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A LATTICE COMPUTATION OF PROTON DECAY AMPLITUDES

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Abstract

We calculate the matrix elements for the decays $p \rightarrow \pi^0 e^+$, $p \rightarrow \pi^+ \bar{\nu}_e$ and $p \rightarrow K^0 + \text{lepton}$ in grand unified theories. The calculations are performed on a $20 \times 10^2 \times 40$ lattice, at $\beta = 6.0$ using Wilson fermions in the quenched approximation. In the minimal SU(5) theory we find that the proton's lifetime divided by its branching ratio into $\pi^0 e^+$ is $(1.7 \pm 0.3 \text{ (statistical)}) \times 10^{30}$ years, for a grand unified mass of 2×10^{14} GeV.

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1. INTRODUCTION

In this paper we present the results of a lattice computation of matrix elements of the form,

$$\langle \pi | 0^B | p \rangle \quad (1.1)$$

where 0^B is one of the baryon number violating operators present in the low energy effective action of grand unified theories.¹ Such matrix elements contain the non-perturbative QCD effects in proton decay amplitudes. Up to now these matrix elements have been estimated by using models and approximations which include the non-relativistic quark model, the bag model, QCD sum rules and the chiral Lagrangian with soft pion relations, (for a review and references to the original literature see ref (1)). In previous lattice calculations^(2,3), the matrix elements

$$\langle 0 | 0^B | p \rangle$$

were computed and these were then used to estimate the elements (1.1), assuming the validity of soft-pion theorems for proton decay amplitudes. We have now computed the matrix elements (1.1) directly on the lattice.

The computations were performed on a $20 \times 10^2 \times 40$ lattice at a value of the bare coupling constant corresponding to $\beta = 6/g_0^2 = 6.0$ and using Wilson fermions⁽⁴⁾ in the quenched approximation. The results are presented in detail in the next section. For the minimal grand unified theory based on the SU(5) gauge group they imply that

$$\tau(p \rightarrow \pi^0 e^+) = (1.7 \pm 0.3) \times 10^{30} \left[\frac{M_X}{2 \times 10^{14} \text{ GeV}} \right]^4 \text{ years.} \quad (1.2)$$

where $\tau(p \rightarrow \pi^0 e^+)$ is the proton's lifetime divided by its branching ratio into $\pi^0 e^+$, and M_X is the unification scale. The error quoted in (1.2) is the statistical error only; the systematic errors are discussed in detail in section 2. The value in (1.2) was obtained using a unified coupling constant $\alpha_5(M_X) = 0.024$ and a short distance

¹In addition we shall also determine the matrix element for the decay $p \rightarrow K^0 + \text{lepton}$ in the limit of equal quark masses

enhancement factor of 10 (at a renormalisation scale of about 2 GeV). (1.2) can be compared to the value obtained by Marciano⁽⁵⁾,

$$\tau(p \rightarrow e^+ \pi^0) = 4 \times 10^{29 \pm 0.7} \left[\frac{M_X}{2 \times 10^{14} \text{ GeV}} \right]^4 \text{ years}, \quad (1.3)$$

a result which was based on a survey of previous estimates of the matrix elements (1.1). As expected (1.2) together with the allowed range for M_X , ($M_X = (2.0^{+2.1}_{-1.0} \times 10^{14} \text{ GeV})^{(5)}$) is inconsistent with the experimental bound $\tau(p \rightarrow e^+ \pi^0) > 2.7 \times 10^{32} \text{ years}^{(6)}$. Thus our results confirm that the minimal SU(5) theory is ruled out by the non-observation of proton decay. The matrix elements presented in section 2 are, however, relevant for all grand unified theories.

The plan of the remainder of this paper is as follows. In the next section, after a brief outline of the calculation we present our results for the matrix elements (1.1). In section 3 we compute the matrix elements $\langle 0 | O^B | p \rangle$ and discuss the precision of the soft pion theorems in proton decay amplitudes. Section 4 contains a brief summary and our conclusions.

2. CORRELATION FUNCTIONS AND MATRIX ELEMENTS

In order to be able to calculate the decay rate for the process $p \rightarrow e^+ \pi^0$ in any grand unified theory, we need to evaluate the two matrix elements

$$\epsilon^{abc} \langle \pi^0 | (u^a C d^b) u_y^c | p \rangle \equiv A_1 N_y \quad (2.1)$$

and

$$\epsilon^{abc} \langle \pi^0 | (u^a C \gamma^5 d^b) (\gamma^5 u^c)_y | p \rangle \equiv A_2 N_y \quad (2.2)$$

where a, b and c are colour labels, y is a spinor index, C is the charge conjugation matrix and N is the proton's spinor. As will be discussed below, from our calculations of these two matrix elements we are also able to determine the decay rate for the processes $p \rightarrow \pi^+ \bar{\nu}_e$ and $p \rightarrow K^0 + \text{lepton}$.

The matrix element for a proton at rest to decay into a pion with momentum p is determined by evaluating the three point correlation function

$$C(t_x, t_y) \equiv \sum_{y=y'=1,2} \sum_{\underline{x}, \underline{y}} \langle 0 | T \{ \bar{J}_{p,y'}(0) 0_y^B(y) J_\pi(x) \} | 0 \rangle \cdot e^{ip \cdot \underline{x}} e^{-ip \cdot \underline{y}} \quad (2.3)$$

where y and y' are spinor labels and t_x and t_y are the time components of the four-vectors x and y . 0_y^B is one of the operators in (2.1) and (2.2), and J_p and J_π are interpolating operators for the proton and pion respectively for which we make the choice,

$$J_{p,\alpha} \equiv \epsilon^{abc} (u^a C \gamma^5 d^b) u_\alpha^c \quad (2.4)$$

and

$$J_{\pi^0} \equiv \frac{1}{\sqrt{2}} (\bar{u}^a \gamma^5 u^a - \bar{d}^a \gamma^5 d^a). \quad (2.5)$$

Choosing $t_x > t_y > 0$ and inserting a complete set of states between each pair of operators in (2.3), we find that for sufficiently large t_y and $t_x - t_y$,

$$C(t_x, t_y) = - \frac{1}{E_\pi} e^{-m_p t_y} e^{-E_\pi(t_x - t_y)} \sqrt{Z_p Z_\pi} A_1 \quad (2.6)$$

where $E_\pi = \sqrt{p^2 + m_\pi^2}$, m_π and m_p are the masses of the pion and proton respectively and A_i ($i=1$ or 2) are defined in equations (2.1) and (2.2). Z_p and Z_π are given by

$$\langle p | \bar{J}_{p,\alpha}(0) | 0 \rangle \equiv \sqrt{Z_p} \bar{N}_\alpha \quad (2.7)$$

and

$$\langle 0 | J_\pi(0) | \pi \rangle \equiv \sqrt{Z_\pi} \quad (2.8)$$

where in (2.7) N is the proton's spinor. The masses of the proton and pion, as well as Z_π and Z_p have been evaluated by calculating two point correlation functions⁽⁷⁻⁹⁾. For our calculation of proton decay amplitudes we use the same gluon configurations as were used in refs (7-9) for the computation of the masses and Z_π and Z_p . Hence by computing $C(t_x, t_y)$, the A_i 's can be determined.

In the quenched approximation $C(t_x, t_y)$ is given by the diagram in Fig.1(a), averaged over gluon configurations. We evaluate this diagram by using the set of quark propagators, $\{S(x,0)\}$, and "generalised" propagators, $\{S_1(y,0;t_x,p)\}$, which had been generated previously for studies of the pion's structure function and electromagnetic form factor⁽⁷⁾ and weak matrix elements⁽⁹⁾. Here $S(x,0)$ is the quark propagator, from an arbitrary lattice point x to the origin, in the gluon background field and

$$S_1(y,0;t_x,p) \equiv \sum_{\underline{x}} S(y,x) \gamma^5 S(x,0) e^{ip \cdot x} \quad (2.9)$$

After redrawing the diagram in Fig 1(a) as that in Fig 1(b), it can be seen that these two sets of propagators are sufficient to compute the A_i 's.

From the previous projects we have the S_1 's for 15 gluon configurations at $t_x = 15$ with $p = (\pi/10, 0, 0)$, the smallest non-zero momentum compatible with our lattice⁽⁷⁾, and for 13 different configurations with $t_x - t_y = 6$ and the pion at rest⁽⁹⁾². In both cases all the propagators were computed at three different quark masses

² Note that in one case it is t_x which is fixed and in the other it is $t_x - t_y$.

corresponding to the three values of the Wilson parameter $K = 0.1515$, 0.1530 , 0.1545 . All the results then have to be extrapolated to the chiral limit, (which is at $K = K_C = 0.1563(2)$)⁽⁷⁾.

Momenta on a lattice can only take discrete values and hence the four-momentum transfer, q , between the proton and the pion will not in general satisfy $q^2 = m_{e^+}^2$, where m_{e^+} is the mass of the positron, (or $q^2 = 0$ in the case $p \rightarrow \pi^+ \bar{\nu}_e$). However with the proton at rest and the pion with momentum $(\pi/10, 0, 0)$, q^2 is very small (indeed it is consistent with zero) and we correctly reproduce the kinematics of the processes $p \rightarrow \pi^0 e^+$ or $p \rightarrow \pi^+ \bar{\nu}_e$. The main results of this paper are therefore obtained by using the 15 configurations for which we have the pion with this non-zero momentum. When the proton and the pion are both at rest q^2 is larger ($q^2/m_p^2 \sim 0.25 - 0.4$), and the corresponding results interpolate between those at the physical value of q^2 ($q^2/m_p^2 \approx 0$) and those calculated in the soft pion limit^(10,11) ($q^2/m_p^2 = 1$). In table 1 we present the values of $q^2 a^2$ and q^2/m_p^2 for each value of K , when the pion has momentum $(\pi/10, 0, 0)$ and when the pion is at rest.

We now present our results for A_1 and A_2 . We write $A_i = A_i^u + A_i^d$ ($i = 1, 2$) where A_i^u and A_i^d are the contributions to A_i from the diagrams in which the spectator quark is an up quark (Fig. 2a) or down quark (Fig. 2b) respectively. In table 2 we tabulate our results for the dimensionless quantities A_i^q/m_p^2 ($q = u$ or d) for two values of (t_x, t_y) , (always keeping $t_x - t_y \geq 6$ and $t_y \geq 8$ so as to isolate the pion and proton states). The matrix elements presented in table 2 are renormalised in the "Lattice Renormalisation Scheme" at a renormalisation scale of about 2 GeV (i.e. $a^{-1} \simeq 2\text{GeV}$)^(7,9). The errors in the matrix elements are computed by dividing the configurations into three clusters and treating the results from each cluster as independent.

K	Proton at rest Pion with $\underline{p} = \left[\frac{\pi}{10}, 0, 0 \right]$		Proton at rest Pion with $\underline{p} = \underline{0}$	
	$\frac{q^2}{m_p^2}$	$q^2 a^2$	$\frac{q^2}{m_p^2}$	$q^2 a^2$
0.1515	$(0.7 \pm 4.1)10^{-2}$	$(0.6 \pm 3.9)10^{-2}$	0.26 ± 0.02	0.28 ± 0.04
0.1530	$(-2.1 \pm 5.3)10^{-2}$	$(-1.4 \pm 4.0)10^{-2}$	0.31 ± 0.03	0.28 ± 0.05
0.1545	$(-4.1 \pm 6.8)10^{-2}$	$(-2.3 \pm 4.5)10^{-2}$	0.40 ± 0.02	0.30 ± 0.02

TABLE 1. Values of $q^2 a^2$ and $\frac{q^2}{m_p^2}$ for the two cases in which the pion has momentum

$\underline{p} = (\pi/10, 0, 0)$ and $\underline{p} = 0$. q is the four-momentum transfer from the proton to the pion.

$\frac{A_i^q}{2m_p}$	K	$p = (\frac{\pi}{10}, 0, 0)$		$p = 0$	
		$t_x = 15, t_y = 8$	$t_x = 15, t_y = 9$	$t_x = 14, t_y = 8$	$t_x = 15, t_y = 9$
$\frac{A_1^u}{2m_p}$	0.1515	$-(4.3 \pm 0.4)10^{-2}$	$-(4.3 \pm 0.3)10^{-2}$	$-(5.7 \pm 2.2)10^{-2}$	$-(5.9 \pm 2.5)10^{-2}$
	0.1530	$-(4.6 \pm 0.6)10^{-2}$	$-(4.5 \pm 0.6)10^{-2}$	$-(7.1 \pm 3.8)10^{-2}$	$-(7.7 \pm 4.6)10^{-2}$
	0.1545	$-(4.5 \pm 0.7)10^{-2}$	$-(4.5 \pm 0.6)10^{-2}$	$-(9.4 \pm 4.2)10^{-2}$	$-(11.7 \pm 5.5)10^{-2}$
	K_c	$-(4.7 \pm 0.9)10^{-2}$	$-(4.7 \pm 0.8)10^{-2}$	-	-
$\frac{A_1^d}{2m_p}$	0.1515	$(4.0 \pm 0.2)10^{-2}$	$(4.0 \pm 0.2)10^{-2}$	$(5.9 \pm 2.0)10^{-2}$	$(6.2 \pm 2.3)10^{-2}$
	0.1530	$(3.9 \pm 0.3)10^{-2}$	$(3.8 \pm 0.2)10^{-2}$	$(6.8 \pm 2.7)10^{-2}$	$(7.3 \pm 3.2)10^{-2}$
	0.1545	$(3.1 \pm 0.4)10^{-2}$	$(2.6 \pm 0.5)10^{-2}$	$(7.6 \pm 3.1)10^{-2}$	$(8.7 \pm 4.0)10^{-2}$
	K_c	$(2.8 \pm 0.5)10^{-2}$	$(2.6 \pm 0.5)10^{-2}$	-	-
$\frac{A_2^u}{2m_p}$	0.1515	$-(7.0 \pm 0.6)10^{-2}$	$-(7.0 \pm 0.6)10^{-2}$	$-(8.9 \pm 1.6)10^{-2}$	$-(9.2 \pm 1.6)10^{-2}$
	0.1530	$-(7.8 \pm 1.2)10^{-2}$	$-(7.8 \pm 1.1)10^{-2}$	$-(10.0 \pm 2.4)10^{-2}$	$-(10.6 \pm 2.5)10^{-2}$
	0.1545	$-(7.9 \pm 1.5)10^{-2}$	$-(8.0 \pm 1.4)10^{-2}$	$-(11.5 \pm 4.2)10^{-2}$	$-(13.5 \pm 5.0)10^{-2}$
	K_c	$-(8.6 \pm 2.1)10^{-2}$	$-(8.8 \pm 1.9)10^{-2}$	-	-
$\frac{A_2^d}{2m_p}$	0.1515	$-(6.4 \pm 0.9)10^{-3}$	$-(5.3 \pm 1.2)10^{-3}$	$-(1.2 \pm 0.1)10^{-2}$	$-(1.2 \pm 0.1)10^{-2}$
	0.1530	$-(7.6 \pm 1.2)10^{-3}$	$-(5.5 \pm 1.0)10^{-3}$	$-(1.4 \pm 0.2)10^{-2}$	$-(1.6 \pm 0.2)10^{-2}$
	0.1545	$-(9.3 \pm 3.5)10^{-3}$	$-(6.2 \pm 2.1)10^{-3}$	$-(2.0 \pm 0.8)10^{-2}$	$-(2.3 \pm 1.3)10^{-2}$
	K_c	$-(10.4 \pm 3.5)10^{-3}$	$-(6.4 \pm 1.3)10^{-3}$	-	-

TABLE 2. Values of $\frac{A_i^q}{2m_p}$ ($q = u, d$; $i = 1, 2$). The matrix elements are evaluated in the

lattice renormalisation scheme at a renormalisation scale of about 2 GeV.

As discussed above, the kinematics of the process $p \rightarrow \pi^0 e^+$ requires the pion to have momentum $(\pi/10, 0, 0)$ and we now concentrate on the results for this case. Among the pleasing features of these results are:-

(i) that the statistical errors are small,

(ii) that the dimensionless quantities $\frac{A_i^q}{m_p^2}$ depend very mildly (if at all) on K . In table 2 we also present the values obtained by extrapolating the results linearly in $1/K$ to the chiral limit, $K_c \simeq 0.1563^{(7)}$.

and

(iii) that the results for $t_y = 8$ and 9 are equal³, implying that the dependence of the correlation function (2.3) on t_y is given correctly by (2.6).

The matrix elements for the process $p \rightarrow \pi^+ \bar{\nu}_e$ are simply a factor $\sqrt{2}$ times those in table 2, i.e.

$$\langle \pi^+ | \epsilon^{abc} (u^a C \Gamma_1 d^b) (\Gamma_1 d^c) | p \rangle = \sqrt{2} \langle \pi^0 | \epsilon^{abc} (u^a C \Gamma_1 d^b) (\Gamma_1 u^c) | p \rangle \quad (2.10)$$

where $\Gamma_1 = I$ and $\Gamma_2 = \gamma^5$.

From table 2 we are also able to deduce matrix elements for the process $p \rightarrow K^0 + \text{lepton}$ in the $SU(3)$ limit,

$$\langle K^0 | \epsilon^{abc} (u^a C \Gamma_1 s^b) (\Gamma_1 u^c) | p \rangle = -\sqrt{2} A_1^d N_y \quad (2.11)$$

where again $\Gamma_1 = I$ and $\Gamma_2 = \gamma^5$. The operator in (2.11) is present in

³ this is also the case for $t_y = 7$ and 10 .

the effective action of some supersymmetric grand unified theories. From table 2 we are not able, however, to obtain the matrix elements for the process $p \rightarrow K^+ \bar{\nu}$. These will be presented in a future publication.

The effective action for proton decay can be written in the form $\sum_i C_i(\mu) O_i^B(\mu)$, where the sum is over all baryon number violating operators O_i^B , and μ is the renormalisation scale. The Wilson coefficient functions $C_i(\mu)$ are calculable in perturbation theory. We need, therefore, to compute the matrix elements of the operators $O_i^B(\mu)$, renormalised in the same renormalisation scheme (the $\overline{\text{MS}}$ scheme say) as that in which the $C_i(\mu)$ were calculated. This requires a perturbative calculation in order to translate the matrix elements computed in the "Lattice Renormalisation Scheme" into those in the $\overline{\text{MS}}$ scheme. This calculation was performed to one loop order in ref.(12). The two operators in eqn. (2.1) and (2.2) mix under renormalisation on the lattice, however this mixing is numerically negligible. To good precision $A_1^{\overline{\text{MS}}}/A_1^{\text{LATT}} \simeq 0.80$ and $A_2^{\overline{\text{MS}}}/A_2^{\text{LATT}} \simeq 0.75$, where the labels $\overline{\text{MS}}$ and LATT indicate that the matrix elements have been computed in the corresponding renormalisation scheme.

Table 2 contains the main results of this paper. The entries can be used to estimate proton decay rates in all grand unified theories. The theory which has been studied most fully is the minimal theory with gauge group $\text{SU}(5)$ and we now determine the rate relevant for the decay $p \rightarrow \pi^0 e^+$ in this theory. The effective action for this decay is given by

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{-g_5^2(M_X)}{M_X^2} F_S \epsilon^{abc} [2(u^a C d_L^b) (\bar{e}^+ u_R^c) + \\ & (u^a C d_R^b) (\bar{e}^+ u_L^c)] + \text{H.C} \end{aligned} \quad (2.12)$$

where $g_5(M_X)$ is the coupling constant at the grand unified scale M_X , F_S is the short distance enhancement factor at a renormalisation

scale of about 2 GeV, and the subscripts L and R denote the left and right handed components of the fields. We have neglected the small difference ($\sim 10\%$) in the short distance enhancement factors for the two operators in equation (2.12). The two matrix elements relevant for this decay can be readily evaluated from table 2 using

$$\langle \pi^0 | \epsilon^{abc} (u^a C d_L^b) (u_R^c)_y | p \rangle = \frac{1}{4} (A_1 - A_2) ((1 + \gamma^5)N)_y \quad (2.13)$$

and

$$\langle \pi^0 | \epsilon^{abc} (u^a C d_R^b) (u_L^c)_y | p \rangle = \frac{1}{4} (A_1 - A_2) ((1 - \gamma^5)N)_y \quad (2.14)$$

The partial width for this decay mode is given by

$$\Gamma(p \rightarrow \pi^0 e^+) = \frac{5\pi}{8} |F_S|^2 \alpha_5^2(M_X) \frac{|A_1 - A_2|^2}{m_p^4} \frac{m_p^5}{M_X^4} \quad (2.15)$$

where $\alpha_5(M_X) = \frac{g_5^2(M_X)}{4\pi}$. We now evaluate (2.15) using the values $\alpha_5(M_X) \simeq 0.024^{(5)}$, $|F_S|^2 = 10$, and the matrix elements from table 2 with the pion at momentum $p = (\pi/10, 0, 0)$. A_1 and A_2 correspond to the matrix elements (2.1) and (2.2) renormalised in the $\overline{MS}$ scheme. We obtain the following values for the lifetime of the proton divided by its branching ratio into $\pi^0 e^+$,

$$\tau(p \rightarrow \pi^0 e^+) = (1.3 \pm 0.5) \times 10^{30} \left[\frac{M_X}{2 \times 10^{14} \text{ GeV}} \right]^4 \text{ years} \quad (2.16)$$

when we use the results for $t_y = 8$, $t_x = 15$ and

$$\tau(p \rightarrow \pi^0 e^+) = (1.7 \pm 0.3) \times 10^{30} \left[\frac{M_X}{2 \times 10^{14} \text{ GeV}} \right]^4 \text{ years} \quad (2.17)$$

when we use the results for $t_y = 9$, $t_x = 15$. The errors in (2.16) and (2.17) are only the statistical errors in our evaluation of

$\frac{A_1 - A_2}{m_p^2}$. The two results in (2.16) and (2.17) are correlated, so

that we cannot combine them in quadrature and we take (2.17) to be our best estimate.

How large are the systematic errors inherent in such a lattice computation? Recently a large class of hadronic matrix elements has been computed⁽⁷⁻⁹⁾, and in general the difference between the lattice results and the experimental measurements is less than about 30-50%. In the present case perhaps the largest uncertainty is in the determination of the mass scale in the dimensionful quantities A_1 and A_2 . Throughout the above discussion we have used the proton's mass to set this scale (i.e. we compute A_1/m_p^2). In common with all other quenched computations we find that the ratio of the proton's mass to the rho mass is larger than the physical value. Had we used the rho mass to set the scale our matrix elements would be about a factor of 2 larger than those in table 2. Thus, in summary, we expect the systematic error to consist of a general uncertainty of about 30-50% superimposed with a possible increase of the matrix elements A_1 and A_2 up to a factor of 2 due to the uncertainty in the scale. The total uncertainty is thus small (compared to the error in eqn. (1.3) for example), particularly remembering that there are no model assumptions or free parameters in the calculation.

Comparing our result (2.16) with that obtained from a survey of previous estimates of the matrix elements, equation (1.3), (where the same values for the other parameters were used), we see that our results are in the expected range, (but towards the top of this range).

3. CHIRAL LAGRANGIANS, SOFT PION THEOREMS AND PROTON DECAY

Chiral symmetry and soft pion theorems have been useful tools in studying low energy hadronic amplitudes. However the precision with which they can be applied to proton decay matrix elements is not clear, since the final state meson carries a significant fraction of the proton's momentum. Nevertheless they have been used in this context^(10,11), and we are now in a position to check the accuracy of the soft pion relation for the decay $p \rightarrow \pi^0 e^+$ (10),

$$\langle \pi^0 | \epsilon^{abc} (u^a C d_L^b) u_R^c | p \rangle = \frac{i}{\sqrt{2} f_\pi} \langle 0 | \epsilon^{abc} (u^a C d_L^b) u_R^c | p \rangle (1+F+D) \quad (3.1)$$

and a similar equation with $L \leftrightarrow R$. The left-hand side of equation (3.1) can be read off from the entries in table 2 and we now evaluate the right hand side. We write

$$\langle 0 | \epsilon^{abc} (u^a C d_L^b) (u_R^c)_y | p \rangle \equiv \alpha (N_R)_y \quad (3.2)$$

and evaluate α by computing the two point correlation function with the operator in (3.2) at one point and J_p (see equation (2.4)) at the other. We have evaluated α using 30 gauge field configurations, and the results for the dimensionless quantity α/m_p^3 are presented in table 3. Since the left hand side of (3.1) was evaluated using 15 configurations, we also tabulate the results obtained for α/m_p^3 using this subset of configurations. We also present the result of a linear extrapolation in $1/K$ to the chiral limit K_c . From table 3 we obtain

$$|\alpha|_{\text{LATT}} = (7.3 \pm 0.9) 10^{-3} \text{ GeV}^3, \quad (3.3)$$

which, using one loop perturbation theory, translates to

$$|\alpha|_{\overline{\text{MS}}} = (5.6 \pm 0.8) 10^{-3} \text{ GeV}^3 \quad (3.4)$$

at a renormalisation scale of about 2 GeV.

K	$\frac{ \alpha }{3m_p}$	
	30 CONFIGURATIONS	SUBSET OF 15 CONFIGURATIONS
0.1515	$(6.5 \pm 0.3) \times 10^{-3}$	$(5.9 \pm 0.4) \times 10^{-3}$
0.1530	$(7.4 \pm 0.5) \times 10^{-3}$	$(6.7 \pm 0.7) \times 10^{-3}$
0.1545	$(7.6 \pm 0.9) \times 10^{-3}$	$(7.4 \pm 1.0) \times 10^{-3}$
K_c	$(8.8 \pm 1.1) \times 10^{-3}$	$(8.4 \pm 1.4) \times 10^{-3}$

TABLE 3. Values of $\frac{|\alpha|}{3m_p}$ obtained using 30 gluon configuration, and also using only the subset of 15 configuration used in evaluating the L.H.S. of (3.1)

This result is in very good agreement with most calculations of α using QCD sum rules or the bag model (we reproduce the compilation of these results from ref. (11) in Table 4). The result in (3.3) is lower than that obtained in a lattice computation by Hara et al⁽²⁾, who found $|\alpha|_{\text{LATT}} \approx 0.03$ with about a 20% error. Much of the discrepancy between these two results can be accounted for by the fact that we use the proton's mass to set the scale for $|\alpha|$, whereas in ref. (2) the rho mass is used. Since in lattice computations the ratio of the proton's mass to the rho's mass is always found to be too large, the results for $|\alpha|$ obtained using the proton's mass to set the scale are smaller than those obtained using the rho mass. At present this remains an inherent systematic uncertainty of lattice computations. We also extrapolate differently to the chiral limit than Hara et al. (we extrapolate the results for $|\alpha|/m_p^3$ which vary rather mildly with K, whereas Hara et al extrapolate the results for $|\alpha|^2$). These two differences in procedure account fully for the difference between the result obtained in ref. (2) and that in (3.3).

Ioffe ⁽¹³⁾	0.009	Ioffe & Smilga ⁽¹⁸⁾	0.006
Tomozawa ⁽¹⁴⁾	0.006	Tadic' et al ⁽¹⁹⁾	0.005
Krasnikov et al ⁽¹⁵⁾	0.006	Brodsky et al ⁽¹¹⁾	0.03
Donoghue et al ⁽¹⁶⁾	0.003	Hara et al ⁽²⁾	0.03
Thomas and McKeller ⁽¹⁷⁾	0.02	Bowler et al ⁽³⁾	0.013

This paper 0.006

TABLE 4. Previously determined values of α taken from the compilation of ref (11), and updated to include some more recent calculations. All values are in units of GeV^3 .

Another earlier lattice computation⁽³⁾ but at $\beta = 5.7$ found $|\alpha| \sim 0.013$ (at a renormalisation scale of about 1 GeV) having used the rho mass to set the scale and extrapolated the results for $|\alpha|a^3$ linearly to the chiral limit.

Using the physical values of f_π and $F + D$, ($f_\pi = 132$ MeV, $F + D = 1.26$) we find that the ratio of the right hand side to the left hand side of equation (3.1) is,

$$\frac{\text{R.H.S. of (3.1)}}{\text{L.H.S. of (3.1)}} = 2.4 \pm 0.6 \quad (3.5)$$

using results for $t_x = 15$, $t_y = 8$ and

$$\frac{\text{R.H.S. of (3.1)}}{\text{L.H.S. of (3.1)}} = 2.8 \pm 0.8 \quad (3.6)$$

using those for $t_x = 15$, $t_y = 9$. Thus it seems that by using the chiral Lagrangian and soft pion limit (eqn. (3.1)) one significantly overestimates the amplitude. Note also that had the rho mass rather than the proton mass been used to set the scale, the values in equations (3.5) and (3.6) would have been even larger.

The tendency of the matrix elements to increase as the momentum transfer is increased, (i.e. as we go towards the soft pion limit) is confirmed by the entries in the columns with $p = 0$ in Table 2. For example, using the entries with $t_y = 8$ for both $p = 0$ and $p = (\pi/10, 0, 0)$ we find that the ratio of the matrix element

$\langle \pi^0 | \epsilon^{abc} (u^a C d_L^b) u_R^c | p \rangle$ at $p = 0$ to that at $p = (\pi/10, 0, 0)$ is given by

$$\frac{\text{Matrix Element at } p = 0}{\text{Martrix Element at } p = (\pi/10, 0, 0)}$$

$$= 1.4 \pm 0.1 \quad \text{at } K = 0.1515$$

$$= 1.4 \pm 0.2 \quad \text{at } K = 0.1530 \quad (3.7)$$

$$= 1.6 \pm 0.5 \quad \text{at } K = 0.1545$$

The values of the corresponding momentum transfers are given in table 1.

In supersymmetric grand unified theories, a second parameter, β , is present in the chiral Lagrangian. This is defined by

$$\langle 0 | \epsilon^{abc} (u^a C d_R^b) u_R^c | p \rangle \equiv \beta N_R \quad (3.8)$$

We find that $\beta = -\alpha$ to very good precision, (typically around 1%), confirming earlier observations that the matrix element

$\langle 0 | \epsilon^{abc} (u^a C d^b) u_R^c | p \rangle$ is very small.

4. SUMMARY AND CONCLUSIONS

In this paper we have presented the results of the first direct lattice computation of the matrix elements for the decays $p \rightarrow \pi^0$ or $K^0 + \text{lepton}$ and $p \rightarrow \pi^+ \bar{\nu}$ in grand unified theories. The results are presented in table 2. The statistical errors are small, and the major systematic uncertainty is associated with the determination of the mass scale. Had we used the rho mass, rather than the proton mass, to set this scale the elements in table 2 would increase by a factor of about 2. Given that there are no model assumptions or free parameters, we believe that this calculation gives the most reliable determination of proton decay amplitudes. Our results lie in the range of previous estimates of these matrix elements.

We have also shown that by using the Chiral Lagrangian and soft-pion theorems one overestimates the proton decay amplitude by a factor of more than 2.

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REFERENCES

1. M. Goldhaber and W.J. Marciano, Comments Nucl.Part.Phys. 16, (1986), 23.
2. Y. Hara et al. Phys. Rev. D34, (1986), 3399.
3. K.C. Bowler et al., Nucl. Phys. B296, (1988), 431.
4. K.G. Wilson, Phys. Ref. D10, (1974), 2445; "New Phenomena in Subnuclear Physics", A. Zichichi ed., Erice 1975 (Plenum Press, New York 1977).
5. W.J. Marciano, Proceedings of the Summer Study on the Physics of the Superconducting Supercollider, Snowmass 1986, p.726 (R. Donaldson and J. Marx eds.; published by the American Physical Society).
6. D.S. Ayres et al., Proceedings of the Summer Study on the Physics of the Superconducting Supercollider, Snowmass 1986, p.719 (R. Donaldson and J. Marx eds.; published by the American Physical Society).
7. G. Martinelli and C.T. Sachrajda, CERN Preprint TH.4909/87 (Nucl. Phys. B, in press).
8. G. Martinelli and C.T. Sachrajda, CERN Preprint TH.5042/88.
9. M.B. Gavela et al., CERN Preprint TH.4905/87 (Nucl. Phys. B, in press).
10. M. Claudson, L.J. Hall and M.B. Wise, Nucl. Phys. B195, (1982), 297.
11. S.J. Brodsky et al., Nucl. Phys. B238, (1984), 581.
12. D.G. Richards, C.T. Sachrajda and C.J. Scott, Nucl. Phys. B286, (1987), 683.
13. B.L. Ioffe, Nucl. Phys. B188, (1981), 317.
14. Y. Tomozawa, Phys. Rev. Lett. 46, (1981), 463.
15. N.V. Krasnikov, A.A. Pivovarov and N.N. Tavheliidze, Z. Phys. C19, (1983), 301.
16. J. Donoghue and E. Golowich, Phys. Rev. D26, (1982), 3092.
17. A.W. Thomas and B.H.J. McKellar, Nucl. Phys. B227, (1983), 206.
18. B.L. Ioffe and A.V. Smilga, Nucl. Phys. B232, (1984), 109.
19. M. Milosevic, D. Tadic and J. Trampetic, Nucl. Phys. B207, (1982) 461.

FIGURE CAPTIONS

- 1a) Feynman Diagram for proton decay. The cross, open circle and closed circle represent the insertion of 0^B , J_π and $\bar{J}_p$ respectively.
- b) The same diagram redrawn to demonstrate the role of the "generalised quark propagator", $S_1(y,0; t_x,p)$, represented by the double line.
- 2 Feynman diagrams for proton decay in which the spectator quark is
 - a) an up quark and
 - b) a down quark.

$\equiv$

(a)

(b)

Fig. 1.

(a)

(b)

Fig. 2