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DEFORMATIONS OF COMPLEX STRUCTURES AND
REPRESENTATIONS OF KRICHEVER-NOVIKOV ALGEBRAS

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A B S T R A C T

By applying the conformal Ward identities we study the representations of the Krichever-Novikov algebras associated to conformal field theories on compact Riemann surfaces. We compute the matrix elements between primary states of the KN generators corresponding to deformations of the complex structure. We show that these matrix elements depend on the derivatives of the partition function with respect to the moduli. The effects of this dependence on the highest weight representations are discussed.

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It is well known that a genus $p \geq 1$ compact Riemann surface Σ_p admits inequivalent complex structures parametrized by the moduli space

$$\mathcal{M}_p \equiv \frac{Met(\Sigma_p)}{Diff(\Sigma_p) \otimes \mathcal{W}} \quad . \quad (1)$$

Here $Met(\Sigma_p)$ is the set of all metrics on Σ_p , while $Diff(\Sigma_p)$ and $\mathcal{W}$ denote the groups of diffeomorphisms and Weyl rescalings respectively. Conformal field theories on Σ_p in general depend on the moduli, namely feel the effect of deformations of the complex structure. In this letter we pursue further the analysis, started in [1], of the role of complex structure deformations in representing the conformal algebra.

Let us start by recalling some basic properties of any conformal field theory on the sphere. As a consequence of the conformal Ward identities [2], the state space $\mathcal{J}$ of such a theory carries a representation of two commuting (conventionally called left and right) Virasoro algebras [3]. This fact has been established already in the early days of conformal field theory [4]. More recently [2] it has been found that the representation under consideration is of the form

$$\mathcal{J} = \oplus_{\alpha, \beta \geq 0} M_{\alpha\beta} \mathcal{V}_L(h_\alpha, c) \otimes \mathcal{V}_R(\bar{h}_\beta, c) \quad . \quad (2)$$

Here $\mathcal{V}_{L(R)}(h, c)$ is a highest weight representation [5] (called also Verma module) of the left (right) Virasoro algebra characterized by central charge c and highest weight, h and $M_{\alpha\beta}$ is the matrix of multiplicities with non-negative integer entries. Some fundamental properties of (2) are:

(i) The Verma module is an universal, meaning model-independent, structure characterized completely by h and c .

(ii) The model dependence is carried exclusively by the matrix $M_{\alpha\beta}$. The representation $\mathcal{J}_{vac} \equiv \mathcal{V}_L(0, c) \otimes \mathcal{V}_R(0, c)$, called vacuum sector, appears exactly once in the direct sum (2).

(iii) $\mathcal{V}(h, c)$ is unitary provided that $h \geq 0$ and $c \geq 1$. For $0 < c < 1$, unitarity implies discrete values for h and c - the celebrated minimal series [6].

Consider now a conformal field theory on a Riemann surface Σ_p of genus $p \geq 1$. A preliminary problem to be solved is to find a generalization of the Virasoro algebra

(*Vir*). The most natural candidate has been proposed by Krichever and Novikov [7,8]. The Krichever-Novikov (*KN*) algebra is defined as the central extension of the algebra of meromorphic vector fields on Σ_p , which are allowed to have poles only in two fixed generic points $P_{\pm} \in \Sigma_p$. $P_{\pm}$ are the counterparts of the south ($z = 0$) and north ($z = \infty$) poles for *Vir* on the sphere [2]. For $p = 0$ the *KN* algebra collapses to *Vir*. An important feature of *KN* is that it carries some global information about Σ_p ; the structure constants as well as the central extension of *KN* depend explicitly on the genus and the chosen complex structure.

The conformal Ward identities imply [9] that the state space $\mathcal{J}$ carries a representation of a left and a right *KN* algebras. Unfortunately, the representation theory of *KN* algebras is under development (see [8]) and is far from being completed. The aim of this letter is to study the impact of complex structure deformations on a particular class of representations of *KN* - those stemming from an energy-momentum tensor. Our results show that the properties (i-iii) listed above for the $p = 0$ case do not hold in general for $p \geq 1$.

Before starting the technical discussion, we have to fix some notations and conventions. We shall assume that Σ_p is equipped with a metric g of constant curvature. In this case the atlas $\{U_{\alpha}\}$ of local complex coordinates on Σ_p can be chosen in such a way that the transition functions between overlapping patches are isometries. The existence of such an atlas is crucial for defining the theory globally on Σ_p , because for $c \neq 0$, only the isometries are unitarily implementable on $\mathcal{J}$. In the above coordinates the conformal Ward identities with one energy-momentum insertion read [9]:

$$\begin{aligned}
& \langle T_{zz} \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle = \\
& \sum_{j=-p_0+2}^{p_0-2} h_{zz}^j(z) \int d^2w \sqrt{g} g^{w\bar{w}} \eta_{\bar{w},j}^w(w, \bar{w}) \langle T_{ww} \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle \\
& - \sum_{k=1}^n \left[h_k \nabla_{z_k} G^{z_k}_{zz}(z_k, z) + G^{z_k}_{zz}(z_k, z) \nabla_{z_k}^{(s_k)} \right] \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle \quad . \quad (3)
\end{aligned}$$

Here the notation is as follows: h_k is the conformal weight of ϕ_k , defined in terms of the dimension d_k and the spin s_k by $h_k = \frac{1}{2}(d_k + s_k)$; $p_0 \equiv \frac{3}{2}p$; $j \in \mathbb{Z}$ for p even, while

$j \in Z + \frac{1}{2}$ for p odd. The covariant derivative $\nabla_z^{(s)}$ is given by

$$\nabla_z^{(s)} = \partial_z + is\omega_z \quad ,$$

ω_z being the spin connection. The function G^w_{zz} is the Green kernel for the operator ∇^w and satisfies

$$\nabla^w G^w_{zz}(w, z) = \frac{1}{\sqrt{g}} \delta(z - w) - \sum_{j=-p_0+2}^{p_0-2} g^{w\bar{w}} \eta^w_{\bar{w},j}(w, \bar{w}) h_{zz}^j(z) \quad , \quad (4)$$

where $\{h_{zz}^j\}$ is a basis of holomorphic quadratic differentials on Σ_p , while $\{\eta^z_{\bar{z},j}\}$ are the Beltrami differentials dual to h_{zz}^j , i.e.

$$\int d^2z \sqrt{g} g^{z\bar{z}} \eta^z_{\bar{z},i}(z, \bar{z}) h_{zz}^j(z) = \delta_i^j \quad . \quad (5)$$

Consider now the vacuum expectation value $\langle T_{zz} \rangle$. From (3) one obtains

$$\langle T_{zz} \rangle - \sum_{j=-p_0+2}^{p_0-2} h_{zz}^j \int d^2w \sqrt{g} g^{w\bar{w}} \eta^w_{\bar{w},j} \langle T_{ww} \rangle = 0 \quad , \quad (6)$$

which represents an integral equation for $\langle T_{zz} \rangle$. We denote by y a system of local complex coordinates on the Teichmüller space

$$T_p \equiv \frac{Met(\Sigma_p)}{Diff_0(\Sigma_p) \otimes \mathcal{W}} \quad ,$$

where $Diff_0(\Sigma_p)$ is the component connected to the identity of $Diff(\Sigma_p)$. The general solution of (6) reads

$$\langle T_{zz} \rangle = \sum_{j=-p_0+2}^{p_0-2} h_{zz}^j(z) V_j(y, \bar{y}) \quad , \quad (7)$$

where V_j are the components of a vector field on T_p . In the case $p = 0$ one gets from (7) that $\langle T_{zz} \rangle = 0$, which is extensively used in deriving the familiar properties of the representation (2) of the Virasoro algebra on the sphere. For $p \geq 1$, a simple computation [9] gives

$$V_j(y, \bar{y}) = \frac{\partial}{\partial y^j} \log Z(y, \bar{y}) \quad , \quad (8)$$

provided that $Z(y, \bar{y}) \neq 0$. In the points of $\mathcal{M}_p$ where Z vanishes, $\langle T_{zz} \rangle$ requires a separate investigation [1].

Let us now concentrate on eqs.(7) and (8). The net result is that T_{zz} develops in general a non-trivial vacuum expectation value $\langle T_{zz} \rangle$, which physically can be interpreted as a sort of Casimir effect due to handles. We think that this effect is quite general and is present also for systems which are not necessarily conformal invariant. Heuristically it can be explained [10] as a result of an extra cooperation via paths on Σ_p that cannot be shrunk to a point. It is worth stressing that in our case $\langle T_{zz} \rangle$ is not related to the central charge and in this sense is not the effect considered in [11]; the phenomenon we are describing occurs even when the central charge vanishes.

From the point of view of field theory, condensates like $\langle T_{zz} \rangle$ usually parametrize unitarily inequivalent field representations; field theories corresponding to non-equivalent complex structures on Σ_p are in general unitarily inequivalent. Indeed, if one associates with the complex deformation

$$\delta g_{\bar{z}\bar{z}} = 2 \sum_{j=-p_0+2}^{p_0-2} g_{\bar{z}z} \eta_{\bar{z},j}^z \delta y^j \quad (9)$$

the "charge" operator [12]

$$Q_j \equiv \int d^2w \sqrt{g} g^{w\bar{w}} \eta_{\bar{w},j}^w T_{w\bar{w}} \quad , \quad (10)$$

from (7) one gets

$$\langle Q_j \rangle = V_j(y, \bar{y}) \quad . \quad (11)$$

From the above discussion it should be evident that the condensate $\langle T_{zz} \rangle$ represents an intrinsic property of the system and that it cannot be absorbed in a field redefinition compatible with conformal invariance. For the proof of this statement we refer the reader to [1], where some necessary conditions for the vanishing of $\langle T_{zz} \rangle$ are also discussed. Concluding the discussion devoted to the energy-momentum tensor, we note that eq.(3) can be given the form

$$\langle T_{zz} \phi_1 \cdots \phi_n \rangle = \langle T_{zz} \rangle \langle \phi_1 \cdots \phi_n \rangle$$

$$\begin{aligned}
&= - \sum_{k=1}^n \left[h_k \nabla_{z_k} G^{z_k}_{zz}(z_k, z) + G^{z_k}_{zz}(z_k, z) \nabla_{z_k}^{(s_k)} \right] \langle \phi_1 \cdots \phi_n \rangle \\
&\quad + \sum_{j=-p_0+2}^{p_0-2} h_{zz}^j \frac{\partial}{\partial y^j} \langle \phi_1 \cdots \phi_n \rangle \quad , \tag{12}
\end{aligned}$$

where the last term (called Teichmüller term) makes explicit the contribution of deformations. We shall need also the Ward identity with two insertions of T_{zz} . It reads

$$\begin{aligned}
&\langle T_{zz} T_{ww} \phi_1 \cdots \phi_n \rangle - \langle T_{zz} \rangle \langle T_{ww} \phi_1 \cdots \phi_n \rangle \\
&= -\frac{c}{12} (\nabla_w)^3 G^w_{zz}(w, z) \langle \phi_1 \cdots \phi_n \rangle \\
&\quad - [2 \nabla_w G^w_{zz}(w, z) + G^w_{zz}(w, z) \nabla_w] \langle T_{ww} \phi_1 \cdots \phi_n \rangle \\
&\quad - \sum_{k=1}^n \left[h_k \nabla_{z_k} G^{z_k}_{zz}(z_k, z) + G^{z_k}_{zz}(z_k, z) \nabla_{z_k}^{(s_k)} \right] \langle T_{ww} \phi_1 \cdots \phi_n \rangle \\
&\quad + \sum_{j=-p_0+2}^{p_0-2} h_{zz}^j \frac{\partial}{\partial y^j} \langle T_{ww} \phi_1 \cdots \phi_n \rangle \quad . \tag{13}
\end{aligned}$$

Now we turn to the KN algebra, considering the case $p \geq 2$. The torus would need a separate investigation [7,8]. Our first step is to complement the basis of holomorphic quadratic differentials $\{h_{zz}^j : -p_0+2 \leq j \leq p_0-2\}$ to a basis of meromorphic quadratic differentials $\{h_{zz}^j : -\infty < j < \infty\}$ allowed to have poles only at the two fixed points $P_{\pm} \in \Sigma_p$. According to Riemann-Roch's theorem, the elements of this latter basis are determined up to a constant factor, provided that one fixes the order of zeros and poles in $P_{\pm}$. In a neighbourhood of $P_{\pm}$ one has

$$h_{zz}^j(z_+) = \sum_{m=0}^{\infty} \beta_{j,m}^+ z_+^{p_0+m-j-2} \quad , \quad \beta_{j,0}^+ = 1 \tag{14}$$

$$h_{zz}^j(z_-) = \frac{1}{\varepsilon_j} \sum_{m=0}^{\infty} \beta_{j,m}^- z_-^{p_0+m+j-2} \quad , \quad \beta_{j,0}^- = 1 \tag{15}$$

respectively. The local coordinates $z_{\pm}$ are chosen so that $z_+(P_+) = z_-(P_-) = 0$. It is important to keep in mind that $\beta_{j,m}^{\pm}$ depend on the given complex structure. In particular, their dependence on the period matrix is rather involved [13].

Following [7], the next step is to introduce the notion of "time" on Σ_p . One constructs an one-parameter family of contours $\{C_\tau : -\infty < \tau < +\infty\}$ on Σ_p as the level lines of a harmonic function $\text{Re } \omega(Q)$, where

$$\omega(Q) = \int_{Q_0}^Q d\omega \quad .$$

Here $Q_0 \in \Sigma_p$ is arbitrary and $d\omega$ is a suitably normalized [7] third kind differential on Σ_p . For $\tau \rightarrow \pm\infty$, C_τ represents a small circle around $P_\mp$. The basis of meromorphic vector fields $\{e_i^z : -\infty < i < +\infty\}$ dual to $\{h_{zz}^j : -\infty < j < +\infty\}$ is fixed now by

$$\oint_{C_\tau} e_i^z h_{zz}^j dz = \delta_i^j \quad . \quad (16)$$

In order to avoid a cumbersome notation, we omit the normalization factor $(2i\pi)^{-1}$ associated with any contour integral. The vector fields $\{e_i^z\}$ are holomorphic outside $P_\pm$; in a neighbourhood of $P_\pm$ they behave like

$$e_i^z(z_+) = \sum_{m=0}^{\infty} \alpha_{i,m}^+ z_+^{m+i-p_0+1} \quad , \quad \alpha_{i,0}^+ = 1 \quad (17)$$

$$e_i^z(z_-) = \varepsilon_i \sum_{m=0}^{\infty} \alpha_{i,m}^- z_-^{m-i-p_0+1} \quad , \quad \alpha_{i,0}^- = 1 \quad . \quad (18)$$

The coefficients $\alpha_{i,m}^\pm$ are uniquely determined by the duality conditions (16) in terms of $\beta_{j,m}^\pm$. Indeed eqs.(16) imply that for any i and for any $M > 0$

$$\sum_{m=0}^M \alpha_{i,m}^\pm \beta_{M+i,M-m}^\pm = 0 \quad . \quad (19)$$

Eqs.(19) can be solved for $\alpha_{i,m}^\pm$. One has

$$\begin{aligned} \alpha_{i,1}^\pm &= -\beta_{1+i,1}^\pm \quad , \\ \alpha_{i,2}^\pm &= \beta_{2+i,1}^\pm \beta_{1+i,1}^\pm - \beta_{2+i,2}^\pm \quad , \end{aligned} \quad (20)$$

and so on.

Now we formulate a working hypothesis. In the spirit of the reconstruction theorem we shall assume that the state space $\mathcal{J}$ of the system is equipped with an inner product $\langle, \rangle$ which can be inferred from the correlation functions $\langle T_{zz} \cdots \phi(w, \bar{w}) \cdots \rangle$. Furthermore we assume that

$$T_{zz} = \sum_{j=-\infty}^{+\infty} \mathcal{L}_j h_{zz}^j, \quad (21)$$

where $\mathcal{L}_j$ are well-defined operators on $\mathcal{J}$ and the series has convergent matrix elements. Combining this hypothesis with the Ward identities (13) one deduces that $\mathcal{L}_j$ satisfy the commutation relations:

$$[\mathcal{L}_i, \mathcal{L}_j] = \sum_{k=-p_0}^{p_0} C_{ij}^k \mathcal{L}_{i+j-k} + \frac{c}{12} \chi_{ij}, \quad (22)$$

where C_{ij}^k and χ_{ij} are the structure constants (with respect to the basis e_i^z) and the central extension of the KN algebra respectively. Indeed

$$[\mathcal{L}_i, \mathcal{L}_j] = \oint_{C_{-\infty}} e_i^w(w) \oint_{C_{\tau(w)}} e_j^z(z) T_{zz} dz T_{ww} dw, \quad (23)$$

where, due to the analytic properties of h^j and e_i , the r.h.s. depends only on the singular part in $z - w$ of the operator product $T_{zz} T_{ww}$. Consequently the line integrals are easily performed; one gets (22) and as a byproduct the explicit form of the structure constants and central extension. They read:

$$\begin{aligned} C_{ij}^k = \frac{1}{2} \sum_{m=0}^{2p_0} \beta_{i+j-k, p_0-m-k}^+ \sum_{n=0}^m [\alpha_{i,n}^+ \alpha_{j,m-n}^+ (i-j+m-2n) \\ + \alpha_{i,m-n}^+ \alpha_{j,n}^+ (i-j-m+2n)] \quad , \end{aligned} \quad (24)$$

$\chi_{ij} = 0$ for $|i+j| > 2p_0$ and

$$\begin{aligned} \chi_{ij} = \frac{1}{2} \sum_{m=0}^{2p_0-i-j} \{ \alpha_{i,m}^+ \alpha_{j, 2p_0-i-j-m}^+ [(p_0-i-m)^3 - (p_0-i-m)] \\ - \alpha_{i, 2p_0-i-j-m}^+ \alpha_{j,m}^+ [(p_0-j-m)^3 - (p_0-j-m)] \} \end{aligned} \quad (25)$$

for $|i + j| \leq 2p_0$. Analogously C_{ij}^k and χ_{ij} can be expressed in terms of $\alpha_{i,m}^-$ and $\beta_{i,m}^-$. By means of eqs.(20), C_{ij}^k and χ_{ij} can be written entirely in terms of $\beta_{i,m}^\pm$, which are the basic quantities we started with.

Summarizing the above discussion, we have shown that the state space $\mathcal{J}$ carries a representation of the KN algebra. Before analysing this representation, we spend two words about the general algebraic structure of KN . It is easily seen that for $s \geq -1$, any of the subspaces $A_\pm^{(s)} = \{\mathcal{L}_j : \pm j \geq p_0 + s\}$ is a subalgebra of KN . The set of $3p - 3$ generators $D = \{\mathcal{L}_j : -p_0 + 2 \leq j \leq p_0 - 2\}$, which are outside the maximal subalgebras $A_\pm^{(-1)}$, is related to complex deformations. A simple geometrical explanation of this fact is given in [7]. In the specific representation we are dealing with, the fact that D generates complex deformations emerges already in the vacuum sector. Indeed from eqs.(7) and (21) one deduces that

$$\langle \mathcal{L}_j \rangle = \theta(j + p_0 - 2)\theta(-j + p_0 - 2)V_j(y, \bar{y}) \quad , \quad (26)$$

which has to be compared with eq.(11). The step function θ in (26) is defined by $\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t \geq 0$. As mentioned above, on the sphere KN and Vir coincide. Consistently for $p \rightarrow 0$ all the $3p + 1$ generators $\{\mathcal{L}_{-p_0}, \dots, \mathcal{L}_{p_0}\}$ collapse to $L_0 \in Vir$ and, in agreement with (26), one has the well-known result $\langle L_j \rangle = 0$ for all j .

The necessity of computing the expectation values of $\mathcal{L}_j \in D$ between states created from the vacuum by primary fields has been emphasized by Krichever and Novikov [8]. The reason is that these expectation values play an essential role for the representation theory of KN . Eq.(26) represents the solution in the vacuum sector. Before computing the general case we have to explain how the notion of Verma module of Vir is generalized to KN .

In analogy with [2], Krichever and Novikov introduce the vacuum states $\Omega_\pm \in \mathcal{J}$ associated with $P_\pm$. If the field ϕ obeys the Ward identity (12) with weight h , one easily obtains

$$\mathcal{L}_j \phi(P_+) \Omega_+ = 0 \quad , j > p_0 \quad , \quad (27)$$

$$\mathcal{L}_{p_0} \phi(P_+) \Omega_+ = h \phi(P_+) \Omega_+ \quad . \quad (28)$$

Analogously

$$\mathcal{L}_j^\dagger \phi(P_-) \Omega_- = 0 \quad , j < -p_0 \quad , \quad (29)$$

$$\mathcal{L}_{-p_0}^\dagger \phi(P_-) \Omega_- = \varepsilon_{-p_0} h \phi(P_-) \Omega_- \quad . \quad (30)$$

where $\dagger$ stands for Hermitian conjugation with respect to the product $\langle, \rangle$ introduced before. The spaces $\mathcal{V}_\pm(h, c)$ which are generated from $|\phi(P_+)\rangle \equiv \phi(P_+) \Omega_+$ and $\langle \phi(P_-)| \equiv \phi(P_-) \Omega_-$ by applying $\{\mathcal{L}_j : j < p_0\}$ and $\{\mathcal{L}_j^\dagger : j > -p_0\}$ respectively, are generalized Verma modules [8]. The problem of computing $\langle \phi(P_-) | \mathcal{L}_j | \phi(P_+) \rangle$ can be solved by using once more the Ward identity (12). The computation gives

$$\begin{aligned} \langle \phi(P_-) | \mathcal{L}_j | \phi(P_+) \rangle &= \\ &= \{(\delta_{j, -p_0} \varepsilon_{-p_0} + \delta_{j, 1-p_0} \varepsilon_{1-p_0} \alpha_{1-p_0, 1}^-) h + (\delta_{j, p_0} + \delta_{j, p_0-1} \alpha_{p_0-1, 1}^+) h \\ &\quad + \theta(j + p_0 - 2) \theta(-j + p_0 - 2) [\frac{\partial}{\partial y^j} + V_j(y, \bar{y})]\} \langle \phi(P_-) | \phi(P_+) \rangle \\ &\quad + \delta_{j, 1-p_0} \varepsilon_{1-p_0} \langle (\nabla^{(*)} \phi)(P_-) | \phi(P_+) \rangle + \delta_{j, p_0-1} \langle \phi(P_-) | (\nabla^{(*)} \phi)(P_+) \rangle \quad . \end{aligned} \quad (31)$$

Besides the contributions from $\mathcal{L}_{\pm p_0}$ (see eqs.(28,30)), one finds contributions coming from $\mathcal{L}_{1-p_0}$, $\mathcal{L}_{p_0-1}$ and $\mathcal{L}_j \in D$. These are all the generators of KN collapsing to $L_0 \in Vir$ on the sphere S^2 where one has:

$$\langle \phi(P_-) | L_j | \phi(P_+) \rangle_{S^2} = \delta_{j, 0} h \langle \phi(P_-) | \phi(P_+) \rangle_{S^2} \quad . \quad (32)$$

In the above formula P_+ and P_- are the south and the north poles, respectively, of S^2 . Remember that, for $h > 0$ the vacuum expectation value $\langle \phi(P_-) | \phi(P_+) \rangle_{S^2}$ is a positive number which can be normalized to 1. On the contrary, on Σ_p , $\langle \phi(P_-) | \phi(P_+) \rangle$ carries a non trivial dependence on $y \in \mathcal{T}_p$. Note also that the right-hand side of (31) involves $V_j(y, \bar{y})$ and derivatives of $\langle \phi(P_-) | \phi(P_+) \rangle$ with respect to y . This is the impact of complex structure deformations on $\langle \phi(P_-) | \mathcal{L}_j | \phi(P_+) \rangle$, which now depend on the partition function, i.e. on the theory. The above features lead us to the conclusion that, contrary to the situation on S^2 (see point (i) in the introduction), the Verma modules $\mathcal{V}_\pm(h, c)$ associated with KN are not universal and depend on the conformal model under consideration.

The reader has noted that eqs.(26) and (31) involve only first derivatives with respect to the Teichmüller parameters. This is not a general feature, however. By means of eq.(13) we may compute the matrix elements $\langle \phi(P_-) | \mathcal{L}_i \mathcal{L}_j | \phi(P_+) \rangle$ which turn out to be dependent on $\partial_i \partial_j \log Z$, due to the last term of (13).

The question of positivity, when p is ≥ 1 , should be also reconsidered. The point is that the range for c and h for which positivity is ensured on S^2 (see (iii) above), does not imply, in general, the positivity of $\langle \phi(P_-) | \phi(P_+) \rangle$ for any $y \in \mathcal{T}_p$. An instructive counter example has been given by Schroer [14], who shows that if ϕ is a conserved vector current, its two-point function on the torus is positive definite only on the line $\text{Re} \tau = 0$ in the fundamental domain of the torus (here τ is the Teichmüller parameter on the torus). From Schroer's investigation it emerges that the domain of positivity of $\langle \phi(P_-) | \phi(P_+) \rangle$ is a complicated, generally unknown, subvariety of $\mathcal{T}_p$.

Finally, the Kac determinant associated with $\mathcal{V}_\pm(h, c)$ should be investigated in the light of the above discussion. The situation is different from the case of S^2 . The Kac matrix has new non-trivial entries, namely entries like $\langle \phi(P_-) | \mathcal{L}_i \cdots \mathcal{L}_j | \phi(P_+) \rangle$, where some of the generators $\mathcal{L}_i$ correspond to deformations. The above entries depend explicitly on the partition function. Moving from S^2 to $\Sigma_p, p > 0$, we obtain in principle different null states.

All the features discussed above show that the new (with respect to S^2) "gravitational" degrees of freedom, the moduli, have an essential contribution. It is reasonable to ask if one could avoid the above problems, by using Vir on Σ_p . Now, following Eguchi and Ooguri [9], one may associate to any coordinate patch U_α of Σ_p a Virasoro algebra Vir_α

$$[L_m(z), L_n(z)] = (m - n)L_{m+n}(z) + \frac{1}{12}c(m^3 - m)\delta_{m+n,0} \quad , z \in U_\alpha \quad (33)$$

The emerging algebraic structure can be globalized; in the atlas considered before, the transition functions are isometries $z \mapsto f(z)$ and one has

$$L_n(z) \mapsto f'(z)^{-n} L_n(f(z)) + \frac{1}{2}(1 - n)f''(z)f'(z)^{-n-2} L_{n+1}(f(z)) + \dots \quad , \quad (34)$$

where the dots stand for terms proportional to $L_m(f(z))$ with $m \geq n + 2$. As is shown in [1] however, the representation of Vir_α on $\mathcal{J}$ is not equivalent to the representation

of Vir associated to the theory on S^2 . Indeed there exists $s \geq 2$ such that

$$\langle L_{-s}(z) \rangle \neq 0 \quad , \quad z \in U_\alpha \quad . \quad (35)$$

In the Eguchi-Ooguri picture eq.(35) is the counterpart of (26). Effects like (35) should be expected, because a simple computation shows that one can express the $\mathcal{L}_j$'s in terms of the $L_j(P_\pm)$'s and viceversa. Indeed one has:

$$\mathcal{L}_i = \sum_{m=0}^{\infty} \alpha_{i,m}^+ L_{i+m-p_0}(P_+) \quad , \quad (36)$$

$$L_i(P_+) = \sum_{m=0}^{\infty} \beta_{i+m+p_0,m}^+ \mathcal{L}_{i+m+p_0} \quad , \quad (37)$$

and analogous formulae for P_- . This argument confirms that one cannot avoid the problems related to the global structure of Σ_p by using a local Virasoro algebra. Our investigation shows that the KN algebra is much more convenient in describing the effects of complex deformations.

In conclusion, we note that there is a recent progress in constructing the superanalogue of KN [15]. Our results may be generalized to this case by means of the superconformal extension [16] of the Ward identities (12,13).

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