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N=2 SUPERSTRINGS IN MORE THAN TWO DIMENSIONS?

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Abstract

For a string propagating in a Parisi-Sourlas superspace the critical dimension equals the difference in the number of positive and negative dimensional coordinates. In this way the dimension of the Minkowski subspace can be increased. Here we apply this to the N=2 superstring, with $D_c = 2$ and find anomaly-free N=2 superstrings in all positive even dimensions. Nontrivial theories can be constructed from these N=2 theories by truncation: in a Parisi-Sourlas superspace with a ten-dimensional Minkowski subspace we find the N=1 NSR superstring, and with a four-dimensional Minkowski subspace we find a N=1 superstring, classically related to the D=10 NSR superstring by a canonical transformation.

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In Hamiltonian BRST quantization [1-3] ghost variables are negative dimensional phase space coordinates that cancel unphysical gauge degrees of freedom by the Parisi-Sourlas mechanism [4,5]. For string theories this implies that the string propagates in a Parisi-Sourlas superspace rather than a Minkowski space [6,3].

In this letter we shall consider string theories directly in Parisi-Sourlas superspaces. Besides the ghost variables, there are now additional negative dimensional coordinates that appear in orthosymplectic combinations with the Minkowskian coordinates. As a consequence the critical dimension does not determine the dimensionality of the Minkowski subspace. Rather, it determines the difference in the number of positive and negative dimensional coordinates, which provides a way to construct string theories above the critical dimension.

We are particularly interested in the N=2 superstring [7], a highly symmetric model obtained from the N=1 NSR superstring by distributing an Abelian U(1) color charge along the string. For this model the critical dimension is two, hence there are no transverse directions and only the ground states propagate. However, if instead we formulate this model in a Parisi-Sourlas superspace, transverse degrees of freedom can be excited and nontrivial results obtained by truncations. In particular, we find that in a superspace with ten-dimensional Minkowski subspace the N=2 superstring truncates to the N=1 NSR superstring. With four dimensional Minkowski subspace we find a N=1 superstring, related to the ten-dimensional NSR superstring by a canonical transformation.

A Parisi-Sourlas superspace with coordinates y_i emerges when the D-dimensional Minkowski space with coordinates x_μ is extended by 2M anticommuting coordinates θ_α and Lorentz invariance generalized into $\text{Osp}(D-1,1|2M)$ invariance with inner product

$$y_1 \cdot y_2 = g_{ij} y_1^i y_2^j = \eta_{\mu\nu} x_1^\mu x_2^\nu + \gamma_{\alpha\beta} \theta_1^\alpha \theta_2^\beta ; \quad \gamma_{\alpha\beta} = -\gamma_{\beta\alpha} \quad (1)$$

By considering integrals of $\text{Osp}(D-1,1|2M)$ invariant functions (with $y_1 = y_2$) we identify the coordinates θ_α as negative dimensional [4,5],

$$\int d^D x d^{2M} \theta F(x^2 + \theta^2) \sim \int d^{D-2M} x F(x^2) \quad (2)$$

where for D-2M nonpositive we define the *r.h.s.* by analytic continuation. In particular, (2) implies [4,5] that $\text{Osp}(D-1,1|2M)$ invariant quantum field theories are equivalent to corresponding D-2M dimensional ordinary quantum field theories. Furthermore, if we

expand the superspace scalar propagator

$$G(y_1 - y_2) = \int \frac{d^D k d^{2M} a}{(2\pi)^{D+2M}} e^{ik \cdot (x_1 - x_2) + ia \cdot (\theta_1 - \theta_2)} \frac{-i}{k^2 + a^2 - m^2 + i\epsilon} \quad (3)$$

in the anticommuting coordinates, we get

$$G(y_1 - y_2) = G^0(x_1 - x_2) \delta^{2M}(\theta_1 - \theta_2) + \dots + G^{2M}(x_1 - x_2) \quad (4)$$

where $G^0(x_1 - x_2)$ is the propagator of the D-dimensional Minkowski space field theory, $G^2(x_1 - x_2)$ (with x_1, x_2 restricted to a D-2 dimensional subspace) the propagator of the D-2 dimensional Minkowski space field theory, and so on. Hence the $\text{Osp}(D-1, 1|2M)$ field theory contains information on all ordinary D, D-2, ..., D-2M dimensional field theories.¹

Here we are interested in string propagation in a Parisi-Sourlas superspace. We inquire *e.g.* whether relations such as (4) also exist between string theories with different critical dimensions. We are particularly interested in the N=2 superstring with critical dimension two, and its Parisi-Sourlas extensions to four and ten dimensions.

The matter fields of the N=2 model are a doublet x_μ, y_μ ($\mu = 0, 1$) of $\text{O}(1, 1)$ bosonic coordinates and a doublet ψ_μ of Dirac spinors. The model has an internal $\text{O}(2, 2)$ symmetry in the bosonic and fermionic sectors but only x_μ can be viewed as spacetime coordinates, y_μ should be viewed as internal coordinates.

It is sufficient to consider only the left-moving sector of the theory; the right-moving sector can be obtained essentially by complex conjugation.² For this we introduce complex coordinates $\sqrt{2}z_\mu = x_\mu + iy_\mu$, and conjugate momenta $\sqrt{2}\pi_\mu = \pi_\mu^x - i\pi_\mu^y$. With λ_μ the left moving components of the Dirac spinors ψ_μ , the nontrivial Poisson brackets are

$$\{\pi_\mu(\sigma), z_\nu(\sigma')\} = -\eta_{\mu\nu} \delta(\sigma - \sigma') ; \quad \{\lambda_\mu(\sigma), \lambda_\nu^\dagger(\sigma')\} = -\eta_{\mu\nu} \delta(\sigma - \sigma') \quad (5)$$

and the supercharges are

$$Q = \frac{1}{\sqrt{2}} \lambda \cdot \mathcal{D} \quad \& \quad \bar{Q} = \frac{1}{\sqrt{2}} \lambda^\dagger \cdot \mathcal{D}^* ; \quad \mathcal{D}_\mu = \frac{1}{\sqrt{2}} (\pi_\mu + \partial z_\mu^*) \quad (6)$$

The remaining generators of the (left-moving) N=2 algebra are obtained from

$$\{Q(\sigma), \bar{Q}(\sigma')\} = -\mathcal{H}(\sigma) \delta(\sigma - \sigma') + \frac{1}{2} [N(\sigma) + N(\sigma')] \delta'(\sigma - \sigma') \quad (7)$$

¹This can be extended to interacting theories by introducing proper projections at the vertices.

²In the following we shall mostly discuss only left-moving coordinates, and consequently subscripts L, R for left *resp.* right movers are usually deleted.

$$\mathcal{H} = \frac{1}{2} \mathcal{D} \cdot \mathcal{D}^* - \frac{1}{2} \lambda \overleftrightarrow{\partial} \lambda^\dagger \quad \& \quad N = \lambda \cdot \lambda^\dagger \quad (8)$$

A representation of the N=2 algebra in the Parisi-Sourlas superspace is obtained as follows: consider anticommuting coordinates θ_α and conjugate momenta π_α , and define the graded bracket of two observables A and B by

$$\{A, B\} = -\gamma_{\alpha\beta} \left\{ \frac{\partial^r A}{\partial \pi_\alpha} \frac{\partial^l B}{\partial \theta_\beta} + \frac{\partial^r A}{\partial \theta_\beta} \frac{\partial^l B}{\partial \pi_\alpha} \right\} \quad (9)$$

This bracket satisfies the graded Jacobi identity and transforms covariantly under general graded coordinate transformations. Hence it determines a symplectic structure on the Parisi-Sourlas phase space, and in particular

$$\{\pi_\alpha, \theta_\beta\} = -\gamma_{\alpha\beta} \quad (10)$$

The Parisi-Sourlas representation of the (left-moving) N=2 algebra is constructed by introducing a doublet of anticommuting coordinates $\theta_\alpha^x, \theta_\alpha^y$ with $\alpha = 1, \dots, 2M$, conjugate momenta π_α^x and π_α^y , and commuting complex spinors ϕ_α . Defining complex variables $\sqrt{2}\theta_\alpha = \theta_\alpha^x + i\theta_\alpha^y$ and $\sqrt{2}\pi_\alpha = \pi_\alpha^x - i\pi_\alpha^y$ the nontrivial Poisson brackets are

$$\{\pi_\alpha(\sigma), \theta_\beta(\sigma')\} = \{\phi_\alpha(\sigma), \phi_\beta^*(\sigma')\} = -\gamma_{\alpha\beta} \delta(\sigma - \sigma') \quad (11)$$

and the N=2 supercharges are

$$\Omega = \frac{1}{\sqrt{2}} \phi \cdot \mathcal{R} \quad \& \quad \bar{\Omega} = \frac{1}{\sqrt{2}} \phi^* \cdot \mathcal{R}^* ; \quad \mathcal{R} = \frac{1}{\sqrt{2}} (\pi_\alpha + \partial \theta_\alpha^*) \quad (12)$$

From the corresponding relation (7)

$$\{\Omega(\sigma), \bar{\Omega}(\sigma')\} = -\mathcal{K}(\sigma) \delta(\sigma - \sigma') + \frac{1}{2} [T(\sigma) + T(\sigma')] \delta'(\sigma - \sigma') \quad (13)$$

we get the remaining N=2 generators

$$\mathcal{K} = \frac{1}{2} \mathcal{R} \cdot \mathcal{R}^* - \frac{1}{2} \phi \overleftrightarrow{\partial} \phi^* \quad \& \quad T = \phi \cdot \phi^* \quad (14)$$

The representation of the N=2 algebra in a Parisi-Sourlas superspace is the direct sum of (6), (8), (12), (14),

$$\mathcal{L} = \mathcal{H} + \mathcal{K} \quad \& \quad \mathcal{G} = Q + \Omega \quad \& \quad \bar{\mathcal{G}} = \bar{Q} + \bar{\Omega} \quad \& \quad \mathcal{T} = N + T \quad (15)$$

and we shall now proceed with its BRST quantization.

In addition to the matter fields, the BRST extended phase space includes the following canonical coordinates [1-3]: a pair of commuting variables λ, π where λ can be viewed as a Lagrange multiplier that enforces the reparametrization constraint $\mathcal{L} \sim 0$ in the Dirac approach, and two pairs of anticommuting variables ξ, ρ and $\bar{\xi}, \bar{\rho}$ where ξ *resp.* $\bar{\xi}$ can be viewed as the Lagrange multipliers that enforce the constraints $\mathcal{G} \sim 0, \bar{\mathcal{G}} \sim 0$ of super reparametrizations in the Dirac approach, and finally a pair of commuting variables A, E where A can be viewed as the Lagrange multiplier that enforces the constraint $\mathcal{T} \sim 0$ of U(1) gauge invariance in the Dirac approach. The canonically conjugate ghost variables necessary for unitarity are anticommuting $\eta, \mathcal{P}$ for reparametrization $\mathcal{L}$ and $\bar{\eta}, \bar{\mathcal{P}}$ for gauge fixing this reparametrization. Commuting ghosts z, q and $\bar{z}, \bar{q}$ for the two super reparametrizations $\mathcal{G}, \bar{\mathcal{G}}$ and commuting ghosts r, s and $\bar{r}, \bar{s}$ for gauge fixing these super reparametrizations, and finally an anticommuting pair a, b of ghosts for the U(1) gauge symmetry $\mathcal{T}$ and the corresponding ghosts $\bar{a}, \bar{b}$ for gauge fixing this U(1) symmetry. The N=2 BRST operator is

$$\Omega_{BRST} = Q_{BRST} + Q_{gauge} \quad (16)$$

where

$$\begin{aligned} Q_{BRST} = & \eta \mathcal{L} + z \mathcal{G} + \bar{z} \bar{\mathcal{G}} + a \mathcal{T} + \eta \partial \eta \mathcal{P} + \frac{3}{2} \eta \partial z q + \frac{1}{2} \eta z \partial q + \frac{3}{2} \eta \partial \bar{z} \bar{q} + \frac{1}{2} \eta \bar{z} \partial \bar{q} - z \bar{z} \mathcal{P} \\ & + \frac{1}{2} z \partial \bar{z} b - \frac{1}{2} \bar{z} \partial z b + \eta \partial a b - z a q + \bar{z} a \bar{q} \end{aligned} \quad (17)$$

is the gauge invariant part of the BRST operator and

$$Q_{gauge} = \bar{\eta} \pi + r \rho + \bar{r} \bar{\rho} + \bar{a} E \quad (18)$$

is the gauge fixing part; see [3]. The quantum theory is determined by the phase space path integral

$$\int [d(\text{Liouville})] \exp(i \int \mathcal{A} - \{\Psi, \Omega_{BRST}\}) \quad (19)$$

where the integration measure is the Liouville measure in the extended phase space, and $\mathcal{A}$ is the corresponding symplectic one-form - the $p\dot{q}$'s. Different gauge fermions Ψ correspond to different gauge fixed versions of the theory and from [1-3], (19) is independent of the functional form of Ψ *i.e.* gauge invariant provided the symmetries of the theory are anomaly free.

We shall now compute the critical dimension. For this we select the gauge fermion $\Psi_L = n_L \mathcal{P}_L$ in the left-moving sector, with $n_L(\sigma, \tau)$ an arbitrary function on the world sheet. It can be viewed as the left-moving lapse function corresponding to an arbitrary

world sheet background metric: if in the right moving sector we select $\Psi_R = n_R \mathcal{P}_R$ with $n_R(\sigma, \tau)$ another arbitrary function on the world sheet, we can identify the world sheet background metric as

$$g_{ab} = \begin{pmatrix} n_L n_R & \frac{1}{2}(n_L - n_R) \\ \frac{1}{2}(n_L - n_R) & -1 \end{pmatrix} \quad (20)$$

The n_L , n_R independence of the functional integral ensures conformal invariance and determines the critical dimension. For this we integrate over the canonical momenta π_μ and π_α . Combining the contribution from the left and right moving sectors we find the following matter action

$$\frac{1}{2} \int \sqrt{-g} \{ \eta_{\mu\nu} \partial x^\mu \partial x^\nu + \eta_{\mu\nu} \partial y^\mu \partial y^\nu + \gamma_{\alpha\beta} \partial \theta_\alpha^\alpha \partial \theta_\alpha^\beta + \gamma_{\alpha\beta} \partial \theta_\alpha^\alpha \partial \theta_\alpha^\beta \} + \int |e| \{ \eta_{\mu\nu} \bar{\psi}^\mu i \not{\partial} \psi^\nu + \gamma_{\alpha\beta} \phi^{*\alpha} i \not{\partial} \phi^\beta \} \quad (21)$$

The gravitational anomaly from both x_μ and y_μ is proportional to D . Similarly the gravitational anomaly from the Dirac fermions ψ_μ is proportional to D , while the gravitational anomaly from each of the scalar fermions θ_α^x , θ_α^y is proportional to $-2M$, as is the anomaly from the complex scalar spinor ϕ_α .

The ghost sector is not affected by the Parisi-Sourlas coordinates. Consequently $\eta, \mathcal{P}$ contribute -26 , z, q and $\bar{z}, \bar{q}$ each contribute 11 and finally a, b contribute -2 to the conformal anomaly. Combining the matter and ghost contributions we then conclude that in both left- and right-moving sectors local gravitational anomalies vanish, *i.e.* we have conformal invariance if

$$3D - 6M - 26 + 11 + 11 - 2 = 0 \Rightarrow D = 2 + 2M \quad (22)$$

Similarly, we have verified that (22) ensures independence of the gauge fermion $\Psi = \zeta \rho + \bar{\zeta} \bar{\rho}$, where $\zeta, \bar{\zeta}$ can be identified as components of the background Dirac gravitino. Hence world sheet superconformal anomalies cancel. Finally, selecting a gauge fermion $\Psi = BE$ with B a background $U(1)$ gauge field, (22) ensures that the world sheet $U(1)$ gauge anomalies cancel. All these can be summarized by demanding nilpotency of the quantum Q_{BRST} , which also yields the condition (22). With M an arbitrary positive integer we then have consistent free $N=2$ superstring theories in all even positive dimensions.

The physical states are constructed by first extending the BRST operator (16) into the full set of $Osp(1,1|2)$ operators [3] and then demanding that physical states are $Osp(1,1|2)$ singlets. For this it is convenient to introduce the oscillator representation. For example

in the bosonic open string sector we have for the gauge invariant BRST operator (17)

$$Q_{BRST} = \eta^n \mathcal{L}_n + z^r \mathcal{G}_r + \bar{z}^r \bar{\mathcal{G}}_r + a^n T_n + n\eta^n \eta^m \mathcal{P}_{n+m} + \left(\frac{n}{2} - r\right) \eta^n \bar{z}^r \bar{q}_{n+r} - m\eta^n a^m b_{n+m} \\ + z^r \bar{z}^s \mathcal{P}_{r+s} + a^n z^r q_{n+r} - a^n \bar{z}^r \bar{q}_{n+r} + \frac{1}{2}(r-s) z^r \bar{z}^s b_{r+s}, \quad (23)$$

where, by a local U(1) gauge transformation we have set r, s half-integers. Similarly, the remaining Osp(1,1|2) generators can be constructed [3,8], and we find that in this sector the Osp(1,1|2) singlets *i.e.* physical states $|p\rangle$ satisfy

$$\mathcal{L}_n |p\rangle = T_n |p\rangle = \mathcal{P}_n |p\rangle = b_n |p\rangle = 0 \quad \text{for } n \geq 0 \quad (24.a)$$

$$\eta^n |p\rangle = a^n |p\rangle = 0 \quad \text{for } n < 0 \quad (24.b)$$

$$\mathcal{G}_r |p\rangle = \bar{\mathcal{G}}_r |p\rangle = q_r |p\rangle = \bar{q}_r |p\rangle = 0 \quad \text{for } r > 0 \quad (24.c)$$

$$z^r |p\rangle = \bar{z}^r |p\rangle = 0 \quad \text{for } r < 0 \quad (24.d)$$

in addition to conditions for the pertinent transverse projectors and the gauge fixing ghosts, along the lines of [3,8].

We shall now proceed to verify modular invariance. For this we select the superconformal gauge with $g_{ab} = \delta_{ab}$ in (20), with both background gravitino and U(1) gauge field gauged away. We then introduce the real components

$$\lambda_\mu = \frac{1}{\sqrt{2}}(\mathcal{X}_\mu + i\bar{\mathcal{X}}_\mu) \quad \& \quad \phi_\alpha = \frac{1}{\sqrt{2}}(\varphi_\alpha + i\bar{\varphi}_\alpha) \quad (25)$$

with $\mathcal{X}_\mu$ and $\bar{\mathcal{X}}_\mu$ real anticommuting variables, and $\varphi_\alpha, \bar{\varphi}_\alpha$ real commuting variables. In the ghost sector we define similarly

$$z = \frac{1}{\sqrt{2}}(\beta + \bar{\beta}), \quad q = \frac{1}{\sqrt{2}}(\gamma + \bar{\gamma}), \quad \bar{z} = \frac{1}{\sqrt{2}}(\beta - \bar{\beta}), \quad \bar{q} = \frac{1}{\sqrt{2}}(\gamma - \bar{\gamma}) \quad (26)$$

For consistency, the BRST operator (16) must be single valued on a torus. Demanding that x_μ must also be single-valued then yields the following eight different boundary conditions, independently in both left and right moving sectors and to each directions on the torus (P stands for periodic and A for antiperiodic)

	y_μ	χ_μ	$\bar{\chi}_\mu$	β	$\bar{\beta}$	a	θ_α^x	θ_α^y	φ_α	$\bar{\varphi}_\alpha$
1 :	P	P	P	P	P	P	P	P	P	P
2 :	P	P	P	P	P	P	A	A	A	A
3 :	P	A	A	A	A	P	P	P	A	A
4 :	P	A	A	A	A	P	A	A	P	P
5 :	A	P	A	P	A	A	P	A	P	A
6 :	A	P	A	P	A	A	A	P	A	P
7 :	A	A	P	A	P	A	P	A	A	P
8 :	A	A	P	A	P	A	A	P	P	A

(27)

The one-loop functional integral is a sum of eight different spin structures independently to each direction on a torus. This can be reduced to four different spin structures by demanding that the variables θ_α^x are periodic. This is consistent with multiloop factorization, and ensures that the Parisi-Sourlas supersymmetry between x_μ and θ_α^x remains unbroken. Evaluating the functional integral we observe that for each spin structure, the Fermi and Bose determinants exactly cancel each other. Summing over different spin structures we then find that the one-loop functional integral vanishes. Consequently it is manifestly modular invariant. Note that this implies that the one-loop cosmological constant also vanishes. These results can be directly generalized to multiloop functional integrals, with the conclusions that these functional integrals are modular invariant, and multiloop corrections to the cosmological constant also vanish.

Besides periodic θ_α^x orbits, there are also other projections that are consistent with multiloop modular invariance. In particular, from (21) we conclude that the global $\text{Osp}(D-1,1|2M) \times \text{Osp}(D-1,1|2M)$ can be extended into global $\text{Osp}(2D-2,2|4M)$. Modular invariant functional integrals then emerge also with periodic θ_α^y orbits.

Multiloop modular invariance, and in particular the triviality of various spin structure contributions to the functional integral can be seen as a direct consequence of the Parisi-Sourlas equivalence between the present theories and the standard $D=2$, $N=2$ superstring. In the $D=2$ theory there are no transverse directions, only the ground states propagate. Hence only the ground states contribute to the functional integral which ensures multiloop modular invariance. In the $D=2+2M$ dimensional theories transverse oscillations can be excited but provided the Parisi-Sourlas supersymmetry is unbroken at the multiloop level, (2) implies that only ground states contribute in $D=2+2M$: The other states cancel each other by the Parisi-Sourlas mechanism. This explains triviality of the individual spin structure contributions to the functional integral. In this regard our $D=2+2M$

superstrings are rather trivial.

Nontrivial results can be obtained by projections, which means the following. From (17), (21) the full gauge fixed action has a large global symmetry: the $\text{Osp}(2D-2,2|4M)$ symmetry in the bosonic and fermionic Lagrangians in (21), the $N=2$ supersymmetries in the matter and ghost sectors, and the $\text{Osp}(1,1|2)$ symmetries extending the BRST invariance can all be combined into a $\text{Osp}(2D+2M+2,2|2D+2M+4)$ invariance.³ This global symmetry can be broken by its subgroups G , by identifying states in the Hilbert space $\mathcal{H}$ on which G acts nontrivially. In this way we get theories defined in Hilbert spaces $\mathcal{H}_G$,

$$\mathcal{H}_G \sim \mathcal{H}/G \quad (28)$$

A representative of these equivalence classes can be constructed by truncating variables on which G acts nontrivially. The truncations in which we are interested are similar to those discussed in [1-3], to establish the equivalence between the BRST quantized theory (19) and the corresponding Dirac quantized theory. Essentially, we look for changes of variables of the form $q \rightarrow \epsilon q$, (sometimes also $q \rightarrow \epsilon^{-1} q$) in both Bose and Fermi sectors, carefully selected so that the Liouville measure in (19) remains intact *i.e.* the super Jacobian is trivial. We also require that the limit $\epsilon \rightarrow 0$ of the action is finite. Provided these truncations are consistent with both local and global anomaly structures we then have confidence that these are consistent quantum truncations and yield realizations of quantum theories in the equivalence classes (28) as $\epsilon \rightarrow 0$.

We first consider a ten dimensional truncation, defined by the change of variables

$$\begin{pmatrix} y_\mu \\ \pi_\mu^y \end{pmatrix}, \bar{\chi}_\mu, \begin{pmatrix} \theta_\alpha^{x,y} \\ \pi_\alpha^{x,y} \end{pmatrix}, \begin{pmatrix} \varphi_\alpha \\ \bar{\varphi}_\alpha \end{pmatrix}, \bar{\beta}, a \rightarrow \epsilon \begin{pmatrix} y_\mu \\ \pi_\mu^y \end{pmatrix}, \epsilon \bar{\chi}_\mu, \epsilon \begin{pmatrix} \theta_\alpha^{x,y} \\ \pi_\alpha^{x,y} \end{pmatrix}, \epsilon \begin{pmatrix} \varphi_\alpha \\ \bar{\varphi}_\alpha \end{pmatrix}, \epsilon \bar{\beta}, \epsilon a \quad (29)$$

in both left- and right-moving sectors. The Liouville measure is invariant under (29), and in the $\epsilon \rightarrow 0$ limit (17) reduces to

$$Q_{BRST} \xrightarrow{\epsilon \rightarrow 0} \frac{1}{4} \eta (\pi_x + \partial x)^2 - \frac{1}{2} \eta \chi \partial \chi + \frac{1}{2} \beta \chi (\pi_x + \partial x) + \eta \partial \eta \mathcal{P} + \frac{3}{2} \eta \partial \beta \gamma + \frac{1}{2} \eta \beta \partial \gamma - \frac{1}{2} \beta \beta \mathcal{P} \quad (30)$$

This we identify as the BRST operator of the $N=1$ supersymmetry algebra [10]. Similarly, the gauge fixing part (18) of the BRST operator can be truncated into that of the $N=1$ theory. Furthermore, restricting to the (modular invariant) periodic y_μ , periodic θ_α^x sector in (27), we find the spin structures of the $N=1$ NSR superstring. In this manner the quantum NSR superstring can be viewed as part of our $N=2$ Parisi-Sourlas superstrings.

³Here it is sufficient to consider only these symmetries, even though the classical action has further symmetries as discussed in [9].

Of particular interest are truncations with x_μ coordinates of a four-dimensional Minkowski space. As an example we consider here

$$\begin{pmatrix} y_\mu \\ \pi_\mu^y \end{pmatrix}, \bar{\mathcal{X}}_\mu, \begin{pmatrix} \theta_\alpha^{x,y} \\ \pi_\alpha^{x,y} \end{pmatrix}, \begin{pmatrix} \varphi_\alpha \\ \bar{\varphi}_\alpha \end{pmatrix} \rightarrow \epsilon \begin{pmatrix} y_\mu \\ \pi_\mu^y \end{pmatrix}, \epsilon \bar{\mathcal{X}}_\mu, \epsilon \begin{pmatrix} \theta_\alpha^{x,y} \\ \pi_\alpha^{x,y} \end{pmatrix}, \epsilon \begin{pmatrix} \varphi_\alpha \\ \bar{\varphi}_\alpha \end{pmatrix} \quad (31)$$

with a trivial super Jacobian in the Liouville measure in (19). If in addition we perform the canonical transformation

$$\bar{\beta} \rightarrow \epsilon \bar{\beta}, \quad \bar{\gamma} \rightarrow \frac{1}{\epsilon} \bar{\gamma}; \quad a \rightarrow \epsilon a, \quad b \rightarrow \frac{1}{\epsilon} b \quad (32)$$

the $\epsilon \rightarrow 0$ limit of the action is finite, and in this limit the gauge invariant BRST operator (17) becomes

$$\begin{aligned} Q_{BRST} &\xrightarrow{\epsilon \rightarrow 0} \eta \left\{ \frac{1}{4} (\pi_x + \partial x)^2 - \frac{1}{2} \mathcal{X} \partial \mathcal{X} + \frac{3}{2} \partial \bar{\beta} \bar{\gamma} + \frac{1}{2} \bar{\beta} \partial \bar{\gamma} + \partial ab \right\} \\ &+ \beta \left\{ \frac{1}{2} \mathcal{X} (\pi_x + \partial x) - \frac{1}{2} \partial \bar{\beta} b - \frac{1}{4} \bar{\beta} \partial b - \frac{1}{2} \bar{\gamma} a \right\} + \eta \partial \eta \mathcal{P} + \frac{3}{2} \eta \partial \beta \gamma + \frac{1}{2} \eta \beta \partial \gamma - \frac{1}{2} \beta \beta \mathcal{P} \end{aligned} \quad (33)$$

which we identify as the gauge invariant BRST operator of the N=1 superconformal algebra corresponding to the representation

$$\mathcal{H} = \frac{1}{4} (\pi_x + \partial x)^2 - \frac{1}{2} \mathcal{X} \partial \mathcal{X} + \frac{3}{2} \partial \bar{\beta} \bar{\gamma} + \frac{1}{2} \bar{\beta} \partial \bar{\gamma} + \partial ab \quad (34.a)$$

$$\mathcal{S} = \frac{1}{2} \mathcal{X} (\pi_x + \partial x) - \frac{1}{2} \partial \bar{\beta} b - \frac{1}{4} \bar{\beta} \partial b - \frac{1}{2} \bar{\gamma} a \quad (34.b)$$

One can verify that (33), (34) is indeed nilpotent in D=4. Provided the theory is also modular invariant we can view (33), (34) as a version of the D=10 NSR superstring, compactified to four dimensions by substituting for the internal degrees of freedom a N=1 superconformal QFT constructed from the ghost fields $\bar{\beta}, \bar{\gamma}, a, b$.

The modular invariance of the quantum theory (33), (34) follows when we restrict to the periodic y_μ, θ_α^x orbits: for each spin structure the $\eta, \mathcal{P}$ and a, b determinants cancel those of x_μ , and the determinants of β, γ and $\bar{\beta}, \bar{\gamma}$ cancel those of $\mathcal{X}_\mu$ (in both left- and right-moving sectors). Summing over different spin structures then yields zero.

We recall that in a given spin structure, the individual contributions can be related to the degeneracy of the energy levels. Consequently it appears that in (33), (34) the ground states are the only physical degrees of freedom. The situation is then quite peculiar. Even though the string is four dimensional there are no transverse degrees of freedom. But we also observe that $\bar{\beta}, \bar{\gamma}$ and a, b do not determine unitary representations of the Virasoro

algebra. Consequently it is not clear whether the ground states really correspond to a four-dimensional unitary supersymmetric field theory. However, these considerations are incomplete in the sense that we have also overlooked the fact that (33), (34) admits further conditions from the original N=2 theory: the physical states of (33), (34) must correspond to truncations of physical states in the original N=2 theory. From (24) we then conclude that the ghost oscillators $\bar{\beta}, \bar{\gamma}$ and a, b should not be excited in the unitary subspace.

This result can also be seen as follows. From [10], the *full* (*i.e.* the sum of gauge invariant and gauge fixed) N=1 BRST operator has the form (in the left-moving sector)

$$\Omega_{BRST} = \eta\mathcal{H} + z\mathcal{S} + \eta\partial\eta\mathcal{P} + \frac{3}{2}\eta\partial zq + \frac{1}{2}\eta z\partial q - \frac{1}{2}zz\mathcal{P} + \bar{\eta}\pi + r\rho \quad (35)$$

with $\mathcal{H}, \mathcal{S}$ a representation of the N=1 algebra. If in (35) we implement a unitary canonical transformation generated by $\Phi = \eta\partial\xi s$,

$$\Omega_{BRST} \rightarrow e^{-\Phi}\Omega_{BRST}e^{\Phi} \quad (36)$$

followed by the nonunitary canonical transformation (see [3])

$$sdr \rightarrow \frac{3}{2}sdr + \frac{1}{2}rds \quad (37)$$

with the identifications $r = \bar{\beta}, s = \bar{\gamma}, \rho = b, \xi = a$ we then conclude that (33) can be viewed as a canonical transformation of the BRST operator (35), *i.e.* as the BRST operator of the N=1 super Virasoro algebra with $\bar{\beta}, \bar{\gamma}, a, b$ identified as ghosts of gauge fixing. Implementing the full $\text{Osp}(1,1|2)$ invariance in addition to BRST invariance then implies that the $\bar{\beta}, \dots, b$ oscillators indeed cannot be excited. Hence these oscillators do not correspond to physical degrees of freedom, leaving the transverse oscillations of $x_\mu, \mathcal{X}_\mu$ as the only true physical degrees of freedom. The truncated theory (33), (34) is thus classically canonically equivalent to the NSR superstring (30), and the shift in the critical dimensionality can be understood by observing that the generator of the canonical transformation in (37) becomes a singular operator product in the quantum theory; see [3] for a discussion on similar phenomena in bosonic strings.

Finally, we wish to draw attention to a parallel between the present work and the discussion in [11]. Here we have established that the two dimensional N=2 superstring can be extended into any even positive dimension by adding Parisi-Sourlas coordinates in orthosymplectic (*i.e.* essentially the BRST in [11]) combinations with other variables. Before truncations the spacetime admits a natural complex structure, with the modular

properties of the functional integral revealing that only ground states can propagate. This parallels closely the discussions in [11].

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