CERN-TH-5186/88

**Improved Fermion Actions and
Hadronic Matrix Elements in Lattice QCD**

A. Patel

Theory Division, CERN, CH-1211 Genève 23, Switzerland

*Talk presented at the XXIV International Conference on High Energy Physics
Munich, Federal Republic of Germany, August 4-10, 1988*

Abstract

I give a quick summary of the derivation of an improved lattice fermion action, followed by Monte Carlo results obtained with this action. Preliminary results on the hadron spectrum including the effects of dynamical quarks are also presented.

CERN-TH-5186/88

September 1988

1. An Improved Lattice Fermion Action

The Renormalisation Group is a powerful framework for extracting infra-red properties of lattice theories. Its basic principle is to integrate out the high frequency modes within a block on a fine lattice, leaving behind an effective theory of the low lying modes on a coarse block lattice. The long distance behaviour of the theory is preserved, but at the expense of creating a lattice action with more complicated couplings. Such an action is expected to be less dependent on lattice artifacts and is generically referred to as an improved action. Though this methodology has been extensively applied to scalar and gauge fields using both linear and non-linear block transformations, progress with fermions has not been as rapid. Fermions, it turns out, are easier to deal with than scalar or gauge fields, the essential ingredient being that anti-commuting Grassmann variables allow only linear block transformations.

The application described below is aimed at an efficient computation of the infra-red behaviour of lattice QCD. The technique was first advocated under the name "approximate block-diagonalisation" [1]. Here I shall follow the language of the Real Space Renormalisation Group and outline the analysis [2] stepwise.

Step 1 : Select a block transformation. Specifically, we use the scheme where each block (in $d = 4$) is made up of 9 sites : the central block site and its 8 nearest neighbours. The scale factor for this scheme is then $b = 9^{1/d} = \sqrt{3}$, and the links on the block lattice are directed along the diagonals of 3-cubes on the fine lattice.

Step 2 : Rewrite the lattice action separating the degrees of freedom within the block from those between the blocks. The fermion matrix on the original lattice for the Wilson action is (with the superscript denoting the blocking level) :

$$\Delta(x^{(0)}, x^{(0)}) = 2m\delta_{x^{(0)}, x^{(0)}} - \sum_{\mu=1}^4 [c_{\mu}^{(0)}(x^{(0)}, x^{(0)}) - \gamma_{\mu} s_{\mu}^{(0)}(x^{(0)}, x^{(0)})] \quad (1)$$

Here s_{μ} and c_{μ} (sine and cosine) represent the nearest neighbour lattice discretisation of first and second derivatives respectively. The tedious though straightforward task of reformulating this action in terms of variables on the block lattice is accomplished by introducing an extra internal degree of freedom on the block lattice :

Degrees of freedom	sites	internal	spin	colour
Original lattice	L^4	—	4	3
Block lattice	$L^4/9$	9	4	3

To maintain explicit gauge covariance, the 9 fermion degrees of freedom inside each block, χ_j , $j \in \{-4, \dots, 0, \dots, 4\}$, are constructed by parallel transporting the original fields to the central site using the gauge connection $U(x^{(1)}, x^{(0)})$.

The Wilson fermion matrix on the block lattice thus takes the form :

$$\begin{aligned} \Delta(x'^{(1)}j', x^{(1)}j) = & (M + \Gamma)_{j'j} \delta_{x'^{(1)}, x^{(1)}} - \sum_{i=1}^8 (T_i^+ - \Gamma_i^+)_{j'j} C_{ij'j}(x'^{(1)}, x^{(1)}) \\ & - \sum_{i=1}^8 (T_i^- - \Gamma_i^-)_{j'j} S_{ij'j}(x'^{(1)}, x^{(1)}) . \end{aligned} \quad (2)$$

Here the matrix $(M + \Gamma)$ represents the interactions within the block, while the interactions between the blocks are described by the rest of the terms [2].

Step 3 : Identify the light and the heavy modes within the block. The mass matrix for the 9 fermion modes, $(M + \Gamma)/2$, has the simple structure :

$$M = (2m+9)\mathbf{1} - E, \quad \Gamma^2 = -9\cdot\mathbf{1} + E, \quad E_{j'j} = 1 \quad \forall j', j \in \{-4, \dots, 0, \dots, 4\}. \quad (3)$$

The matrix $(M + \Gamma)$ therefore has one “zero mode” with eigenvalue $2m$ and four pairs of “heavy modes” with complex conjugate eigenvalues $2m + 9 \pm 3i$. The real part of the eigenvalues arises from the mass term and the Wilson coupling, while the Dirac coupling produces the imaginary part. We choose the eigenvector for the zero mode as the block fermion field :

$$\psi^{(1)} = \frac{1}{3} \sum_{j=-4}^4 \chi_j . \quad (4)$$

Step 4 : Integrate out (formally) the heavy modes. The fermion action for the 9 fermion modes can be broken up using the identity :

$$\det(\Delta) = \det \begin{pmatrix} \Delta_{00} & \Delta_{0h} \\ \Delta_{h0} & \Delta_{hh} \end{pmatrix} = \det(\Delta_{hh}) \det(\Delta_{00} - \Delta_{0h} \Delta_{hh}^{-1} \Delta_{h0}) . \quad (5)$$

The first factor on the r.h.s. contributes to the zero-point free energy. The second factor representing the effective interaction on the block lattice contains all sorts of couplings on the block lattice due to the presence of Δ_{hh}^{-1} :

$$\Delta_{hh}^{-1} = ((M + \Gamma)_{hh} + \tilde{\Delta}_{hh})^{-1} = (M + \Gamma)_{hh}^{-1} \sum_{n=0}^{\infty} (-1)^n (\tilde{\Delta}_{hh} (M + \Gamma)_{hh}^{-1})^n . \quad (6)$$

Higher order terms in this expansion contain longer range interactions. However, the heavy mass denominators suppress their coefficients exponentially with the distance.

Step 5 : Find a truncation scheme. The first term (“zeroth order truncation”) in the expansion of the effective fermion action is

$$\text{Tr}[\Delta_{00}] = 2m \delta_{x^{(1)}, x^{(1)}} - \sum_{i=1}^8 r_i c_i^{(1)}(x'^{(1)}, x^{(1)}) + \sum_{i=1}^8 \Gamma_i s_i^{(1)}(x'^{(1)}, x^{(1)}) , \quad (7)$$

$$r_i = \frac{2}{3}, \quad \Gamma_i = \frac{2}{9} \gamma_i^{(1)}, \quad \text{if } i \in \{1, \dots, 4\} ; \quad r_i = \frac{1}{9}, \quad \Gamma_i = \frac{1}{27} \gamma_i^{(1)}, \quad \text{if } i \in \{5, \dots, 8\} . \quad (8)$$

The $\gamma_i^{(n)}$ here are the Dirac matrices with respect to the axes of the block lattices.

This zeroth order effective action contains two types of couplings on the block lattice : nearest neighbours ($i=1,\dots,4$) and 3-space diagonals ($i=5,\dots,8$). The effective gauge fields implicit in the combinations $c_i^{(1)}, s_i^{(1)}$ are just the block links.

Step 6 : Look for a fixed point. The truncated blocking procedure can easily be iterated, yielding

$$\Delta^{(n)} = 2m \cdot 1 - A_1^{(n)} \sum_{i=1}^4 c_i^{(n)} - A_3^{(n)} \sum_{i=5}^8 c_i^{(n)} + A_2^{(n)} \sum_{i=1}^4 \gamma_i^{(n)} s_i^{(n)} + A_4^{(n)} \sum_{i=5}^8 \gamma_i^{(n)} s_i^{(n)}, \quad (9)$$

where the couplings satisfy the recurrence relations :

$$A_1^{(n+1)} = \frac{2}{3} A_1^{(n)} + A_3^{(n)}, \quad A_3^{(n+1)} = \frac{1}{9} A_1^{(n)}, \quad A_2^{(n+1)} = \frac{2}{9} A_2^{(n)} + A_4^{(n)}, \quad A_4^{(n+1)} = \frac{1}{27} A_2^{(n)}. \quad (10)$$

This system of equations is closed, and the recurrence relations for the ratios of couplings have only one attractive fixed point solution :

$$A_1^{(\infty)} / A_3^{(\infty)} = 3(1 + \sqrt{2}), \quad A_2^{(\infty)} / A_4^{(\infty)} = 9. \quad (11)$$

Choosing the relative strength of the Wilson type to Dirac type couplings, by imposing the constraint that the fixed point fermion action be the same as the standard Wilson action in the naive continuum limit, we arrive at the final form of the zeroth order improved action :

$$\begin{aligned} \Delta(x', x) = 2m \delta_{x', x} - \sum_{\mu=1}^4 U(x', x) \left[\frac{1}{\sqrt{2}} c_\mu(x', x) - \frac{3}{4} \gamma_\mu s_\mu(x', x) \right] \\ - \sum_{\mu=1}^4 \tilde{U}(x', x) \left[\frac{\sqrt{2}-1}{3\sqrt{2}} \tilde{c}_\mu(x', x) - \frac{1}{12} \tilde{\gamma}_\mu \tilde{s}_\mu(x', x) \right]. \end{aligned} \quad (12)$$

This action can be systematically improved to desired accuracy by including higher order terms in the expansion (6). The dimensionless expansion parameter that governs the corrections to the zeroth order truncation scheme is the ratio of the inverse lattice spacing to heavy mode masses. The magnitude of this expansion parameter reflects the efficiency of the renormalisation group transformation — the smaller its value the faster the convergence. Since the sum of all the eigenvalues of the effective mass matrix $(M + \Gamma)$ is fixed to be $(2m + 2d)b^d$, we have an optimal situation here; all the heavy mode masses are degenerate in absolute value. This is a consequence of the symmetry of the $\sqrt{3}$ blocking scheme, which results in the appearance of the matrix E in the formulae.

The fixed point action we have arrived at is to be viewed as a renormalisation group improved lattice discretisation of the continuum action, and not as an approximation to the action on the fine lattice. The important question is whether this action involving more complicated couplings is really better (i.e., approaches the continuum limit faster) than the standard Wilson action, and if so in what manner. To answer it we put this action to the test in Monte Carlo calculations.

Fig. 1. Monte Carlo results for the nucleon to rho mass-ratio in the valence quark approximation (a), and including the effect of dynamical quarks (b)

2. Monte Carlo Results

We first used the fixed point fermion action (12) (within the valence approximation) to compute the hadron spectrum [3]. The background gauge fields were obtained by blocking $24^3 \times 48$ lattice configurations at $\beta = 6.3$ twice with $\sqrt{3}$ geometry. The quark propagators on the resulting $8^3 \times 16$ configurations were then evaluated with the improved fermion action and finite size reduction technique. The promising features of the results were :

- The signal to noise ratio is significantly improved compared to computations with the standard Wilson action at similar couplings on similar size lattices.
- Smaller quark masses are reached at comparatively smaller correlation lengths.

This reduces finite size effects and allows us to go closer to the chiral limit.

The results for M_N/M_ρ are shown in Fig. 1a, together with recent high statistics data from APE [4] and expectations of quark model with hyperfine splitting [5].

Subsequently, we investigated hadronic matrix elements of scalar densities to study how the hadron masses shift as the bare quark mass is changed [6]. With exact isospin symmetry, the mass term in the continuum Hamiltonian is

$$H_m = \bar{m} \bar{\Psi} \mathbf{1} \Psi - \frac{m_s - m}{\sqrt{3}} \bar{\Psi} \lambda_8 \Psi, \quad m = m_u = m_d, \quad \bar{m} = \frac{2m + m_s}{3}. \quad (13)$$

In the case of the baryon octet, H_m gives rise to three independent parameters characterising the mass shifts. Two of them are the reduced matrix elements (evaluated in the $SU(3)$ symmetric limit), F and D :

$$\langle B_1 | \bar{\Psi} \lambda_i \Psi | B_2 \rangle = F \text{Tr}(B_1^\dagger [\lambda_i, B_2]) + D \text{Tr}(B_1^\dagger \{\lambda_i, B_2\}). \quad (14)$$

They control the strength of the flavour $SU(3)$ breaking effects within the baryon octet and are related to the matrix elements of scalar densities according to

$$\begin{aligned} F &= (A - C)/2 , \quad D = (A - 2B + C)/2 , \\ A &= \langle p | \bar{u}u | p \rangle , \quad B = \langle p | \bar{d}d | p \rangle , \quad C = \langle p | \bar{s}s | p \rangle . \end{aligned} \quad (15)$$

The third parameter is the nucleon σ -term which measures the effect of changing the mean quark mass. It can be expressed as

$$\sigma = m(A + B) = m(3F - D + 2C) = \sigma_{val} + 2mC . \quad (16)$$

For the computation of the non-singlet quantities F and D , only the insertion of a scalar density on the valence quarks is relevant. σ includes a contribution from the sea quarks as well, but we are unable to estimate it in the valence approximation ($C = 0$). Assuming first order $SU(3)$ flavour symmetry breaking to be valid for the masses of the baryon octet, one estimates

$$D/F \approx -0.32 , \quad \frac{\sigma_{val} M_N}{M_\pi^2} = \frac{3M_N(M_8 - M_N)}{2(M_K^2 - M_\pi^2)} \approx 1.32 , \quad (17)$$

where M_8 is the average mass of the baryon octet.

We compute the physical quantities involving scalar densities using the operator structure $m_q \frac{\partial}{\partial m_q}$ so as to avoid renormalisation Z -factors. The matrix element of a scalar density sandwiched between proton states was extracted from the ratio of 3-point and 2-point correlation functions :

$$R_q(t) = \frac{\sum_{\vec{x}} \langle \mathcal{P}(\vec{x}, t) | \sum_z \bar{q}q(z) | \mathcal{P}(\vec{0}, 0) \rangle}{\sum_{\vec{x}} \langle \mathcal{P}(\vec{x}, t) | \mathcal{P}(\vec{0}, 0) \rangle} \xrightarrow{t \text{ large}} \text{const.} + f \langle p | \bar{q}q | p \rangle t . \quad (18)$$

Here $\mathcal{P}$ denotes the proton operator, and the extra factor in front of the matrix element, $f \equiv \frac{\partial m_q}{\partial(1/\kappa)}$, converts our lattice results to the continuum normalisation.

As shown in Fig. 2a, both F and D increase in absolute magnitude at smaller quark masses, D much more rapidly than F . Qualitatively, such a behaviour is anticipated on the basis of the quark model [5] (m_{eff} is the constituent quark mass):

$$\frac{D}{F} = -\frac{3x}{2+x} , \quad x = \left(\frac{223 \text{ MeV}}{m_{eff}} \right)^3 . \quad (19)$$

Our signals are good enough to bring us down to the $SU(3)$ symmetric point, avoiding the nuisance of extrapolations from large quark masses. Estimates of F and consequently the $\Xi - N$ mass-splitting agree with their experimental values. But both D and D/F turn out to be too large, by roughly a factor of two, compared to the values inferred from the baryon octet masses (17). However, rapid variation of D in the range $m_q < m_s$ implies that the $\Sigma - \Lambda$ mass-splitting cannot be treated as a first order flavour symmetry breaking effect. A realistic estimate of this splitting would therefore necessitate combining quark propagators of unequal masses.

Figure 2b shows our results for σ_{val} plotted in the combination $\sigma_{val} M_N / M_\pi^2$, which is more suitable for extrapolation to the chiral limit. Our data, though consistent with the result in (17), indicates the possibility of a rise at small quark masses. This again signals a breakdown of first order flavour symmetry breaking and suggests that the sea quark contribution required to reach the $\pi - N$ scattering result need not be as large as often claimed.

Fig. 2. (a) The F and D matrix elements of the $\frac{1}{2}^+$ baryon octet vs. the lattice quark mass. (b) The valence quark contribution to the σ -term vs. M_π/M_ρ . The results extracted from experimental data are plotted at the $SU(2)$ symmetric point

3. Effect of Dynamical Fermions on the Hadron Spectrum

Finally, I present an interesting feature of the hadron spectrum when the effect of dynamical quarks is included. These results were obtained using the Hybrid Monte Carlo algorithm with $N_f = 2$ at $\beta = 5.3$ on an 8^4 lattice tripled to $8^3 \times 24$ before calculating the hadron propagators. The preliminary results for M_N/M_ρ are shown in Fig. 1b and the details will be published elsewhere [7]. There are many sources of errors at this stage — insufficient statistics, strong coupling lattice artifacts, finite volume effects, fitting biases, algorithmic uncertainties etc. — which can shift the absolute values of the numbers. However, the relative order of the results, particularly the three points at $M_\pi/M_\rho \approx 0.85$, is less sensitive to such errors and unambiguously shows that lowering the sea quark mass decreases M_N/M_ρ .

I am grateful to the CERN DD-division for their assistance with the computational facilities. It is a pleasure to acknowledge enjoyable collaborations with Stephan Güsken, Rajan Gupta, Greg Kilcup, Karl-Heinz Mütter, Klaus Schilling, Stephen Sharpe and Rainer Sommer.

References

1. K.-H. Mütter and K. Schilling: Nucl.Phys. **B230** [FS10], 275 (1984)
2. S. Güsken, K. Schilling, R. Sommer, R. Gupta, K.-H. Mütter, and A. Patel: CERN preprint, CERN-TH-5094/88 (1988)
3. Ph. de Forcrand, R. Gupta, S. Güsken, K.-H. Mütter, A. Patel, K. Schilling and R. Sommer: Phys.Lett **B200**, 143 (1988).
4. APE collaboration: Rome preprint, ROM2F-88-021 (1988)
5. S. Ono: Phys. Rev. **D17**, 888 (1978)
6. S. Güsken, K. Schilling, R. Sommer, K.-H. Mütter, and A. Patel: CERN preprint, CERN-TH-5027/88 (Revised, 1988)
7. C. Baillie, R. Gupta, G. Guralnik, G. Kilcup, A. Patel and S. Sharpe: In preparation