemented by the Weyl 0-forms $C$ in accordance with the structure of the linearized chains of §2):

1. All massless fields of even integer spins in the (second-rank) symmetric representation of $SO(n)$, i.e. the fields $\omega_{i_j}^{AA}(k, \ell)$ with $\frac{1}{2}(k+\ell)_{2}=1$ restricted by the condition $\omega_{i_j}^{AA}(k, \ell) = \omega_{j_i}^{AA}(k, \ell)$.

2. All massless fields of odd integer spins in the antisymmetric (adjoint) representation of $SO(n)$, i.e. the fields $\omega_{i_j}^{AA}(k, \ell)$ with $\frac{1}{2}(k+\ell)_{2}=0$ and $\omega_{i_j}^{AA}(k, \ell) = -\omega_{j_i}^{AA}(k, \ell)$.

3. Massless fermionic fields $\omega_{i_j}^{AA}(k, \ell)$ of spins $s = 1 + \frac{1}{2}(k+\ell)_{2} = \frac{3}{2} - |b+A|_{2} + 2p$ \( (p=0,1,...) \), belonging to the symmetric representation of $SO(n)$.

4. Massless fermionic fields $\omega_{i_j}^{AA}(k, \ell)$ of spins $s = 1 + \frac{1}{2}(k+1)_{2} = \frac{3}{2} - |b+A+1|_{2} + 2p$ \( (p=0,1,...) \), belonging to the adjoint representation of $SO(n)$.

5. Bosonic auxiliary fields $\omega_{i_j}^{AB}(k, \ell)$ \( (A+B=1) \) in the symmetric representation of $SO(n)$, such that the parameter $u=k-1$ takes on values $u=2(1+a)+4p$ \( (p=0,1,2,...) \).

6. Bosonic auxiliary fields $\omega_{i_j}^{AB}(k, \ell)$ \( (A+B=1) \) in the adjoint representation of $SO(n)$, such that the parameter $u=k-1$ takes on values $u=2a+4p$ \( (p=0,1,2,...) \).

7. Fermionic auxiliary fields $\omega_{i_j}^{AB}(k, \ell)$ \( (A+B=1) \) in the symmetric representation of $SO(n)$, such that the parameter $u=k-1$ takes on values $u=1+2(a+b+A)+4p$ \( (p=0,1,2,...) \).

8. Fermionic auxiliary fields $\omega_{i_j}^{AB}(k, \ell)$ \( (A+B=1) \) in the adjoint representation of $SO(n)$ such that the parameter $u=k-1$ takes on values $u=-1+2(a+b+A)+4p$ \( (p=0,1,2,...) \).

As follows from 1. and 2., bosonic massless fields emerge in pairs in the both symmetric and antisymmetric representations of $SO(n)$. This is another manifestation of the doubling of massless bosons mentioned previously. Some potential physical im-
plications of this doubling were discussed briefly in ref. /19/.

Note that speaking about symmetric representation of SO(n), we assume second-rank symmetric tensors \( \omega_{ij} \) which form reducible representation of SO(n) containing the singlet component originating from the unit element of the initial associative algebra A, \( \omega_{ij} \sim \delta_{ij} \). This singlet (trace) component is very important because we identify the gravitational field (vierbein and Lorentz connection) with the singlet massless spin-2 field, \( \omega^{(0)}_{ij} \) (n,m) at n+m=2. It is this field which is only allowed to have a nontrivial zero-order background component describing background AdS space when constructing a linearization procedure generalizing that of §2 to an arbitrary associative algebra A (which is therefore required to possess a unit element). Specifically, such a linearization procedure will be used in §7 when analysing a cubic action based on the generalization of shsa(1) to arbitrary A.

It follows from 2. that, for all truncations (6.15), massless spin-1 fields \( \omega_{ij}^{AA} \) with A=0 and 1 belong both to the adjoint representation of SO(n). On the other hand, it follows from eq.(6.1) that in the framework of the original (untruncated) FDA, both types of massless spin-1 fields belong to the adjoint representation of U(n) (antihermitian matrices). This implies that the Yang-Mills gauge group related to massless spin-1 fields is U(n)xU(n) for the untruncated FDA and SO(n) x SO(n) for all its truncations (6.15) (in fact, the gauge fields corresponding to each product factor in U(n)xU(n) and SO(n)xSO(n) are not \( \omega_{ij}^{AA} \) themselves but coincide with \( \omega_{ij}^{(0)}(0,0) \pm \omega_{ij}^{(0,0)}(0,0) \). It can readily be seen that corresponding Yang-Mills transformations remain undeformed in the framework of the FDA (5.21),(5.22) (see also below). Therefore, we denote the full FDA (5.21),(5.22) as shsa(u(n)) when the initial complex associative algebra A coincides with Mat\(_n\)(C). (as already mentioned, the parameter \( \delta \) is irrelevant.) Analogously, its truncations (6.15) are denoted as shsa(so(n)|a). All these FDA's may be of physical importance as nontrivial dynamical systems describing various consistent interactions of massless and auxiliary fields including their gravitational and Yang-Mills interactions.

Further truncations of shsa(so(n)|a) can be constructed with the aid of the
automorphisms $f$, $k$ and various automorphisms of $\text{Mat}_n(O)$ (the latter automorphisms can also be used for generalizations of eqs. (6.15), and some examples of such generalizations are discussed in ref. 12/ for the specific case of Clifford associative algebras $A$ leading to gauge groups $U(2^k)$). In particular, by means of the automorphisms $f$ and $k$ one can extract out bosonic sector of $\text{shas}(\text{so}(n)|a)$, sector of massless fields of $\text{shas}(\text{so}(n)|a)$ (and its bosonic subsector) and the part of $\text{shas}(\text{so}(n)|a)$ containing bosonic massless fields and fermionic auxiliary fields.

On the other hand, by using automorphisms of $\text{Mat}_n(O)$ one can easily construct truncations leading to various semi-simple gauge subgroups of the form $[U(n_1)xU(n_2)x...xU(n_k)]^2$ or $[(\text{SO}(m_1)x\text{SO}(m_2)x...x\text{SO}(m_k))^2$ of the initial gauge groups $(U(n))^2$ or $(\text{SO}(n))^2$ (analogous results can also be obtained if one starts with semi-simple associative algebras $A$).

To conclude this section, we discuss which symmetries remain undeformed in the framework of the deformation under consideration. According to the general relation (3.5), the transformation laws with some parameters $E^A$ remain undeformed (i.e. coincide with those corresponding to the Lie superalgebra which governs the structure of the FDA in the zero order) if the corresponding gauge fields $W^A$ are not present in non-zero orders of the deformation under investigation. It is a trivial consequence of eq. (5.21) that all first-order ($C$-dependent) terms in eq. (5.21) do not contain $Z$-independent part of $W^A_{ij}(Z)$ because of factors of $\alpha_{12}, \alpha_{23}, b_{12}$ and $b_{23}$. As a result, one finds that all symmetries with the parameters $E^A_{ij}, a(o), \beta(o)$ remain undeformed. By choosing a new basis

$$
\frac{\bar{\phi}}{3} \delta_{ij}, \ a(o), \ b(o) = \sum_{A,B=0} \frac{1}{2} (-1)^{A+B} E^A_{ij}, a(o), \beta(o)
$$

(6.16)

one can easily see that corresponding gauge group coincides with $U(n)xU(n)x\text{GL}(n; C)$ (the parameters $\frac{3}{2}^{00}_{ij}$ and $\frac{3}{2}^{11}_{ij}$ correspond to two components $U(n)$, while the parameters $\frac{3}{2}^{02}_{ij}$ and $\frac{3}{2}^{10}_{ij}$ are conjugate and describe $\text{GL}(n; C)$). One can also make sure that the Yang-Mills gauge group $U_1(n)xU_2(n)$, corresponding to massless spin-1 fields, is a product of the diagonal subgroup of $U(n)xU(n)$ and of a maximal compact subgroup of $\text{GL}(n; C)$. It is also worth mentioning that the whole group
with the parameters \( \tilde{F}^{AB}_{ij}, \tilde{a}(0), \tilde{b}(0) \) is non-compact but, as emphasized in ref. /19/, this is not dangerous because this non-compactness is only related to auxiliary (non-physical) sector, while the Yang-Mills subgroup corresponding to the physical (massless) fields is compact.

Another important symmetry which remains undeformed is the local Lorentz symmetry with the parameters \( \tilde{E}^{ao}_{ic}, \tilde{a}(2), \tilde{b}(0) \) and \( \tilde{E}^{ao}_{ic}, \tilde{a}(0), \tilde{b}(2) \) belonging to the center of Mat\(_n\)(\( \mathbb{C} \)). The fact that the r.h.s. of eq. (5.21) is independent on the fields \( \psi^{ao}_{ic}(2,0) \) and \( \psi^{ao}_{ic}(0,2) \) requires for an explicit verification which is however simple enough and reduces to collecting all those terms on the r.h.s. of eq. (5.21) which may contain the Lorentz connection, completing an explicit integration over some integration variables\( \phi \), and commuting some factors containing the Lorentz connection (the latter is possible because, by its very definition, the Lorentz connection belongs to the center of \( \mathbb{A} \')).

Thus, the local symmetry group, which is shown to remain undeformed in the framework of the deformation (5.21), coincides with \( U(n) \times U(n) \times U(n; \mathbb{C}) \times U_3(3,1) \). It seems very likely that this is a maximal subgroup which remains undeformed in the framework of FDA (5.21).

We believe that the general properties of this section are valid not only for the first-order deformation elaborated in the present paper but also for the FDA as a whole which is expected to exist in all orders.

\[ 7. \quad \text{The Cubic Action} \]

In this section, we demonstrate briefly that the results of refs. /1,2/ on cubic interactions of massless higher-spin fields admit a natural generalization to the systems of fields corresponding to \( \text{shsa}(1) \) and its further extensions based on associative algebras admitting a non-degenerate invariant quadratic form. The approximation used in this section is weaker considerably as compared to that used in the rest of the paper\(^x\).

\(^x\) In this section, we are working in the lowest order of an expansion over the gauge 1-forms \( \omega \), while nontrivial effects due to Weyl 0-forms are in fact disregarded here. This is to be compared with the expansion over Weyl 0-forms used in preceding sections where a contribution of the gauge 1-forms \( \omega \) is taken into account completely in the first nontrivial order over Weyl 0-forms.
Let us consider an action of the form

\[
S = \frac{1}{2} \tau \int \sum_{A, B = 0}^{\infty} \left( \frac{1}{n! m!} \right) R^{AB} \psi \left( \psi \right) (n-m) R^{AB} \psi \left( \psi \right) \right. 
\]

where the curvatures \( R \) are defined in eq. (2.29) but now all quantities are assumed
to belong to an arbitrary associative algebra \( A \) admitting some trace operation, \( \text{tr} \),
such that the bilinear form \( \text{tr}(ab) \) is non-degenerate and symmetric,

\[
\text{tr}(ab) = \text{tr}(ba) .
\]

(7.2)

For \( A = \text{Mat}_n(\mathbb{C}) \), \( \text{tr} \) is the ordinary matrix trace.

To avoid ghosts in the action (7.1), the quadratic form \( \text{tr}(ab) \) should be sign-
definite when \( a \) and \( b \) are restricted by appropriate hermiticity conditions extracting
a real form under consideration out the initial complex algebra. Specifically,
this is the case for \( A = \text{Mat}_n(\mathbb{C}) \) when eqs. (6.1), (6.2) hold.

As follows from the results of refs. /10,18/, the quadratic part of the action
(7.1) reduces to the sum of free actions for all massless spin \( s \geq 3/2 \) fields and all
auxiliary fields with non-vanishing parameters \( u \) (\( u = n-m \) is the difference between
the numbers of undotted and dotted spinorial indices). There are two species of any
massless (auxiliary) field for any allowed value of \( s(u) \). Let us note once again
that the expansion procedure used in this section is analogous to that of refs. /1,2/:
only the gravitational fields \( \omega^{\hat{0} \hat{0}}_{\hat{A} \hat{B}} \) (\( n+m=2 \), which belong to the center
of \( A \), are allowed to have nontrivial zero-order background AdS components, while all
other fields are assumed to be of the first order.

The action (7.1) is hermitian and P-invariant. An important property of this
action is that the two species of massless fields, \( \omega^{A A} \), with \( A = 0 \) or \( 1 \), have cor-
correct relative signs of their kinetic terms. To see this, one can introduce new fi-

\[
\left( \omega^{A B} \right)_{i j} \dot{\alpha}(n), \dot{\beta}(m) = \left( \omega^{A B} \right)_{i j} \dot{\alpha}(n), \dot{\beta}(m) .
\]

(7.3)

The hermitian conjugation (6.1) reads for these fields

\[
\left( \omega^{A B} \right)_{i j} \dot{\alpha}(n), \dot{\beta}(m) = - \left( \omega^{A B} \right)_{i j} \dot{\beta}(m), \dot{\alpha}(n) .
\]

(7.4)
Simultaneously, the factor of $(-1)^{m_1 + m_2}$ disappears in the action (7.1) and now it is obvious that the both types of massless fields, $\omega_{AA}$ with $A=0$ and $1$, have coinciding signs of their kinetic terms. Note also that although the coefficients against free actions for two independent real combinations of auxiliary fields are of opposite signs, this is not dangerous because auxiliary fields cannot lead to ghosts (see also ref./19/).

Using the linearized constraints suggested in refs./10,18/, which lead to the representation (2.33) on the mass shell of free fields, and acting along the lines of refs./1,2/, one can easily make sure that the action (7.1) is gauge invariant at least in the first nontrivial order in interaction.

In fact, the action (7.1) is incomplete because it does not contain spin $s<1$ massless fields which are expected to be necessary for constructing a closed theory (this follows from the structure of the FDA described in preceding sections). In addition, this action does not contain kinetic terms for massless spin-1 fields and for auxiliary fields with $n=0$. As demonstrated in ref./2/, kinetic terms for spin-1 massless fields can be added separately without spoiling the gauge invariance. As for auxiliary fields with $n=0$, it was shown in ref./18/ that the representation (2.33) takes place in the sector of auxiliary fields with $u=0$ when these fields are pure gauge (at the linearized level). Requiring the auxiliary fields with $n=0$ to be trivial (pure gauge), one needs no kinetic terms for these fields. Although such a procedure is awkward enough in the sector of auxiliary fields with $u=0$ and can lead to some problems in higher orders, it is sufficient when analysing cubic interactions. In the author's opinion, the problems above are not too serious and can be avoided in higher orders as well. (The trivial possibility is to disregard auxiliary fields at all taking into account that this is a consistent truncation.)

In conclusion of this section, we would like to emphasize that the results above strongly support a belief that some full consistent actions exist for the systems of fields corresponding to (consistent truncations of) FDA's based on sha(1) and its generalizations constructed by means of associative algebras. On the other hand, an important fact, which follows from the results of the present paper, is
that most probably the pure higher-spin superalgebras discussed in refs./1,2/ cannot lead to consistent interacting theories beyond the cubic order (in presence of fermions) because these superalgebras do not correspond to consistent truncations of the constructed FDA's. Note that there is no contradiction between this conclusion and the results of refs./1,2/ because, as emphasized in these references, a complete set of fields needed for constructing a full theory cannot be determined from an analysis of cubic interactions. Note also that the results of ref./2/ can easily be derived from those of this section in the specific case when A is some Clifford algebra C_m.

§8. Conclusion

There are two main results in the paper.

The first is a re-formulation of free equations for massless and auxiliary fields suggested in refs./10,18/ in terms of an appropriate Free Differential Algebra (FDA). One of important advantages of this approach is that it enables us to describe quite uniformly not only spin s≥3/2 massless fields as in refs./10-12,18,19,1,2/ but lower spin (s≤1) massless fields as well. Another advantage is that this approach is adequate for introducing interaction in terms of a deformation of the original (undeformed) FDA related to the superalgebra shsa(1) of ref./19/ and Weyl 0-forms in its adjoint representation. Such an approach gives a hope to construct full equations of motion of massless fields in a closed form.

The second result consists of explicit construction of the deformation above in the first nontrivial order in Weyl 0-forms. It is worth mentioning that this expansion in 0-forms (in fact, in those components of curvatures which are non-vanishing on the mass shell) is considerably more powerful than that used previously in refs./10-12,18,19,1,2/ which reduced to an expansion in both curvatures and connection 1-forms themselves. Within the expansion procedure used in the present paper, all terms, containing connection 1-forms explicitly, are accounted completely in each order in the Weyl curvatures. However presently, the price for this is that we are working only on the mass shell while in refs./1,2/ the analysis was carried out on the action level. Nevertheless, we believe that the analysis of the present
paper admits some off-mass-shell generalization that however requires for introducing additional sets of fields which trivialize on the mass shell (note that some action is proposed in §3 leading to the FDA under consideration but it is unlikely that this action possesses all necessary physical properties, and some less trivial approach seems to be needed).

In any case, the results of the present paper form a good starting point for constructing both full consistent motion equations and an action for interacting massless higher-spin fields. The problem of existence of a second-order deformation receives crucial importance in this situation because, in principle, there is a possibility that the constructed first order deformation is "accidental" and cannot be extended to higher orders. This problem is now under investigation and our preliminary results are sufficiently promising.

An important result of the paper is that the pure higher-spin superalgebras $\mathfrak{sh}_{p}(1)$ and $\mathfrak{sh}(N,4)$ suggested in refs./11,12/ turn out to be insufficient for constructing closed motion equations, and broader superalgebras of massless and auxiliary fields of ref./19/ are needed in presence of fermions, which lead to doubling of at least some part of massless bosonic fields (including spin-2 fields). Let us stress that this fact does not contradict to the results of refs./1,2/. Indeed, as emphasized in refs./1,2/ (see also ref./21/), it is impossible to conclude definitely which set of fields is needed for constructing a closed theory in all orders when analysing a gauge invariance only in the lowest order in interactions (which order was analysed in refs./1,2/). On the other hand, the approximation procedure applied in the present paper is much stronger, and it is likely that this procedure fixes a complete set of fields already in its first nontrivial order (i.e. a first-order deformation does not exist if a set of fields is chosen incorrectly).

Another interesting point is that, at least in the first nontrivial order analysed in the paper, all quantities (such as connection 1-forms, Weyl 0-forms etc.) are allowed to take their values in an arbitrary associative algebra $A$ without spoiling the consistency of the deformation constructed. If $A=\text{Mat}_{n}(\mathbb{C})$ (in fact, these are the only interesting finite-dimensional complex associative algebras) then, af-
ter imposing appropriate hermiticity conditions, this leads to systems of fields which take values in the adjoint representation of $U(n)$. In this case, massless fields $c^\dagger$ any spin emerge twice (each in the adjoint representation of $U(n)$), while the Yang-Mills gauge group corresponding to massless spin-1 fields is $U(n) \times U(n)$. Other consistent systems of massless fields with the Yang-Mills gauge groups $SO(n) \times SO(n)$ arise as truncations of the $U(n) \times U(n)$ systems above. (All these systems of massless fields can be regarded as higher-spin counterparts of the usual extended supergravitational supermultiplets.) An additional argument indicating that one can assume that all quantities belong to an arbitrary associative algebra (with an appropriate trace operation) is provided by the analysis of corresponding actions in §7 which is analogous to that of refs./1,2/ for pure higher-spin superalgebras. Note that, in fact, extended higher-spin superalgebras used in refs./12,2/ arise naturally in the specific case when an associative algebra $A$ coincides with some Clifford algebra $C_{2M}$ (equivalently, $A \cong \text{Mat}_{2M}(\mathbb{C})$). An interesting question, which remains to be investigated, is to analyse various possibilities related to infinite-dimensional associative algebras (e.g., $\text{Mat}_\infty(\mathbb{C})$, tensorial products of Heisenberg algebras, etc.) which lead to systems with infinite numbers of fields of any spin, the situation analogous to that familiar for massive excitations of (super)strings.

It is worth mentioning that the same as in refs./1,2/, the AdS background is essential in the present paper because any attempt to take a flat limit in the constructed FDA either spoils linearized chains of §2 or leads to a meaningless interaction. To see this, one has to introduce an arbitrary inverse AdS radius $\Lambda \neq 0$ acting along the lines of refs./10-12,18/. After this, it is a matter of simple analysis to show that no sensible limit exists with $\Lambda = 0$ for the FDA under consideration. Note also that the necessity of using AdS background in our construction originates from the assumption that Weyl 0-forms are small in expansions of the type (2.33) (for more detail see §2).

We would like to emphasize that although AdS background is essential for massless higher-spin fields, this does not imply that theories under consideration are physically unacceptable. Indeed, we only claim that the cosmological constant should
necessarily be non-vanishing in the phase with unbroken gauge symmetries of massless higher-spin fields. However, similarly to ordinary supersymmetry, gauge symmetries of massless higher-spin fields should be broken in a physical phase.

One can hope that simultaneously an effective cosmological constant will vanish in this phase. In any case, the question of spontaneous breakdown of symmetries of massless higher-spin fields is both very important and interesting. Although its detailed investigation lies far beyond the scope of the present paper, in conclusion we would like to make a few comments on this point.

As usual, we expect that any spontaneous breakdown of (higher-spin) gauge symmetries will give masses to all (originally massless) gauge fields corresponding to broken symmetries. In the present paper, we are working with such systems of fields with infinitely increasing spins, that the numbers \( n_\gamma \) of spin-\( \gamma \) massless fields are restricted by some spin-independent constant, \( n_\gamma \leq n \), \( n \) is twice a dimensionality of the associative algebra \( A \). Evidently, no spontaneous breakdown can exist transforming such a system of massless fields into some system of massive fields. Indeed, a massive spin-\( \gamma \) field contains \( 2s+1 \) helicities \( -s \leq \gamma \leq s \), while a massless spin-\( \gamma \) field contains only two helicities, \( \gamma = \pm s \). As a result, any chain of massive fields of infinitely increasing spins should reduce in a massless limit (i.e., in an unbroken phase) to some set of massless fields with infinite numbers of fields of any spin, the set which differs drastically from those discussed in the paper. As a result, we conclude that the following two possibilities are allowed for spontaneous breakdowns of higher-spin gauge symmetries.

The first is that using some infinite-dimensional algebras \( A \) one would succeed in constructing some spontaneously broken theories starting from pure massless theories (i.e., without adding massive "matter" multiplets). Such "pure massless" theories seem to be most promising from the point of view of their finiteness on the quantum level. However, the question is highly nontrivial whether appropriate theories of this class do actually exist.

The second possibility, which was many times used in ordinary supergravity, is to introduce additional massive matter multiplets containing sufficient numbers of
degrees of freedom to make gauge higher-spin fields massive too. To proceed in this direction, a structure of massive multiplets of higher-spin supersymmetry (which was argued in ref./19/ to coincide with the full higher-spin subalgebra of shsa(1)) should be elaborated as a first step.

In fact, there also exists a third possibility which interpolates between those mentioned above. This is spontaneous breakdown via compactification of extra dimensions. Indeed, after compactifications, pure massless D-dimensional theories containing higher-spin fields should reduce to theories containing both massless and massive fields in d dimensions (with d<D). In particular, all higher-spin fields can acquire masses after an appropriate compactification, that corresponds to a breakdown of the whole gauge group to some its subgroup with lower-spin (massless) gauge fields of spins 1, 3/2 and 2 (i.e., to some ordinary local supersymmetry times Yang-Mills symmetries). In the author's opinion, compactifications of extra dimensions will provide the most elegant and efficient tool for constructing theories with broken higher-spin symmetries. Therefore, the investigation of non-trivial theories of massless higher-spin fields in arbitrary dimensions receives great importance. It is worth mentioning that no barrier of d<11 is expected in this investigation since it originated from the restriction s<2 in ordinary supergravities which has nothing to do with the theories under consideration. Moreover, the results of refs./40,41/, generalizing those of ref./10/ to arbitrary d>4, indicate that there are good perspectives for a generalization of the approach of refs./9-12,18,19,1,2/ and of the present paper to arbitrary dimensions.

Whatever mechanism is used providing spontaneous breakdown of higher-spin gauge symmetries, resulting theories will contain some chains of massive higher-spin fields with infinitely increasing spins. The only theories of this class, which are presently known, are (super)string theories, and one can speculate that string field theories can be regarded as some theories with spontaneously broken symmetries of massless higher-spin fields. In this case, the theories of massless higher-spin fields may become more fundamental than string theories themselves.

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Appendix. Comparison of Linearized Deformations

Here we show that the deformation (5.21) is equivalent at the linearized level to the linearized deformation (2.33). In other words, we prove that the deformation (5.21) can be reduced to that described by eq.(2.33) by an appropriate change of variables.

As a first step, we linearize the deformation (5.21) by disregarding in eq. (5.21) all terms except for those which are quadratic in the vierbein one-form \( h_{\alpha}^\beta \). Taking into account that \( h_{\alpha}^\beta \) is described by those fields \( \omega_{ij}^{\alpha}(q,r) \) which are linear in both \( q \) and \( r \),

\[
\begin{align*}
\hat{h}(Z) &= \frac{1}{2!} h^{\beta}_{\alpha} \eta_{\alpha}^\beta \tilde{\zeta}_\beta, \\
\hat{h}^{\beta}_{\alpha} &= 2i \frac{\partial^2}{\partial q^\alpha \partial \tilde{\zeta}_\beta} \omega_{ij}^{\alpha}(q, \tilde{\zeta}) \bigg|_{q=\tilde{\zeta}=\zeta=0} \quad (A.1)
\end{align*}
\]

the linearized part of the deformation (5.21) can be reduced to the form

\[
\begin{align*}
\tilde{R}^2_{\tilde{Z}}(Z_0) &= -4 \eta_1 \alpha_{12} \exp(-i \theta_0) \times \\
& \times \left[ (b_{12} + b_{23}) \cos(\alpha_{03}) + i(b_{12} + b_{23} + b_{02} b_{23}) \sin(\alpha_{03}) \right] \\
& \times \hat{h}(Z_1) \hat{h}(Z_2) C^{1F+1L26}(Z_3) |_{Z_1=Z_2=Z_3=0 + h. c.}
\end{align*}
\]

where h.c. denotes the terms proportional to \( \tilde{\zeta}_1 \), which ensure \( \tilde{R} \) to obey the correct hermiticity conditions.

To derive eq.(A.2), one has to take into account that the gravitational field \( h \) belongs to the center of the associative algebra \( A \) under consideration and, therefore, \( h \) commutes with all other product factors as usual bosonic 1-form (without matrix indices). In addition, we use the relation

\[
\hat{h}(Z^{NM}) = (-1)^{N+M} \hat{h}(Z) \quad (A.3)
\]

which follows from eq.(A.1). Finally, let us note that the terms quadratic in \( h \) arise if and only if the values 1 and 2 of the indices \( ij \) carried by \( a_{ij} \) emerge once and only once (and similarly for \( b_{ij} \)). In other words, only those terms contribute in the approximation under consideration which contain once either \( a_{12} \) or \( a_{12} \) with some \( i,j \neq 1,2 \), as well as analogous combinations of \( b_{ij} \) (only
such terms will be accounted in the subsequent analysis).

On the other hand, the linearized deformation (2.33) can be re-written in the form

\[ R^{EFG}_{1}(Z) = -i \mu \alpha_{12} \exp(-i \delta_{03}) x \]

\[ x \left[ \delta(1F + G12) b_{12} b_{23} + \delta(1F + G + T12) b_{02} b_{02} \right] x \]

\[ x \left. \phi(Z_1) \phi(Z_2) C^{1F + 4G}(Z_3) \right|_{Z_1 = Z_2 = Z_3 = 0} + h.c. \]  \hfill (A.4)

Now, our aim is to prove that there exists such a trivial deformation that describes at the linearized level the difference between \( R^{4}_{4}(A.4) \) and \( R^{4}_{4}(A.2) \). Comparison of eqs. (A.2) and (A.4) shows that, in fact, we are to look for such trivial deformation that makes the r.h.s. of eq. (A.2) independent on \( \alpha_{03} \).

Let us consider trivial deformations (5.7)-(5.10) with the functions \( \Phi_{I,II} \) of the form

\[ \Phi_{I}(p_0, p_1, p_2) = (1 + i(\alpha_{01} + \alpha_{12})) (\alpha_{01} \psi(\alpha_{02}) + \alpha_{12} \chi(\alpha_{02})) \]  \hfill (A.5)

\[ \Phi_{II}(p_0, p_1, p_2) = (1 + i(\alpha_{02} + \alpha_{12})) (\alpha_{02} \rho(\alpha_{01}) + \alpha_{12} \beta(\alpha_{01})) \]  \hfill (A.6)

where \( \psi, \chi, \rho \) and \( \beta \) are some functions to be determined. The fact that the functions \( \Phi_{I,II} \) can be sought in the form (A.5), (A.6) is due to the following. We are working in the linearized approximation, in which the gravitational 1-forms \( h^{\alpha}(1), \beta^{(1)}, \omega^{\alpha}(2), \rho(0), \beta(0) \) and \( \omega^{\alpha}(0), \beta(2) \) are assumed to have zero-order background components while all other 1- and 0-forms are at least of the first order. A non-vanishing contribution into the linearized deformation under consideration is thus provided by those transformations (5.7) which contain 0-forms \( C \) only in combinations with the gravitational 1-forms above. (It is essential here that the gravitational 1-forms correspond to the proper subalgebra \( sp(4) \) of the whole infinite-dimensional superalgebra under consideration and, therefore, the terms quadratic in the gravitational fields cannot be produced by those terms on the r.h.s. of eq. (5.7) which contain 0-forms \( C \) with non-gravitational 1-forms.
(6) As is easily seen, the terms linear in $\alpha_{01}$, $\alpha_{12}$ in eq. (A.5) and in $\alpha_{02}$ and $\alpha_{12}$ in eq. (A.6), combined with the terms linear in $b_{01}$, $b_{02}$ and $b_{12}$ coming from the factor of $\exp(-i\sum_{m<n} b_{nm})$ in eq. (5.7), describe all possible terms containing the tetradic field $m_{m<n}$. On the other hand, as mentioned in §6, the full deformation (5.21) is Lorentz covariant (i.e. it does not contain the Lorentz connection 1-forms). As a result, the linearization of the deformation (5.21) is also Lorentz covariant. Because the linearized deformation (A.4) is Lorentz covariant too, it seems natural to suppose that the trivial linearized deformation (5.7), which is expected to compensate the difference between the deformations (A.2) and (A.4), should be Lorentz covariant, i.e. that it does not contain the Lorentz connection 1-forms. In fact, the factors of $(1+ia_{01})(1+ia_{13})$ and $(1+ia_{02})(1+ia_{13})$

are introduced in $\Psi_1$ (A.5) and $\Psi_2$ (A.6), respectively, in such a way that they compensate exactly the terms containing the Lorentz connection which originate from the factor of $\exp(-i\sum_{m<n} a_{nm})$ in eq. (5.7). In other words, these factors make the corresponding trivial deformation Lorentz covariant.

By direct substitution of eqs. (A.5), (A.6) into eqs. (5.8)-(5.10), one finds

$$f_1(p_0, p_1, p_2, p_3) = \alpha_{01}(\alpha_{02}\psi'(\alpha_{03}) - i(\alpha_{02} + \alpha_{13})\psi(\alpha_{03}))$$

$$+ \alpha_{02}(\alpha_{13}\psi'(\alpha_{03}) - i(\alpha_{03} + \alpha_{13})\psi(\alpha_{03}))$$

$$+ \alpha_{13}(\alpha_{02}\chi'(\alpha_{03}) - i(\alpha_{02} + \alpha_{13})\chi(\alpha_{03}))$$

$$+ \alpha_{12}(\psi'(\alpha_{03}) + \chi(\alpha_{03})),$$

$$f_2(p_0, p_1, p_2, p_3) = -\alpha_{01}(\alpha_{03} + \alpha_{23})(\psi'(\alpha_{02}) + 2i\psi(\alpha_{02}))$$

$$- \alpha_{12}(\alpha_{03} + \alpha_{23})[\chi'(\alpha_{02}) - 2i\chi(\alpha_{02})] + \alpha_{03}(\alpha_{12} - \alpha_{01})\rho'(\alpha_{02})$$

$$+ \alpha_{23}(\alpha_{12} - \alpha_{03})\rho'(\alpha_{02}) - \alpha_{13}(\psi(\alpha_{02}) + \chi(\alpha_{02})) + \beta(\alpha_{02}) - \rho(\alpha_{02}),$$

$$f_3(p_0, p_1, p_2, p_3) = \alpha_{03}(\alpha_{02}\rho'(\alpha_{01}) - i(\alpha_{02} + \alpha_{12})\rho(\alpha_{01}))$$

$$- \alpha_{02}(\alpha_{13}\rho'(\alpha_{01}) - i(\alpha_{03} + \alpha_{13})\rho(\alpha_{01})) - \rho(\alpha_{01}).$$
- \alpha_{12} (\alpha_{13}'(\alpha_{01}) - i(\alpha_{03} + \alpha_{13})\beta(\alpha_{01})) \\
+ \alpha_{13} (\alpha_{02}'(\alpha_{01}) - i(\alpha_{02} + \alpha_{12})\beta(\alpha_{01})) + \alpha_{23} (\beta(\alpha_{01}) - \rho(\alpha_{01}))

where the prime denotes an ordinary derivative of a function over its argument, and we have disregarded some terms which do not contribute in the linearized approximation (the derivatives above originate from expansions of the form
\Psi(\alpha_{01} + \alpha_{12}) = \Psi(\alpha_{01}) + \alpha_{12} \Psi'(\alpha_{01}) + ... where dots denote those terms which should be neglected in the linearized approximation).

Transforming all linearized terms in eq.(5.2), coming from \( f_{\text{III}} \) (A.7)-(A.9), to the form hh\( \text{C} \) and disregarding some further terms which are irrelevant in the linearized approximation (for example, only the factor of \( \exp(-i\alpha_{03}) \) survives from \( \exp(-i\alpha_{nm}) \) because the rest terms do not contribute in the linearized approximation when the functions \( f_{\text{III}} \) are of the form (A.7)-(A.9)), one reduces the trivial linearized deformation to the form
\[ R^{\ell} F G(Z_0) = -\exp(-i(\alpha_{03} + \beta_{03})) \times \]
\[ x \left[ f_{\text{I}}(p_0, p_1, p_2, p_3)(i \beta_{12} + (\beta_{01} + \beta_{13})(\beta_{02} + \beta_{23})) + \\
+ (-1)^{F+G+1} f_{\text{II}}(p_0, p_2, p_3, p_2)(i \beta_{12} + (\beta_{01} + \beta_{13})(\beta_{02} - \beta_{23}) \\
+ f_{\text{III}}(p_0, p_3, p_1, p_2)(i \beta_{12} + (\beta_{01} - \beta_{13})(\beta_{02} - \beta_{23})) \right] \times \\
x h(Z_1) h(Z_2) C^{F+G} Z_3^{10}|_{Z_1=Z_2=Z_3=0} + (h.c.) \]

where we have used eq.(A.3) once again.

Taking into account that \( h(Z_1) h(Z_2) = -h(Z_2) h(Z_1) \) because \( h \) is a 1-form (exterior product is implied everywhere), we see that a nontrivial contribution into eq.(A.10) is produced only by that part of the expression in the square brackets on the r.h.s. of eq.(A.10) which is antisymmetric with respect to the interchange \( 1 \leftrightarrow 2 \). Then, by noting that both \( \tilde{R}^{\ell}_{\text{I}} \) (A.2) and \( R^{\ell}_{\text{I}} \) (A.4) are antisymmetric with respect to the interchange \( 1 \leftrightarrow 2 \) in \( \alpha_{nm} \), being simultaneously symmetric with respect to the interchange \( 1 \leftrightarrow 2 \) in \( \beta_{nm} \), one concludes that eq.(A.10) may describe the difference between \( \tilde{R}^{\ell}_{\text{I}} \) (A.2) and \( R^{\ell}_{\text{I}} \) (A.4) only when the part
of the expression in the square brackets on the r.h.s. of eq. (A.10) vanishes which is symmetric with respect to the interchange 1 ↔ 2 in $\alpha_{nm}$ and antisymmetric with respect to the interchange 1 ↔ 2 in $\beta_{nm}$. As is easily seen, this gives the following two conditions:

$$f_{\Pi}(p_0, p_1, p_2, p_3) + f_{\Pi}(p_0, p_3, p_1, p_2) + (1\leftrightarrow 2) = 0, \quad (A.11)$$

$$f_{\Pi}(p_0, p_1, p_3, p_2) + (1\leftrightarrow 2) = 0. \quad (A.12)$$

Taking eq. (A.8) into account, one can make sure that eq. (A.12) holds if and only if the following equations are satisfied

$$\Psi' - 2i\Psi + \rho' = 0, \quad (A.13)$$

$$\chi' - 2i\chi - \beta' = 0. \quad (A.14)$$

It is remarkable that eqs. (A.13), (A.14) simultaneously ensure that eq. (A.11) holds.

Now eq. (A.10) reduces to the form

$$R^{\eta F G}(Z_0) = \left[-\frac{i}{2} \text{exp}(-i(d_{03} + d_{03})) \times \right.$$

$$\times \left[ b_{01} b_{02} (f_{\Pi}(p_0, p_1, p_2, p_3) + f_{\Pi}(p_0, p_3, p_1, p_2) -
\frac{1}{2}(F+\delta f_{\Pi}(p_0, p_1, p_2, p_3)) +
+ b_{13} b_{23} (f_{\Pi}(p_0, p_1, p_2, p_3) + f_{\Pi}(p_0, p_3, p_1, p_2)
\frac{1}{2}(F+\delta f_{\Pi}(p_0, p_1, p_2, p_3))
\right] \times
\left[ h(Z_1) h(Z_2) C^{(F+1/2)G}(Z_3)^{10} \right]\left[ Z_1 = Z_2 = Z_3 = 0 \right] - \left[ p_1 \leftrightarrow p_2 \right] + h.c.$$

By using eqs. (A.7)–(A.9), one finds that

$$\frac{1}{2}(f_{\Pi}(p_0, p_1, p_2, p_3) - (1\leftrightarrow 2)) = \frac{1}{2}(d_{01} d_{23} - d_{02} d_{13}) (\Psi'(x) + \chi'(x))$$

$$+ \alpha_{12} (\Psi(x) + \chi(x)), \quad (A.16)$$

$$\frac{1}{2}(f_{\Pi}(p_0, p_1, p_3, p_2) - (1\leftrightarrow 2)) = \frac{1}{2}(d_{01} d_{23} - d_{02} d_{13}) 	imes$$

$$x (\Psi'(x) - 2i\Psi'(x) + \chi'(x) - 2i\chi(x) - \rho'(x) + \beta'(x))$$

$$- \alpha_{12} (\Psi(x) + \chi(x) + \beta(x) - \rho(x)), \quad (A.17)$$
\[
\frac{1}{2} (f_{\Pi}(p_0, p_3, p_1, p_2) - (1 \leftrightarrow 2)) = \alpha_{12}(\beta(x) - \rho(x)) + \frac{1}{2} (\alpha_{01} \alpha_{23} - \alpha_{02} \alpha_{13}) \left[ \rho'(x) - 2i \rho(x) - \beta'(x) + 2i \beta(x) \right]
\]

(A.18)

where the designation \( \alpha_{03} = x \) is introduced which is currently used from now on.

Note that the functions \( \psi, \chi, \rho \) and \( \beta \) enter eqs. (A.16)-(A.18) only via the combinations \( \psi + \chi \) and \( \rho - \beta \). This enables us to set

\[
\chi = \beta = 0
\]

(A.19)

everywhere. This restriction respects eq. (A.14) and therefore cannot lead to any loss of (inequivalent) solutions of the task. (The ambiguity in one arbitrary function, surviving when eq. (A.14) is solved, originates from the deformations "trivial twice" mentioned in §5, which correspond to gauge transformations of the undeformed FDA.)

Now let us use the identity

\[
\alpha_{01} \alpha_{23} - \alpha_{02} \alpha_{13} = -x \alpha_{12}
\]

(A.20)

(expressing \( x = \alpha_{03} \)) which is a particular case of eq. (5.15). The fact that spinorial indices take on only two values. Using also eqs. (A.16)-(A.19), we reduce eq. (A.15) to the form

\[
R^\ell_{\mu} F_G(Z_0) = -\alpha_{12} \exp(-i(x + b_{03})x)
\]

\[
x \left[ b_{01} b_{02} \left[ \psi(x) - \rho(x) + \frac{i}{2} x (\psi'(x) - \rho'(x)) + 2i \rho(x) \right] + (-1)^{F+G} (\psi(x) - \rho(x) + \frac{i}{2} x (\psi'(x) - 2i \psi(x) - \rho'(x)) \right]
\]

\[
+ b_{23} b_{23} \left[ \psi(x) - \rho(x) + \frac{i}{2} x (\psi'(x) - \rho'(x) + 2i \rho(x)) \right] + (-1)^{F+G} (\psi(x) - \rho(x) + \frac{i}{2} x (\psi'(x) - 2i \psi(x) - \rho'(x)) \right]
\]

\[
+ (b_{01} b_{23} + b_{02} b_{13}) \left[ \psi(x) + \rho(x) + \frac{i}{2} x (\psi'(x) + \rho'(x) - 2i \rho(x)) \right] \right] \]

\[
x \left[ \hat{h}(Z_1) \hat{h}(Z_2) C^{1_{F+1_{1_{2}}} G(Z_3)} |_{Z_1 = Z_2 = Z_3 = 0} + \hat{h} c. \right]
\]

Then, with the aid of eq. (A.13) and the simple identity \( 1 + (-1)^n = 2 \delta (|n|_2) \), one fi-
nds from eq. (A.21) that
\[ R_{e}^{e}F_{e}(Z_{0}) = - \alpha_{12} \exp(-i(x + \theta_{03})) \times 
\left[ b_{01} b_{02}[(i \chi(x + \rho(x)) + 2S(1F + G_{l}^{2})(\psi(x) - \rho(x)) - x \rho'(x)) \right] 
+ b_{13} b_{23}[(i \chi(x + \rho(x)) + 2S(1F + G + G_{l}^{2})(\psi(x) - \rho(x)) - x \rho'(x)) \right] 
+ (b_{01} b_{23} + b_{02} b_{13})[(\psi(x) + \rho(x) + i \chi(x + \rho(x)))] \times 
\left[ h(Z_{1}) h(Z_{2}) c^{1F + \gamma_{1}^{2}G} Z_{3}^{10} \right] \right|_{Z_{1} = Z_{2} = Z_{3} = 0} + h. c. \quad (A.22) \]

As already mentioned, in order to reproduce the linearized deformation (A.4) as the sum of the deformations (A.2) and (A.22), this sum should be x-independent. In addition, it follows from eq. (A.4) that the terms proportional to \( b_{01} b_{23} + b_{02} b_{13} \) should cancel in the sum of \( R_{e}^{e} \) (A.2) and \( R_{e}^{e} \) (A.22). As is easily seen, these conditions lead to the following restrictions on \( \psi(x) + \rho(x) + i \chi(x + \rho(x)) = -4i \eta_{1} \exp(i \chi) \sin(x), \quad (A.23) \]
\[ x(\psi(x) + \rho(x)) = 4i \eta_{1} \exp(i \chi) (\cos(x) - 1), \quad (A.24) \]
\[ \psi(x) + \rho(x) - x \rho'(x) = \alpha \exp(i \chi), \quad (A.25) \]
where \( \alpha \) is some constant to be determined. If eqs. (A.23)-(A.25) hold, the sum \( \bar{R}_{e}^{e} + R_{e}^{e} \) takes the desired form
\[ \bar{R}_{e}^{e}F_{e}(Z_{0}) + R_{e}^{e}F_{e}(Z_{0}) = - \alpha_{12} \exp(-i \theta_{03}) \times 
\left[ b_{01} b_{02}(4 \eta_{1} + 2dS(1F + G_{l}^{2})) + b_{13} b_{23}(4 \eta_{1} + 2dS(1F + G + G_{l}^{2})) \right] \times 
\left[ h(Z_{1}) h(Z_{2}) c^{1F + \gamma_{1}^{2}G} Z_{3}^{10} \right] \right|_{Z_{1} = Z_{2} = Z_{3} = 0} + h. c. \quad (A.26) \]

It is remarkable that although the system of equations (A.13), (A.23)-(A.25) seems strongly overdetermined, it possesses the following unique solution
\[ \psi(x) = 2 \eta_{1} \exp(i \chi) x^{-2} [(ix - 1)(\cos(x) - 1) - x \sin(x)], \quad (A.27) \]
\[ \rho(x) = 2 \eta_{1} \exp(i \chi) x^{-2} [(ix + 1)(\cos(x) - 2) + x \sin(x)], \quad (A.28) \]
\[ \lambda = -2 \eta_1 \]  

(A.29)

Finally, one finds that

\[ \widetilde{R}_1^{\text{EF}}(Z_0) + R^{\text{EF}}(Z_0) = -4 \eta_1 \alpha \beta \exp(-i \varphi_0) \times \]

\[ \times \left[ b_{01} b_{02} \mathcal{S}(1F+G+1/2) + b_{21} b_{23} \mathcal{S}(1F+G1/2) \right] \times \]

\[ \times h(Z_1) h(Z_2) C^{1F+1/2 \mathcal{E}}(Z_3^{10}) \bigg|_{Z_1=Z_2=Z_3=0} + (\text{h.c.}) \]

thus completing the proof of the fact that

\[ R_1^{\text{EF}}(Z_0) = \widetilde{R}_1^{\text{EF}}(Z_0) + R^{\text{EF}}(Z_0). \]  

(A.31)

Let us stress that we searched for those solutions of eqs. (A.13), (A.23)-(A.25) which are non-singular in the point \( x=0 \) (only such functions of differential operators make sense when acting on power series of the type (4.13), (4.14) with massless fields as coefficients). In fact, it is this regularity requirement that fixes \( \lambda = -2 \eta_1 \); in which case eq. (A.26) coincides exactly with eq. (A.4) (note that the functions \( \Psi(x) \) (A.27) and \( \rho(x) \) (A.28) are analytical).

Thus it is shown that the deformation (5.21) is equivalent at the linearized level to the linearized deformation (2.33) which has been shown in §2 to describe correctly free motion equations of massless and auxiliary fields. In fact, only minor improvements are needed concerning a more careful analysis of the terms containing the Lorentz connection in order to extend the analysis above to the complete formal proof of the fact that the deformation (5.21) is nontrivial. However, this formal proof is of little interest since it is obvious that the deformation constructed is nontrivial because it describes nontrivial equations for physical massless fields.

The last point we would like to discuss is the role of the deformation parameter \( \eta_1 \) in eq. (5.21). Obviously, one can fix an absolute value of \( \eta_1 \) arbitrarily by rescaling \( C \to \eta C \) with an appropriate real parameter \( \eta \). However, one cannot use the complex parameter \( \eta \) because this spoils the hermiticity conditions (6.2). As a result, one finds that eq. (5.21) describes the class of inequivalent deformations characterized by the phase parameter \( \eta_1 = e^{i \varphi} \) and \( \eta = e^{-i \varphi} \).
lest case $\eta_1 = \eta_2 = 1$ seems to be most promising because the p-reversal automorphisms (6.11) exist in that case. In principle, there is a possibility that the ambiguity in the parameter $\varphi$ can be fixed by the analysis of the deformation in higher orders. However, our preliminary results indicate that this is not the case.

Therefore, we expect that, at least on the level of motion equations, a class of inequivalent theories of massless and auxiliary fields may exist, which correspond to different values of $\varphi$. Perhaps, this ambiguity will be fixed when constructing an off-mass-shell formulation.
References


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