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Correlation functions for Minimal Models on the torus

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ABSTRACT

A Coulomb gas approach to the construction of correlation functions for minimal models on the torus is presented. In particular the conformal blocks for the two point function $\langle \phi_{12} \phi_{12} \rangle$ are discussed in detail and their monodromy properties and modular transformations are derived. We also obtain the modular and monodromy invariant combination of the left and right blocks of the correlation function.

1. INTRODUCTION

Recently, considerable progress has been made in the study of conformal field theories on Riemann surfaces of genus $g \geq 1$. Beginning with the work of Verlinde [1] on the relation between modular transformations and fusion rules a number of papers have studied the properties of conformal field theories on the torus. In particular, correlation functions on the torus can be used to express the Verlinde conjecture in a different form [2]; the differential equations for the correlation functions can be written down while the Wronskian of the solutions can be used to classify allowed highest weights for conformal field theories [3]; the differential equations for characters can also be used for an approach to the classification of rational conformal field theories [4]; the properties of the modular transformations of characters have been studied [5] and finally also polynomial equations provided by the monodromy properties of correlation functions have been investigated [6]. In all these studies, the fundamental role of correlation functions of CFT on the plane combined with the property of factorization of CFT on higher genus surfaces (previously emphasized [7]) has become clear.

However, explicit constructions of these correlation functions on $g \geq 1$ surfaces so far exist only for a very few theories which are basically free (or related to free) theories like the Ising model or the $k = 2$ $SU(2)$ WZW model [8]. The explicit construction of these correlation functions is important not only from the point of view of conformal field theory but also in the study of the perturbation theory of string models a la Gepner. The Gepner models [9], which describe string propagation in Calabi-Yau spaces, are built using non-trivial conformal field theories. The study of string perturbation theory and the study of questions like finiteness etc. require the explicit constructions of the correlation functions of non-trivial CFT on higher genus surfaces.

In this paper we develop a Coulomb gas formalism for the construction of correlation functions for minimal conformal field theories on the torus, in the spirit of the work of Dotsenko and Fateev [10] for the correlation functions on the plane. While we concentrate on the study of two-point functions, the method may be easily generalized to four-point

and higher correlation functions. Further, while we restrict our detailed attentions to the models of the unitary series of FQS [11], it is quite simple to extend these considerations to all the $c < 1$ theories.

We begin the paper with a brief review in sec. 2 of the Coulomb gas formalism on the plane and the nature of its extension to the torus. In sec. 3 we turn to the two-point functions of the fields of the type $\phi_{n,1}$ and $\phi_{1,m}$. Most of the details are illustrated for $\phi_{2,1}$ and $\phi_{1,2}$ fields, and the general extension is discussed. We also describe the screening contours, an important point in the construction. In sec. 4, we run through various checks to ensure that the two-point functions have indeed the right properties; viz. i) that there are no null states propagating round the torus and ii) that in the correct integral representations, as the two operators of the correlation function come close together we recover either the character or the one-point functions on the torus. In sec. 5 we turn to the monodromy properties of the correlation functions and show how to compute the monodromy around the a and b cycles on the torus. In sec. 6 we study the modular transformation properties of the conformal blocks and show how to compute the modular transformation matrix. We also demonstrate that the modular transformation matrix transforms the b -cycle monodromy to the a -cycle monodromy. With the right basis, it is also clear how the Verlinde conjecture is satisfied. In sec. 7 we show how to construct the complete modular and monodromy invariant correlation functions by taking appropriate combinations of left and right sectors. Details of two calculations are relegated to the appendices.

2. THE COULOMB GAS REPRESENTATION

The essential idea of the Coulomb gas representation, introduced in [10], is to represent all the primary fields (or their secondaries) as vertex operators of a bosonic scalar. One also introduces a fixed background charge $-2\alpha_0$ at infinity on the complex plane. This amounts to introducing a charge at the pole which the metric $dzd\bar{z}$ has at infinity. If the

vertex operator is $e^{i\alpha\phi}$, the conformal weight of the operator is given by

$$\Delta_\alpha = \alpha^2 - 2\alpha_0\alpha = \Delta_{2\alpha_0-\alpha} . \quad (2.1)$$

The central charge c (of the Virasoro algebra) is $1 - 24\alpha_0^2$. If we now wish to compute a correlation function of some specified vertex operators, the correlation function will vanish unless the overall charge in the correlator is zero (in a specific theory, which is given by a definite choice of α_0). This is achieved by introducing integrals of vertex operators, V_{α_+} and V_{α_-} , where

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} . \quad (2.2)$$

Thus we get the final correlation function as an integral of a correlation function involving the vertex operators representing the primary fields and the “screening operators” V_{α_+} and V_{α_-} . The choice of the contour of integration has to be specified and typically for the four-point function there is a basis of two contours e.g. one enclosing the points 0 and z and the other enclosing 1 and ∞ (note that three vertex operators can always be brought to 0, 1, ∞ by Mobius invariance). One may choose a different basis, one enclosing the points 1 and z and the other 0 and ∞ . The first basis is a s-channel representation and the second is a t-channel representation of the correlation function.

One may attempt to extend this to the case of the torus. The first point to note is that the metric $dzd\bar{z}$ on the torus has no poles or zeroes. Thus there is no “charge at infinity” on the torus. The second point is that in computing correlators of vertex operators one has to account also for the classical part of the bosonic action, which contributes through the non-zero periods of the holomorphic one-form $\partial_z\phi$ on the torus.

One may, for instance, attempt to compute the partition functions of the minimal models. There are no vertex operator insertions and the only contributions are those arising from the classical part of the bosonic action. However it has been noted that in order to reproduce the correct expression for the characters [12] we need to consider a bosonic field compactified on two radii $R_1 = \sqrt{2}\alpha_0$ and $R_2 = \alpha_-/\sqrt{2}$, and subtract the second contribution from the first. The final expression for the conformal block of characters (after a somewhat delicate computation to separate holomorphic and anti-holomorphic parts of

the partition function) reduces to

$$\frac{1}{\eta(\tau)} \left(\tilde{\Gamma}(r, s) - \tilde{\Gamma}(-r, s) \right) \quad (2.3)$$

where

$$\tilde{\Gamma}(r, s) = \sum_{n=-\infty}^{\infty} \exp \left(\pi i \tau \frac{(r(p+1) - sp + 2np(p+1))^2}{2p(p+1)} \right). \quad (2.4)$$

The values of r and s are in the range $1 \leq r \leq p-1$ and $1 \leq s \leq r$. The lattice terms together serve the purpose of properly subtracting out null states as may be seen by an expansion in $q = e^{2\pi i \tau}$ and examining the coefficients of powers of q .

The primary fields of the theory are labelled by the integers r and s in the above and are represented as vertex operators $\phi_{rs} = V_{\beta} = V_{2\alpha_0 - \beta}$ where

$$\beta = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_- \quad (2.5)$$

which carries a conformal weight

$$\Delta_{rs} = \frac{(r(p+1) - sp)^2 - 1}{4p(p+1)}. \quad (2.6)$$

We would now like to extend this approach to the case of correlation functions.

3. CORRELATION FUNCTIONS (TWO-POINT FUNCTIONS)

We will begin for the sake of simplicity, and to illustrate the details, with the case of primary fields of the type $\phi_{1,n}$; in fact for $\phi_{1,2}$ (which for the case of the Ising model corresponds to the σ field). Consider the correlation function $\langle V_{\alpha} V_{\alpha} \rangle$ where $V_{\alpha} \equiv \phi_{1,2}$ and $\alpha = -1/2\alpha_-$. The correlator will be non-zero only if further screening operators are inserted. The number of screening operators for this case is just one and it is V_{α_-} . Thus the non-zero correlator is

$$\oint_{C_i} dz \langle V_{\alpha}(z_1) V_{\alpha}(z_2) V_{\alpha_-}(z) \rangle \quad (3.1)$$

where C_i is a generic expression for a basis of contours to be specified later.

Let us first consider the evaluation of the integrand. We proceed in a manner similar to the case of the partition function, compactifying the boson on two radii of compactification $R_1 = \sqrt{2}\alpha_0$ and $R_2 = \alpha_-/\sqrt{2}$ and subtracting the second expression from the first. We calculate first the contribution of the quantum part of the bosonic action and it gives the expected contributions involving $\theta_1(z)$. The second part is from the classical solutions and their winding on the homology cycles. We obtain the expression for the integrand where again we have to separate the holomorphic and anti-holomorphic parts.

We can finally extract the expression for the conformal block $G(r, s)$ where (r, s) labels the primary field of weight $\Delta_{(r,s)}$ in whose representation the trace is evaluated:

$$G(r, s) = \frac{1}{\eta} \oint_{C_i} dz \left(\frac{\theta_1(z_1 - z_2)}{\theta'_1(0)} \right)^{2\alpha^2} \left(\frac{\theta_1(z_1 - z)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \left(\frac{\theta_1(z - z_2)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \times (\Gamma(r, s) - \Gamma(-r, s)) \quad (3.2)$$

where

$$\Gamma(r, s) = \sum_{n=-\infty}^{\infty} \exp \left(\pi i \tau \frac{(r(p+1) - sp + 2np(p+1))^2}{2p(p+1)} \right) \times \exp \left(\frac{2\pi i}{\sqrt{2p(p+1)}} (r(p+1) - sp + 2np(p+1))(\alpha z_1 + \alpha z_2 + \alpha_- z) \right) \quad (3.3)$$

The range of r and s are $0 \leq r \leq p-1$ and $0 \leq s \leq r$.

However in analogy with the complex plane, we still have to define the screening contours. The correct intuition is provided by considering the limit $\tau \rightarrow i\infty$. We can evaluate the coefficient of the leading term in q in this limit in the integrand of $G(r, s)$. Torus co-ordinates z are related to the co-ordinates on the plane via $z = \ln x/2\pi i$. In this limit we obtain,

$$\begin{aligned} \theta_1(z_i - z_j) &\longrightarrow \frac{(x_i - x_j)}{\sqrt{(x_i x_j)}} q^{1/8} \\ \Gamma(r, s) &\longrightarrow \left(\frac{x}{\sqrt{(x_1 x_2)}} \right)^{(r\alpha_+ + s\alpha_-)\alpha_-} q^{\frac{1}{4}(r\alpha_+ + s\alpha_-)^2} \end{aligned}$$

Combining all this and denoting by $I(r, s)$ the integrand, we get

$$\int dz I(r, s) \longrightarrow q^{\Delta_{r,s} - c/24} (x_1 - x_2)^{2\alpha^2} x_1^{-\frac{\alpha}{2}} x_2^{-\frac{\alpha}{2}} \left[\frac{\partial z_1}{\partial x_1} \frac{\partial z_2}{\partial x_2} \right]^{-\Delta_{12}}$$

$$\times \int dx x^a (x - x_1)^b (x - x_2)^c \quad (3.4)$$

where

$$\begin{aligned} a &= ((1+r)\alpha_+ + (1+s)\alpha_-)\alpha_- \\ b &= c = -\alpha_-^2 \end{aligned} \quad (3.5)$$

a , b , and c are the branches at 0 , x_1 and x_2 respectively and the branch at ∞ is given by $(a+b+c)$,

$$-2-(a+b+c) = ((1-r)\alpha_+ + (1-s)\alpha_-)\alpha_- . \quad (3.6)$$

The factors $(\partial z_i / \partial x_i)^{\Delta_{12}}$ are just the conformal transformation factors from $\phi_{12}(x) \rightarrow (\partial z / \partial x)^{\Delta_{12}} \phi_{12}(z)$. We see therefore that the integrand above is just the integrand of the four-point function on the plane.

$$\begin{aligned} \langle \phi_{rs}(0) \phi_{12}(x_1) \phi_{12}(x_2) \phi_{rs}(\infty) \rangle &= \langle V_{2\alpha_0-\beta}(0) V_\alpha(x_1) V_\alpha(x_2) V_\beta(\infty) \rangle \\ &= \int dx \langle V_{2\alpha_0-\beta}(0) V_\alpha(x_1) V_\alpha(x_2) V_{\alpha_-}(x) V_\beta(\infty) \rangle \end{aligned} \quad (3.7)$$

where

$$\beta = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_- .$$

Henceforth we shall set $Im z_1 > Im z_2$ so that upon degeneration $|x_1| < |x_2|$.

In order to reproduce the result expected from factorization, one would expect the screening contour on the torus to degenerate to the screening contour on the plane. If we represent the torus as an annulus the two closed contours corresponding to the two contours on the plane $\int_0^{z_1}$ and $\int_1^\infty$ are represented by the curves C_1 and C_2 respectively (see Fig. 1). It is understood that the contour C_1 winds around the point z_1 and the inner radius of the annulus in a manner so as to give a closed curve on a branched cover of the torus. A similar situation holds for the contour C_2 around z_2 and the outer radius. From the integrand, where the τ is scaled effectively by a factor $2p(p+1)$, it is clear that these two radii are not identified and hence C_1 and C_2 are indeed distinct contours. We can also define $\tilde{C}_1$ and $\tilde{C}_2$, the analog of the other basis of contours on the plane (corresponding to the t-channel) $\int_{z_1}^{z_2}$ and $\int_0^\infty$ (see Fig. 2). Here also the contour winds suitably around the inner and outer radii or z_1 and z_2 as the case may be.

Anticipating elaboration in subsequent sections, we point out the simple physical interpretation of these bases. The first basis corresponds to the representation of the correlation function in a basis which is diagonal under the monodromy around the a -cycle (as shown in Fig. 3 for the $\phi_{1,2}$ field). The second basis corresponds to the one described by Verlinde which is more natural for the study of the short distance behaviour, when $z_1 \rightarrow z_2$ (see Fig. 4).

We note that the number of conformal blocks $G(r, s)$ is greater than the number expected from the fusion rules. The $G(r = 0, s)$ and $G(r, s = 0)$ blocks must vanish. This is because only the range $1 \leq r \leq p - 1$ and $1 \leq s \leq r$ sweeps over the space of characters, counting every character once. The first vanishes trivially. The $(r, s = 0)$ contribution does not do so but vanishes on integration. This is easily seen by examining the leading term in q (for $\tau \rightarrow i\infty$) which gives the difference of two integrals of the hypergeometric type that are in fact equal to each other.

More generally one can prove this as follows. The crucial point here is that for $s = 0$ the branches around the origin and x_1 are equal and opposite ($a = -b \bmod \text{integers}$) and similarly the branch around ∞ and x_2 ($a + b + c = -c$). It is clear that C_1 can be continuously deformed to C_2 . The second point is that $\Gamma(r, 0)$ and $\Gamma(-r, 0)$ are identical provided we change $(z_1 + z_2 - 2z) \rightarrow -(z_1 + z_2 - 2z)$ in the latter. This can be accomplished by performing a change in the dummy variable $z \rightarrow z_1 + z_2 - u$. With this transformation the contour $\int_{C_1} dz \rightarrow \int_{C_2} du$. The latter can be brought back to $\int_{C_1} du$ by deforming continuously as noted above. This means that $\Gamma(r, 0)$ and $\Gamma(-r, 0)$ contribute equally and hence they cancel.

Though we could have, of course, by definition taken the range $1 \leq r \leq p$ and $1 \leq s \leq r$, the $(r, s = 0)$ terms in $G(r, s)$ reappear, as will be shown later, under modular transformations. Hence it is important to ensure that they actually vanish on integration.

We now consider the $(r, s = 1)$ blocks. Here only the integral of the $\oint_{C_1}$ type contributes. The other one vanishes, because if we examine the coefficients of q as $\tau \rightarrow i\infty$ in the integrand they have no branch point or single-order pole at infinity. Hence as $\oint_{C_2} \rightarrow \int_0^\infty -\exp(2\pi i n) \int_\infty^0$ the contribution vanishes completely. Therefore in $G(r, s = 1)$

there is no $\oint_{C_2}$ contribution.

Thus the total number of conformal blocks are $(2p(p-1)/2) - (p-1) = (p-1)^2$ which is indeed the right number expected from the fusion rules of the minimal conformal models.

The two-point function for the $\phi_{2,1}$ (which is the ϵ field in the Ising model) can be done by similar methods. Proceeding in a manner identical to the one used earlier we can extract the conformal blocks, the generic $G(r, s)$ being given by

$$G(r, s) = \frac{1}{\eta} \oint_{C_i} dz \left(\frac{\theta_1(z_1 - z_2)}{\theta'_1(0)} \right)^{2\alpha^2} \left(\frac{\theta_1(z_1 - z)}{\theta'_1(0)} \right)^{2\alpha\alpha_+} \left(\frac{\theta_1(z - z_2)}{\theta'_1(0)} \right)^{2\alpha\alpha_+} \times (\Gamma(r, s) - \Gamma(r, -s)) \quad (3.8)$$

where

$$\Gamma(r, s) = \sum_{n=-\infty}^{\infty} \exp \left(\pi i \tau \frac{(r(p+1) - sp + 2np(p+1))^2}{2p(p+1)} \right) \times \exp \left(\frac{2\pi i}{\sqrt{2p(p+1)}} (r(p+1) - sp + 2np(p+1))(\alpha z_1 + \alpha z_2 + \alpha_+ z) \right) \quad (3.9)$$

the range being given by $0 \leq r \leq s-1$ and $0 \leq s \leq p$. Note that here $\alpha = -\alpha_+/2$. The contours are identical to the ones used before. Considerations similar to those for the $\phi_{1,2}$ field allow us to demonstrate that the $(r=0, s)$ and $(r, s=0)$ terms do not contribute. Moreover, only one integral $\oint_{C_1}$ out of the two contours contributes to the case of $(r=1, s)$ blocks. The total number of conformal blocks in the correlation function is different; there are $(p(p-1)) - p = p(p-2)$ blocks, the number expected from the fusion rules.

We note that the generalization to the case of $\phi_{1,n}$ and $\phi_{n,1}$ is straightforward. The combinations of contours follow the pattern of the combinations on the plane when there are only screening operators of one type, as discussed in [10].

4. CHARACTERS, ONE-POINT FUNCTIONS AND NULL STATES

We now have to ensure that these expressions have indeed some of the properties that we expect from two-point functions on the torus. The first check is to ensure that

as we proceed to the limit $z_1 \rightarrow z_2$ the residue over the singularity is the character or a conformal block of a one-point function. To see this let us look, not at $G(r, s)$ but $\tilde{G}(r, s)$ which are the conformal blocks with the contour integrals $\oint_{\tilde{C}_1}$ and $\oint_{\tilde{C}_2}$ with the integrand itself unchanged. Now it is clear that all the characters come from the $\oint_{\tilde{C}_1}$ integral as this winds around z_1 and z_2 , and as the two points approach each other the contour is vanishing. Let us examine this limit in more detail.

Consider $\tilde{G}(r, s)$ for the $\phi_{1,2}$ field

$$\tilde{G}(r, s) = \frac{1}{\eta} \oint_{\tilde{C}_1} dz \left(\frac{\theta_1(z_1 - z_2)}{\theta'_1(0)} \right)^{2\alpha^2} \left(\frac{\theta_1(z_1 - z)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \left(\frac{\theta_1(z - z_2)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \times (\Gamma(r, s) - \Gamma(-r, s)) \quad (4.1)$$

where again $\alpha = -\alpha_-/2$.

Set $z_1 - z_2 = \epsilon$, $z - z_2 = \epsilon u$ and $z_1 - z = \epsilon(1 - u)$. If we now substitute this in the integral we get

$$\frac{1}{N} \int_0^1 du \epsilon \theta_1(\epsilon)^{2\alpha^2} \theta_1(\epsilon(1 - u))^{-4\alpha^2} \theta_1(\epsilon u)^{-4\alpha^2} (\Gamma'(r, s) - \Gamma'(-r, s)) \quad (4.2)$$

where $\Gamma'(r, s)$ is the same as $\Gamma(r, s)$ except that in the summation $(z_1 + z_2 + -2z)$ is replaced by $\epsilon(1 - 2u)$. N is the normalisation consisting of the θ'_1 and η factors. On taking the limit $\epsilon \rightarrow 0$ (4.2) reduces to

$$\frac{1}{\eta} \int_0^1 du \epsilon (\epsilon)^{2\alpha^2} (\epsilon(1 - u))^{-4\alpha^2} (\epsilon u)^{-4\alpha^2} (\tilde{\Gamma}(r, s) - \tilde{\Gamma}(-r, s)) \quad (4.3)$$

where $\tilde{\Gamma}(r, s)$ is the same as in (2.4). The overall power in ϵ is given by $\epsilon^{1-6\alpha^2}$ which is precisely $(z_1 - z_2)^{-2\Delta_{12}}$ (where Δ_{12} is the weight of the $\phi_{1,2}$ field) as expected. We see that apart from the (r, s) and τ independent factor $\int_0^1 du (1 - u)^{-4\alpha^2} u^{-4\alpha^2}$ we indeed recover the character as written in (2.3).

We now look at the $\oint_{\tilde{C}_2}$ type blocks. These appear for (r, s) with $s > 1$ as may be quickly ascertained by a change of basis of contours of the corresponding four-point functions on the plane. For those sectors where it does appear, we get the conformal block

$$\tilde{G}(r, s) = \frac{1}{\eta} \oint_{\tilde{C}_2} dz \left(\frac{\theta_1(z_1 - z_2)}{\theta'_1(0)} \right)^{2\alpha^2} \left(\frac{\theta_1(z_1 - z)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \left(\frac{\theta_1(z - z_2)}{\theta'_1(0)} \right)^{2\alpha\alpha_-}$$

$$\times (\Gamma(r, s) - \Gamma(-r, s)) \quad (4.4)$$

where again $\alpha = -\alpha_-/2$. Now change variable $z_1 - z_2 = \epsilon$ in (4.4). Putting also $z - z_2 = u$ (due to translation invariance on the torus) we get in the limit $\epsilon \rightarrow 0$

$$\frac{1}{\eta} \oint_{\tilde{C}_2} du \epsilon^{2\alpha^2} \left(\frac{\theta_1(-u)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \left(\frac{\theta_1(u)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} (\Gamma''(r, s) - \Gamma''(-r, s)) \quad (4.5)$$

where $\Gamma''(r, s)$ is the same expression as $\Gamma(r, s)$ except that the $(z_1 + z_2 - 2z)$ argument is replaced by $-2u$. But this is precisely a one-point function block. The fusion rule is

$$\phi_{1,2} \times \phi_{1,2} = \phi_{1,1} + \phi_{1,3} \quad .$$

$\phi_{1,1}$ is the identity and we have already produced the characters. If we represent $\phi_{1,3}$ by $V_{\alpha'}$ where $\alpha' = -\alpha_- = 2\alpha$, the one-point function for $\phi_{1,3}$ is given by

$$\langle \phi_{1,3} \rangle = \oint_{\tilde{C}_i} dz \langle \phi_{1,3}(z_1) V_{\alpha_-}(z) \rangle = \oint_{\tilde{C}_i} du \langle \phi_{1,3}(0) V_{\alpha_-}(u) \rangle \quad (4.6)$$

(where we have used the translation invariance of the torus). The expression (4.5) is precisely the result of the computation of eq.(4.6). The factor $\epsilon^{2\alpha^2}$ can be easily recognized as $(z_1 - z_2)^{-2\Delta_{12} + \Delta_{13}}$. These considerations can be similarly extended to the case of the $\phi_{2,1}$ field and the other generalizations.

We now turn to the question of null states. One should ensure that there are in fact no null states propagating around the torus. If we examine $G(r, s)$, where there is a trace in the representation labelled by (r, s) , and expand in q , then there is a term that differs from the leading one by q^Δ that must vanish, where Δ would correspond to the level of the first null state over the primary. In general the lowest terms in q come from $\Gamma(r, s)$ and $\theta_1(z)$ terms and the appropriate q^Δ term from these cancels with the first term from $\Gamma(-r, s)$ and the leading contribution of the $\theta_1(z)$ term. We note first that the correct normalization is important to ensure that this indeed happens. Secondly, and more important, this cancellation does not take place trivially but does so after the integration and involves specific properties of hypergeometric functions (a steady hand with the signs is recommended). A complete check to all orders of the absence of null states is considerably harder.

A few cases have been checked and a simple explicit example for the case $(r = 1, s = 1)$ is sketched in the appendix. We emphasize that despite the brevity of the discussion here, this check is quite important. We would like to point out that we could have obtained the first few four-point functions on the plane, in agreement with those of [10], and still basically have the wrong result.

5. MONODROMY PROPERTIES

In this section we will derive the monodromy $\phi(a)$ and $\phi(b)$ of the correlation functions, when z_1 (say) is moved around a and b cycles of the torus respectively. Before proceeding further we clarify some notation. Henceforth

$$G_i(r, s) = \oint_{C_i} dz \text{Integrand}(r, s)$$

with $i = 1, 2$.

(i) $\phi(a)$:

In the s-channel $\phi^s(a)$ is diagonal. When the contour goes around z_2 , we can easily obtain the phase by transforming $z_1 \rightarrow z_1 + 1$ in the integrand. In order to get the phases coming from the θ -functions, one can go to the degeneration limit $\tau \rightarrow i\infty$, $z \rightarrow \ln x/2\pi i$. z_1 going to $z_1 + 1$ implies that x_1 goes around the origin counterclockwise. The result is

$$G_2(r, s) \longrightarrow e^{2\pi i(-\Delta_{r,s} + \Delta_{r,s-1})} G_2(r, s) . \quad (5.1)$$

Similarly for the contour around z_1 , the contour is also pulled to the second sheet as $z_1 \rightarrow z_1 + 1$. Thus we must also transform $z \rightarrow z + 1$ (this prescription can be seen explicitly by going to the degeneration limit). The result is

$$G_1(r, s) \longrightarrow e^{2\pi i(-\Delta_{r,s} + \Delta_{r,s+1})} G_1(r, s) . \quad (5.2)$$

Note that $(r, s - 1)$ and $(r, s + 1)$ are precisely the intermediate s-channels for the two contours C_2 and C_1 respectively. Thus

$$\phi^s(a)_{(rs)(r's')} = e^{2\pi i(-\Delta_{r,s} + \Delta_{\text{intermediate}})} \delta_{rr'} \delta_{ss'} . \quad (5.3)$$

(ii) $\phi(b)$:

In order to construct $\phi(b)$ it is again easier to work in the s-channel basis. Let us start from the contour 0 to x_1 . We split the action of $\phi(b)$ in two parts: first we move z_1 across the parallelogram to some point z' such that $\text{Im} z' < \text{Im}(z_2 + \tau)$ and then in the second step we move z' to $z_1 + \tau$ (see Fig. 5). The contour around z_1 is, of course, pulled along with the point z_1 as it moves. In the second parallelogram therefore the contour goes from $x' = e^{2\pi i z'}$ to ∞ . More precisely the original contour C consists of a rotation clockwise around the origin going from $0 \rightarrow x_1$, turning around x_1 clockwise $(s+1)$ times and then returning from x_1 to 0. This is a closed contour

$$\oint_C = (e^{-i\pi a} - e^{i\pi a}) \int_0^{x_1} = -2i \sin \pi a \int_0^{x_1} \quad (5.4)$$

where $a = ((1+r)\alpha_+ + (1+s)\alpha_-)\alpha_-$ is the branch around the origin. As we move z_1 across the parallelogram, the effect on the contour is the same as pulling the contour C in the annulus to C' as shown in Fig. 6. In the second parallelogram, therefore, this contour now looks like one from x' to ∞ as shown in Fig. 7. Thus,

$$\begin{aligned} \oint_{C'} &= \left(-e^{-i\pi(a+c)} + e^{i\pi(a+c)} \right) \int_{x'}^{\infty} \\ &= 2i \sin \pi(a+c) \int_{x'}^{\infty} \end{aligned} \quad (5.5)$$

where $c = -\alpha_-^2$ is the branch around x_1 . Thus

$$\int_0^{x_1} \longrightarrow -\frac{\sin \pi(a+c)}{\sin \pi a} \int_{x'}^{\infty} = -\frac{\sin \pi s \alpha_-^2}{\sin \pi(s+1)\alpha_-^2} \int_{x'}^{\infty} \quad (5.6)$$

Note that in the second parallelogram the branch around infinity is $(a+c)$. This is equivalent to having the character in the second sheet correspond to $(\tilde{r}, \tilde{s}) = (r, s+1)$, because the corresponding branch at infinity is $(\tilde{a} + \tilde{b} + c) = (a+c)$. Indeed one can see this directly by considering the integrand and making the transformation $z_1 \rightarrow z_1 + \tau$ and $z \rightarrow z + \tau$ (as the contour is pulled to the second sheet). This gives

$$\begin{aligned} \Gamma(\pm r, s) &\longrightarrow e^{-i\frac{\pi}{2}\tau\alpha_-^2 + 2\pi i\alpha_-^2(z_1+z_2-2z)} \Gamma(\pm r, s+1) \\ (\theta_1(z_1 - z_2))^{\alpha_-^2/2} &\longrightarrow e^{-i\frac{\pi}{2}\tau\alpha_-^2 - \pi i\alpha_-^2(z_1-z_2)} e^{\pm i\pi\alpha_-^2/2} (\theta_1(z_1 - z_2))^{\alpha_-^2/2} \\ (\theta_1(z - z_1))^{-\alpha_-^2} &\longrightarrow (\theta_1(z - z_1))^{-\alpha_-^2} \\ (\theta_1(z - z_2))^{-\alpha_-^2} &\longrightarrow e^{+i\pi\tau\alpha_-^2 + 2\pi i\alpha_-^2(z_1-z_2)} e^{\mp i\pi\alpha_-^2} (\theta_1(z - z_2))^{-\alpha_-^2} \end{aligned}$$

where the $\pm$ sign in the θ -function transformation is for z_1 going past z_2 in the second sheet in a clockwise or counterclockwise way. Combining all the above formulae we obtain for $I(r, s)$ (the integrand) the transformation

$$I(r, s) \longrightarrow e^{\mp i \frac{\pi}{2} \alpha_-^2} I(r, s + 1) \quad (5.7)$$

Returning now to the second step, we have to move z' to $z_1 + \tau$, or equivalently x' to x'_1 in the second sheet. This process however involves x' going around x_2 , as shown in Fig. 8, depending on whether x' goes clockwise or anticlockwise around x_2 .

$$\int_{x'}^{\infty} \longrightarrow \int_{x_1}^{\infty} = e^{\pm i \pi \alpha_-^2} \int_{x_1}^{x_2} + \int_{x_2}^{\infty} \quad (5.8)$$

as the branch cut around x_2 is $(-\alpha_-^2)$.

Similarly one can consider the effect on the contour $\int_{x_2}^{\infty}$ for $s > 1$. In this case it is convenient to consider the transformation $z_2 \rightarrow z_2 - \tau$, $z \rightarrow z - \tau$. The integrand this time transforms as

$$I(r, s) \longrightarrow e^{\mp i \frac{\pi}{2} \alpha_-^2} I(r, s - 1) \quad (5.9)$$

Once again this can be seen by considering the first step of the transformation $z_2 \rightarrow z'_2$ in the second sheet which does not cross $(z_1 - \tau)$. Thus,

$$\begin{aligned} \int_{x_2}^{\infty} &= -\frac{1}{2i \sin \pi(a + b + c)} \oint_C = -\frac{1}{2i \sin \pi(a + b + c)} \oint_{C'} \\ &\longrightarrow -\frac{\sin \pi(a + b)}{\sin \pi(a + b + c)} \int_0^{x'_2} \end{aligned} \quad (5.10)$$

Note that the branch in the second sheet at the origin is $a + c = \bar{a}$ which is equivalent to having the character $(\bar{r}, \bar{s}) = (r, s - 1)$. In fact, this exactly corresponds to an interchange of the character and the intermediate state (note that $\int_{x_2}^{\infty}$ corresponds to the intermediate channel $(r, s - 1)$ and $\int_0^{x_2}$ to $(r, s + 1)$; the same is also true in the previous case of $\int_0^{x_1}$).

In the second step we have to move x'_2 around x_1 clockwise or counterclockwise to x_2 .

$$\int_0^{x'_2} \longrightarrow \left[\int_0^{x_1} + e^{\pm i \pi \alpha_-^2} \int_{x_1}^{x_2} \right] \quad (5.11)$$

Combining all these transformation rules we can get the $\phi(b)$ matrix. The final expressions turn out to be simpler in the t-channel basis. We can use the results of [10] for this change of basis:

$$\begin{aligned}\int_0^z &= \frac{1}{\sin \pi(b+c)} \left(-\sin \pi b \int_z^1 - \sin \pi(a+b+c) \int_0^\infty \right) \\ \int_1^\infty &= \frac{1}{\sin \pi(b+c)} \left(-\sin \pi c \int_z^1 + \sin \pi a \int_0^\infty \right)\end{aligned}\quad (5.12)$$

Thus we have, for example,

$$\begin{aligned}\int_{z_1}^{z_2} &= \frac{1}{\sin \pi(a+c)} \left[-\sin \pi a \int_0^{z_1} - \sin \pi(a+b+c) \int_{z_2}^\infty \right] \\ \text{(first step)} &\longrightarrow \delta_{\tilde{r},r} \left[\delta_{\tilde{s},s+1} \int_{z'_1}^\infty + \delta_{\substack{\tilde{s},s-1 \\ s>1}} \int_0^{z'_2} \right] \\ \text{(second step)} &\longrightarrow \delta_{\tilde{r},r} \left[\delta_{\tilde{s},s+1} \left(\int_{z_2}^\infty + e^{\pm i\pi\alpha_-^2} \int_{z_1}^{z_2} \right) + \delta_{\substack{\tilde{s},s-1 \\ s>1}} \left(\int_0^{z_1} + e^{\pm i\pi\alpha_-^2} \int_{z_1}^{z_2} \right) \right] \\ &= \delta_{\tilde{r},r} \frac{1}{\sin 2\pi\alpha_-^2} \left[e^{\pm 2\pi i\alpha_-^2} \sin \pi\alpha_-^2 (\delta_{\tilde{s},s+1} + \delta_{\substack{\tilde{s},s-1 \\ s>1}}) \int_{z_1}^{z_2} \right. \\ &\quad \left. + (-1)^{r+1} (\delta_{\tilde{s},s+1} \sin \pi(\tilde{s}+1)\alpha_-^2 - \delta_{\substack{\tilde{s},s-1 \\ s>1}} \sin \pi(\tilde{s}-1)\alpha_-^2) \int_0^\infty \right]\end{aligned}\quad (5.13)$$

And similarly one can get the transformation for $\int_0^\infty$. Taking into account also the phase coming from the transformation of the integrand, eqs. (5.7, 9), we write the final expressions for $\phi(b)$ in the t-channel.

$$\begin{aligned}\phi_{(11,11)}^t(b) &= \delta_{\tilde{r},r} e^{\pm \frac{3}{2}\pi i\alpha_-^2} \frac{\sin \pi\alpha_-^2}{\sin 2\pi\alpha_-^2} \left(\delta_{\tilde{s},s+1} + \delta_{\substack{\tilde{s},s-1 \\ s>1}} \right) \\ \phi_{(11,13)}^t(b) &= \delta_{\tilde{r},r} e^{\mp \frac{1}{2}\pi i\alpha_-^2} \frac{(-1)^{r+1}}{\sin 2\pi\alpha_-^2} \\ &\quad \times \left(\delta_{\tilde{s},s+1} \sin \pi(\tilde{s}+1)\alpha_-^2 - \delta_{\substack{\tilde{s},s-1 \\ s>1}} \sin \pi(\tilde{s}-1)\alpha_-^2 \right) \\ \phi_{(13,11)}^t(b) &= \delta_{\tilde{r},r} e^{\pm \frac{3}{2}\pi i\alpha_-^2} (-1)^{r+1} \frac{(\sin \pi\alpha_-^2)^2}{\sin 2\pi\alpha_-^2} \frac{1}{\sin \pi\tilde{s}\alpha_-^2} \left(\delta_{\tilde{s},s+1} - \delta_{\substack{\tilde{s},s-1 \\ s>1}} \right) \\ \phi_{(13,13)}^t(b) &= \delta_{\tilde{r},r} e^{\mp \frac{1}{2}\pi i\alpha_-^2} \frac{\sin \pi\alpha_-^2}{\sin 2\pi\alpha_-^2} \frac{1}{\sin \pi\tilde{s}\alpha_-^2} \\ &\quad \times \left(\delta_{\tilde{s},s+1} \sin \pi(\tilde{s}+1)\alpha_-^2 + \delta_{\substack{\tilde{s},s-1 \\ s>1}} \sin \pi(\tilde{s}-1)\alpha_-^2 \right)\end{aligned}\quad (5.14)$$

Similarly one can analyse the transformation $z_1 \rightarrow z_1 - \tau$. In this case the two steps mentioned above are reversed: first we move z_1 to z'_1 in the same sheet such that $\text{Im} z'_1 < \text{Im} z_2$ and next move z'_1 to $z_1 - \tau$. Once again depending on whether z_1 moves counterclockwise or clockwise around z_2 we get two expressions and as expected they are just the inverses of $\phi^t(b)$ in eq. (5.14) for clockwise and counterclockwise cases respectively.

It is clear from the first equation in (5.14) that the Verlinde conjecture [1] is obeyed by our correlation function. The $\phi(b)$ matrix has non-zero entries precisely as in the fusion rules of the minimal models. In fact, up to an overall normalization factor, the $\phi(b)$ are identical to the N_{ijk} , as defined by Verlinde.

6. MODULAR TRANSFORMATION

In this section we will study the modular transformation properties of the conformal blocks of the correlation function $\langle \phi_{12}, \phi_{12} \rangle$. Under the S transformation, $\tau \rightarrow -1/\tau$, the role of σ and t gets interchanged. There are actually two cases: (i) $(a, b) \rightarrow (b, -a)$ cycles and (ii) $(a, b) \rightarrow (-b, a)$ cycles. Case (i) corresponds to rotating the parallelogram counterclockwise and we denote this transformation by S . Case (ii) corresponds to rotating the parallelogram clockwise and this transformation is, of course, $S^{-1} = S^*$. We will give here the details for the first case. We will further assume that $\text{Im}(-z_1/\tau) > \text{Im}(-z_2/\tau)$ so that after the transformation the time ordering of the points z_1 and z_2 remains unchanged. This, on the other hand, implies that for case (ii) the time ordering would be reversed as $\text{Im}(z_1 + 1)/\tau < \text{Im}(z_2 + 1)\tau$.

We first obtain the action of S and then we verify that S transforms $\phi(a)$ to $\phi(b)$. It is more convenient to use the t -channel for this purpose. It is clear that the two contours $\tilde{C}_1$ and $\tilde{C}_2$ do not mix under S , implying that the conformal blocks corresponding to one-point functions are modular covariant for the identity and ϕ_{13} fields respectively.

Let us first consider the S -transformation of the integrand. The lattice sum after a

Poisson resummation becomes

$$\begin{aligned} \Gamma(r, s) \longrightarrow \alpha_0 \sqrt{\frac{2i}{\tau}} \sum_k \exp \left(-\frac{2i\pi}{\tau} k^2 \alpha_0^2 + 2\pi i 2k \alpha_0 \alpha (z'_1 + z'_2 - 2z') \right) \\ \times \exp(-2\pi i \alpha_0 (r\alpha_+ + s\alpha_-), k) \end{aligned} \quad (6.1)$$

where $z'_i = -z_i/\tau$. Writing $k = r'(p+1) - s'p + 2np(p+1)$ where $1 \leq r' \leq p$ and $1 \leq s' \leq 2(p+1)$ and $n \in Z$, we obtain

$$\begin{aligned} (\Gamma(r, s) - \Gamma(-r, s)) \longrightarrow \\ \alpha_0 \sqrt{\frac{2i}{\tau}} \sum_{r'=1}^p \sum_{s'=1}^{2(p+1)} e^{-i\pi(r'\alpha_+ + s'\alpha_-)(r\alpha_+ + s\alpha_-)} (\Gamma(r', s') - \Gamma(-r', s')) \end{aligned} \quad (6.2)$$

where in the second term we have replaced $r' \rightarrow -r'$ and used the fact that

$$\exp(-i\pi(-r\alpha_+ + s\alpha_-)(-r'\alpha_+ + s'\alpha_-)) = \exp(-i\pi(r\alpha_+ + s\alpha_-)(r'\alpha_+ + s'\alpha_-))$$

The S -transformations of the remaining terms, up to $(rs, r's')$ independent phases are

$$\begin{aligned} \left(\frac{\theta(z, \tau)}{\theta'(0, \tau)} \right) \longrightarrow \left(\frac{\theta(z', -1/\tau)}{\theta'(0, -1/\tau)} \right) \left(\frac{dz'}{dz} \right)^{-1} \\ \eta(\tau) \longrightarrow \sqrt{\frac{i}{\tau}} \eta(-1/\tau) \end{aligned} \quad (6.3)$$

Combining all this we obtain

$$\begin{aligned} \oint_C dz I(r, s) \longrightarrow A(z, z') \oint_{C'} dz' I(r', s') \\ A(z, z') = \left(\frac{dz'_1}{dz_1} \frac{dz'_2}{dz_2} \right)^{\Delta_{12}} \frac{\epsilon}{\sqrt{2p(p+1)}} \sum_{r'=1}^p \sum_{s'=1}^{2(p+1)} e^{-i\pi(r'\alpha_+ + s'\alpha_-)(r\alpha_+ + s\alpha_-)} \end{aligned} \quad (6.4)$$

where ϵ is a phase, independent of $(r, s)(r', s')$, which will be determined later by considering S^2 . The term $(dz'_i/dz_i)^{\Delta_{12}}$ is just the conformal transformation of the primary fields $\phi_{12}(z_i) \longrightarrow \phi_{12}(z'_i)(dz'_i/dz_i)^{\Delta_{12}}$, therefore we shall omit this factor in the following. Note that the ranges of r', s' are not the same as those of r and s , and hence we will have to bring them to the standard range by various symmetries.

(a) Identity Channel: In this case the contour is around z_1 and z_2 . Under the S transformation this contour is obviously unchanged. We bring the ranges of r' and s' to the standard form in the following two steps. Split

$$\sum_{r'=1}^p \sum_{s'=1}^{2(p+1)} = \sum_{r'=1}^p \left(\sum_{s'=1}^{p+1} + \sum_{s'=p+1}^{2(p+1)} \right)$$

In the second sum we define $s' = p+1+s''$ and $r' = p-r''$. Then $\Gamma(\pm r', s') \rightarrow \Gamma(\mp r'', s'')$, and thus the sum reduces to:

$$\sum_{r'=1}^{p-1} \sum_{s'=1}^{p+1} e^{-i\pi s' \alpha_- (r \alpha_+ + s \alpha_-)} \left(2i \sin \pi r' r \frac{(p+1)}{p} \right) (-1)^{r' s} \int dz' \text{Integrand}(r', s')$$

In the next step we split the sum as

$$\sum_{r'=1}^{p-1} \left(\sum_{s'=1}^{r'} + \sum_{s'=r'+1}^{p+1} \right)$$

In the second sum we define $s' = p+1-s''$ and $r' = p-r''$. Now $\Gamma(\pm r', s') \rightarrow \Gamma(\mp r'', -s'')$. By redefining the dummy variable $z' = z'_1 + z'_2 - z''$ the Γ becomes now with $z' \rightarrow z''$, $\Gamma(\pm r'', s'')$. Under this transformation $\theta(z'_1 - z') \rightarrow \theta(z'' - z'_2)$ and $\theta(z' - z'_2) \rightarrow \theta(z'_1 - z'')$ and hence the integrand is unchanged. Taking into account the phases coming from the factors outside the integrand, the result is finally,

$$\begin{aligned} & \frac{-4\epsilon}{\sqrt{2p(p+1)}} \sum_{r'=1}^{p-1} \sum_{s'=1}^{r'} (-1)^{r' s + s' r} \sin \pi r' r \frac{(p+1)}{p} \sin \pi s' s \frac{p}{(p+1)} \\ & \times \int dz' \text{Integrand}(r', s') \end{aligned} \quad (6.5)$$

where we have used the fact that for $s' = 0$, the integrand is zero.

Thus in the identity block

$$S_{rs, r's'}^{(11,11)} = \frac{-4\epsilon}{\sqrt{2p(p+1)}} (-1)^{r' s + s' r} \sin \pi r' r \frac{(p+1)}{p} \sin \pi s' s \frac{p}{(p+1)} \quad (6.6)$$

This is exactly the modular transformation of the characters up to an overall phase ϵ . This phase is determined by considering S^2 . By simple manipulations one can see that

$S^2 = \epsilon^2$. On the other hand S^2 must transform $\tau \rightarrow \tau$ and $z_1 - z_2 \rightarrow -(z_1 - z_2)$ (z_1 going counterclockwise around z_2). If one takes the limit $(z_1 - z_2) \rightarrow 0$ then the correlation function in the (r, s) sector goes to $(z_1 - z_2)^{-2\Delta_{12}} \chi_{(r,s)}$. Then $S^2 = \exp(-2i\pi\Delta_{12})$ implying that $\epsilon = \exp(-i\pi\Delta_{12})$.

(b) ϕ_{13} Channel: In this case the contour is the one described earlier as $\tilde{C}_2$; in the degeneration limit it is the closed contour around 0 and ∞ . More precisely the contour may be defined as follows. We start from the point A (see Fig. 9), go around the origin clockwise by π , go from 0 to ∞ , go around ∞ clockwise once, return from ∞ to 0, go around 0 counterclockwise once, then 0 to ∞ , counterclockwise around ∞ once, again ∞ to 0, and around 0 clockwise returning to the point A . This is a closed contour C . The integral

$$\begin{aligned} \oint_C &= e^{i\pi a} \left[\int_0^\infty + e^{-2\pi i(a+b+c)} \int_\infty^0 + e^{-2\pi i(2a+b+c)} \int_0^\infty + e^{-2\pi i a} \int_\infty^0 \right] \\ &= -4 \sin \pi a \sin \pi(a+b+c) e^{-i\pi(a+b+c)} \int_0^\infty \end{aligned} \quad (6.7)$$

where $a = ((1+r)\alpha_+ + (s+1)\alpha_-)\alpha_-$ is the branch around the origin, and $b = c = -\alpha_-^2$ are the branches around points z_1 and z_2 respectively. $a+b+c = ((r+1)\alpha_+ + (s-1)\alpha_-)\alpha_-$ is the branch around ∞ . Representing the torus as a parallelogram, this contour is as shown in Fig. 10(a).

Under the modular transformation S , the contour will transform into a contour C' shown in Fig. 10(b). The part of the contour C' in the bottom parallelogram is obviously obtained by taking $z' \rightarrow z' - \tau$. By using the methods of sec. 5, this is equivalent to

$$\int dz' I(r', s') \xrightarrow{z' \rightarrow z' - \tau} \int dz' I(r', s' - 2) \quad (6.8)$$

Thus the contour integral is

$$\begin{aligned} \oint_{C'} I(r', s') &= \left[e^{-i\pi a'} \int_\infty^0 I(r', s') + e^{i\pi a'} \int_0^\infty I(r', s') + e^{i\pi a'} e^{-2\pi i(a'+b+c)} \int_0^\infty I(r', s' - 2) \right. \\ &\quad \left. + e^{i\pi a'} e^{-2\pi i(a'+b+c)} e^{2\pi i(a''+b+c)} \int_\infty^0 I(r', s' - 2) \right] \end{aligned} \quad (6.9)$$

where $a' = ((1+r')\alpha_+ + (1+s)\alpha_-)\alpha_-$ and $a'' = ((1+r')\alpha_+ + (1+(s'-2))\alpha_-)\alpha_-$ are the branches at the origin in the two sheets corresponding to (r', s') and $(r', s' - 2)$ characters respectively.

Substituting eq. (6.9) in eq. (6.4), regrouping the sum over r' and s' and using eq. (6.7) we finally obtain the transformation

$$\int_0^\infty I(r, s) \rightarrow \frac{\epsilon}{\sqrt{2p(p+1)}} \sum_{r'=1}^p \sum_{s'=1}^{2(p+1)} F(r', s') \int_0^\infty dz' I(r', s')$$

where

$$F(r', s') = \frac{-i}{2} e^{-\pi i \alpha_-^2} (-1)^{r+r'+r's+s'r} e^{-i\pi r' \alpha_+^2} \times \left[\frac{e^{-i\pi s'(s-1)\alpha_-^2}}{\sin \pi(s-1)\alpha_-^2} - \frac{e^{-i\pi s'(s+1)\alpha_-^2}}{\sin \pi(s+1)\alpha_-^2} \right] \quad (6.10)$$

Next we proceed to reduce the range of (r', s') to the standard one, exactly as before. We obtain finally

$$\int_0^\infty dz I(r, s) = \sum_{r'=1}^{p-1} \sum_{s'=1}^{r'} S_{rs, r's'}^{(13,13)} \int_0^\infty dz' I(r', s')$$

where

$$S_{rs, r's'}^{(13,13)} = \frac{2i\epsilon}{\sqrt{2p(p+1)}} e^{-\pi i p/(p+1)} (-1)^{r+r'+r's+s'r} \sin \pi r r' \frac{(p+1)}{p} \times \left[\frac{\sin \pi s'(s-1)\alpha_-^2}{\sin \pi(s-1)\alpha_-^2} - \frac{\sin \pi s'(s+1)\alpha_-^2}{\sin \pi(s+1)\alpha_-^2} \right] \quad (6.11)$$

A few comments are in order at this point. Firstly we see that for $s' = 1$, $S = 0$, as it should be because for $s' = 1$ there is no ϕ_{13} channel. Secondly using the value $\epsilon = \exp(-i\pi\Delta_{12})$ we see that

$$S^2 = \exp(i\pi(-2\Delta_{12} + \Delta_{13})) \quad (6.12)$$

as expected. Indeed in the limit $z_1 - z_2 \rightarrow 0$ the correlation function in this channel goes to $(z_1 - z_2)^{-2\Delta_{12} + \Delta_{13}} \times (\text{function of } \tau)$. Thirdly we note that $S^* S = 1$.

We now discuss the action of S on the monodromy $\phi(a)$. We assume that $\text{Im}(z_1 + 1)/\tau > \text{Im}(z_2 + 1)/\tau$, so that S^* does not change the ordering. We first make a S^*

transformation, then apply $\phi(a)$ which takes $z'_1 \rightarrow z'_1 + 1$ and finally transform by $(S^*)^{-1} = S$. This is then equivalent to $z_1 \rightarrow z_1 + \tau$, where z_1 goes clockwise around $z_2 + \tau$. Therefore, for consistency, we should obtain $\phi^t(b)$ for the clockwise case. The counterclockwise case would correspond to $\text{Im}(-z_1/\tau) > \text{Im}(-z_2/\tau)$, therefore we first transform by S , apply $\phi(-a)$ which takes $z'_1 \rightarrow z'_1 - 1$, and finally transform back by S^* . This amounts to taking $z_1 \rightarrow z_1 + \tau$, counterclockwise around $z_2 + \tau$. Here we shall give the details only for the first case, though also in the second, one gets the desired result.

The expression for S obtained above is in the t-channel basis. Thus we must transform $\phi(a)$ also into the t-channel basis. Using the equations for the change of basis, eq. (5.12) one can then obtain the expressions for $\phi^t(a)$ in the four blocks as follows:

$$\begin{aligned}
\phi_{11,11}^t(a) &= 2(-1)^r e^{i\pi\frac{3}{2}\alpha_-^2} \frac{\sin \pi\alpha_-^2}{\sin 2\pi\alpha_-^2} \cos \pi s \alpha_-^2 \delta_{r,r'} \delta_{s,s'} \\
\phi_{11,13}^t(a) &= -2ie^{i\frac{\pi}{2}\alpha_-^2} \frac{\sin \pi(s+1)\alpha_-^2 \sin \pi(s-1)\alpha_-^2}{\sin 2\pi\alpha_-^2} \delta_{r,r'} \delta_{s,s'} \\
\phi_{13,11}^t(a) &= 2ie^{\frac{i\pi}{2}\alpha_-^2} \frac{(\sin \pi\alpha_-^2)^2}{\sin 2\pi\alpha_-^2} \delta_{r,r'} \delta_{s,s'} \\
\phi_{13,13}^t(a) &= 2(-1)^r e^{-\frac{i\pi}{2}\alpha_-^2} \cos \pi s \alpha_-^2 \frac{\sin \pi\alpha_-^2}{\sin 2\pi\alpha_-^2} \delta_{r,r'} \delta_{s,s'}
\end{aligned} \tag{6.13}$$

Now we can compute in a straightforward way $S^*\phi^t(a)S$ and the result turns out to be exactly $\phi^t(b)$. We demonstrate the computation for the (13,13) block in Appendix B, the most tricky case. By similar techniques one can compute the $\phi^t(b)$ for all the other blocks and the final results are identical to those given in eq. (5.14). Note that the relative phase between the S^{11} and the S^{13} blocks plays a crucial role in getting the correct expressions for the second and the third equations in (5.14).

7. MODULAR AND MONODROMY INVARIANT CORRELATORS

We now construct the full modular and monodromy invariant correlation functions by taking an appropriate combination of the left and right sectors. This combination turns out to be just that of the monodromy invariant four-point function as found in ref. [10]. We shall change notation, at this point, for convenience: $\tilde{G}(r, s)$ for the contours $\tilde{C}_1$ and

$\tilde{C}_2$ will be named G^{11} and G^{13} .

We first consider the so called *A*-series, namely the diagonal combination $\sum_{r,s} \chi_{rs} \bar{\chi}_{rs}$. Clearly, in the *t*-channel, the character block G^{11} is modular invariant for this combination. The G^{13} block is more tricky. Let us consider $G_{rs}^{13} \bar{G}_{rs}^{13}$. This, after a modular transformation goes to

$$S_{(rs,r's')}^{(13,13)} S_{(rs,r''s'')}^{*(13,13)} G_{r's'}^{13} \bar{G}_{r''s''}^{13}.$$

The problem here is that the $S^{13,13}$ block is not symmetric. Indeed one can explicitly check that

$$\sum_{r=1}^{p-1} \sum_{s=2}^r S_{(rs,r's')}^{(13,13)} S_{(rs,r''s'')}^{*(13,13)} \neq \delta_{r'r''} \delta_{s's''}$$

However if we define the combination to be

$$N_{rs} G_{rs}^{13} \bar{G}_{rs}^{13} \tag{7.1}$$

where

$$N_{rs} = \sin \pi(s-1)\alpha_-^2 \sin \pi(s+1)\alpha_-^2 \tag{7.2}$$

then one can also show, by calculations similar to those of Appendix B, that

$$\sum_{r=1}^{p-1} \sum_{s=2}^r S_{(rs,r's')}^{(13,13)} S_{(rs,r''s'')}^{*(13,13)} N_{rs} = N_{r's'} \delta_{r'r''} \delta_{s's''} \tag{7.3}$$

Thus the combination

$$\alpha \sum_{r,s} |G_{rs}^{11}|^2 + \beta \sum_{r,s} N_{rs} |G_{rs}^{13}|^2 \tag{7.4}$$

is modular invariant for any α, β which are independent of (r, s) . Note that if we define a new basis for conformal blocks as $\tilde{G}_{(rs)}^{(13)} = \sqrt{N_{rs}} G_{rs}^{(13)}$, then $\tilde{S}^{(13,13)}$ is in fact symmetric and $\tilde{S}^{(13,13)} \tilde{S}^{*(13,13)} = 1$.

To consider monodromy invariance, it is best to go to the *s*-channel, as $\phi(a)$ in the *s*-channel is diagonal. In general one would get cross terms $G_{(rs),1} \bar{G}_{(rs),2}$ which would destroy $\phi(a)$ invariance. The condition that such terms do not appear is

$$\alpha = \beta (\sin \pi \alpha_-^2)^2 \tag{7.5}$$

With this condition, clearly, the combination is also $\phi(b)$ invariant as $\phi(b) = S^* \phi(a) S$. Thus, up to an overall proportionality constant

$$G = \beta \sum_{r,s} \left(\sin \pi \alpha_-^2 |G_{rs}^{(11)}|^2 + N_{rs} |G_{rs}^{(13)}|^2 \right) \quad (7.6)$$

the coefficient β can be determined by demanding that in the degeneration limit in the identity sector $(rs) = (11)$ one obtains the correctly normalized two-point function $\langle \phi_{12} \phi_{12} \rangle$ on the plane. Note that upon degeneration the above combination is exactly the monodromy-invariant four-point function for each (r, s) found in ref. [10]. Thus,

$$G = \sum_{r,s} \langle \phi_{12} \phi_{12} \rangle_{rs} \quad (7.7)$$

where (rs) here refers to the diagonal primary fields $\phi_{(rs)\overline{(rs)}}$.

One can carry out a similar analysis for D and E series and one obtains the expected combination of invariant correlation functions. For example, for the D -series, for $p = 4\rho + 1$, it is known that the modular invariant partition function is proportional to [13]

$$\sum_{r=1}^{p-1} \sum_{\substack{s=1 \\ \text{odd}}}^p |\chi_{r,s} + \chi_{r,p+1-s}|^2 \quad (7.8)$$

Likewise, the invariant correlation function is proportional to

$$\sum_{r=1}^{p-1} \sum_{\substack{s=1 \\ \text{odd}}}^p \left[(\sin \pi \alpha_-^2)^2 |G_{rs}^{(11)} + G_{r,p+1-s}^{(11)}|^2 + N_{rs} |G_{rs}^{(13)} + G_{r,p+1-s}^{(13)}|^2 \right] \quad (7.9)$$

8. CONCLUDING REMARKS

We have shown that a natural prescription, within the Coulomb gas approach, enables one to construct the correlation functions for primary fields on the torus, and derive their modular transformation and monodromy properties. We have described in detail the case of $\langle \phi_{12} \phi_{12} \rangle$, and this can readily be extended to the case of $\langle \phi_{1n} \phi_{1n} \rangle$ correlation functions. In the latter case we have $(n - 1) V_{\alpha_-}$ screening operators, and the correlation function

is again given by an expression involving the non-zero modes of the bosonic scalar. This gives rise to appropriate powers of the θ -function, and a lattice contribution (from the solitonic sector) $[\Gamma(r, s) - \Gamma(-r, s)]$, where

$$\begin{aligned} \Gamma(r, s) = & \sum_k \exp \left(i \frac{\pi}{2} \tau (r\alpha_+ + s\alpha_- + \frac{k}{\alpha_0})^2 \right) \\ & \times \exp \left(2\pi i (r\alpha_+ + s\alpha_- + \frac{k}{\alpha_0}) (\alpha z_1 + \alpha z_2 + \alpha_- \sum_{i=1}^{n-1} w_i) \right) \end{aligned} \quad (8.1)$$

$\alpha = -(n-1)/2\alpha_-$ and w_i are the locations of the $(n-1)$ screening operators. The closed contours can again be described by considering the degeneration of the torus and the corresponding contour on the plane for the four-point functions. $\phi(a)$, $\phi(b)$ and S can be obtained by considering the transformations of the contours as described above.

The case $\langle \phi_{nm} \phi_{nm} \rangle$ is slightly more complicated. Now there are $(n-1)V_{\alpha_+}$ and $(m-1)V_{\alpha_-}$ screening operators. In order to construct the $\phi(a)$ -diagonal basis in the s -channel we cannot simply take the combination $\Gamma(r, s) - \Gamma(-r, s)$. However we can modify the prescription as follows: let k and l be the number of contours of α_+ and α_- type of screening operators that go from 0 to x_1 . Then, for $\Gamma(-r, s)$ one can take $(n-1-k)$ contours for V_{α_+} from 0 to x_1 . The intermediate state corresponds to $(r+2k-(n-1), s+2l-(m-1))$. As k and l range from 0 to $(n-1)$ and 0 and $(m-1)$ respectively, one gets all the conformal blocks. Transforming $z_1 \rightarrow \tau - z_1$ (the first step in the construction of $\phi(b)$) one could show again that characters and intermediate states interchange. The construction of $\phi(b)$ and S would then follow as above.

We have not proven here a general theorem that there are no null states propagating on the torus, although we have checked explicitly the non-existence of the lowest null states. However one may give another argument for the correctness of the correlation functions written down above. In ref. [3], Mathur et. al. give a procedure to obtain differential equations for conformal blocks of correlation functions, the coefficients of the equation being given in terms of the Wronskian of the conformal blocks. The crucial property of the Wronskian is that it is invariant under $\phi(a)$ and $\phi(b)$ and has a well defined singularity as $z_1 \rightarrow z_2$. The Wronskians constructed from our expressions for the conformal blocks

have precisely the same properties. The differential equation is then determined up to a few constant coefficients. The latter are further determined uniquely by requiring that in the character block the q -expansion has positive integral coefficients. Since our expressions in the character block reproduce the correct characters (with therefore positive integral coefficients in the q -expansion), they must be the solutions of the corresponding differential equations.

One of the interesting questions is that of extending this formalism to higher genus surfaces. This has been achieved by Zamolodchikov [14], for the case of hyperelliptic curves, by representing these curves as branched covers of the sphere. However we believe that one should be able to carry over the procedure outlined in this paper directly to surfaces of arbitrary genus.

After the completion of this work, we became aware of two related studies. The first, by G. Felder [15], describes a BRST approach to the understanding of the Coulomb gas formalism on the plane and the torus. The second, by C. Crnković, G. Sotkov and M. Stanishkov [16], describes the computation of genus 2 characters of minimal conformal field theories using the picture of the surface as a branched cover of the sphere.

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APPENDIX A

Consider the conformal block $G(r, s)$:

$$G(r, s) = \frac{1}{\eta} \oint_{C_i} dz \left(\frac{\theta_1(z_1 - z_2)}{\theta'_1(0)} \right)^{2\alpha^2} \left(\frac{\theta_1(z_1 - z)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \left(\frac{\theta_1(z - z_2)}{\theta'_1(0)} \right)^{2\alpha\alpha_-} \times (\Gamma(r, s) - \Gamma(-r, s)) \quad (A.1)$$

where

$$\Gamma(r, s) = \sum_{n=-\infty}^{\infty} \exp \left(\pi i \tau \frac{(r(p+1) - sp + 2np(p+1))^2}{2p(p+1)} \right) \times \exp \left(\frac{2\pi i}{\sqrt{2p(p+1)}} (r(p+1) - sp + 2np(p+1))(\alpha z_1 + \alpha z_2 + \alpha_- z) \right) \quad (A.2)$$

If we consider the case $(r = 1, s = 1)$, the θ -functions are the same as above and we merely have to specialize $\Gamma(r, s)$.

$$\Gamma(1, 1) = \sum_{n=-\infty}^{\infty} \exp \left(\pi i \tau \frac{((p+1) - p + 2np(p+1))^2}{2p(p+1)} \right) \times \exp \left(\frac{2\pi i}{\sqrt{2p(p+1)}} ((p+1) - p + 2np(p+1))(\alpha z_1 + \alpha z_2 + \alpha_- z) \right) \quad (A.3)$$

We can study the degeneration limit in the same way as the derivation of eq. (3.4). It is easy to see that the leading term in q comes from the leading terms of the θ -functions and $\Gamma(r, s)$. This is what gave eq. (3.4). To check the null state cancellation for the case we have, we have to examine the q^1 term after the leading term. These clearly come from the following: the first non-leading term of one of the θ -functions and the leading terms of the rest including $\Gamma(1, 1)$ (and thus for all the θ -functions and Γ) and the leading term of $\Gamma(-1, 1)$ combined with the leading terms of the θ -functions. If one collects all these terms together, a straightforward exercise, one gets, (up to overall factors) the following integral expression

$$\int_0^{x_1} dx (x - x_1)^{-p/(p+1)} (x_2 - x)^{-p/(p+1)} x^{-2/(p+1)}$$

$$\times \left[1 - \frac{p}{2(p+1)} \frac{(x_2 - x_1)^2}{x_1 x_2} + \frac{p}{p+1} \left(\frac{(x_2 - x)^2}{x_2 x} + \frac{(x - x_1)^2}{x_1 x} \right) - \frac{x^2}{x_1 x_2} \right] \quad (A.4)$$

It is easy to see that the above expression may be reduced to

$$\begin{aligned} & \left(1 - \frac{p}{2(p+1)} \frac{(1 - x_1)^2}{x_1} \right) I \left(\frac{p-1}{p+1} - 1; \frac{-p}{p+1}; \frac{-p}{p+1} \right) \\ & + \frac{p}{p+1} I \left(\frac{p-1}{p+1} - 2; \frac{-p}{p+1} + 2; \frac{-p}{p+1} \right) \\ & + \frac{p}{p+1} \frac{1}{x_1} I \left(\frac{p-1}{p+1} - 2; \frac{-p}{p+1}; \frac{-p}{p+1} + 2 \right) \\ & - \frac{1}{x_1} I \left(\frac{p-1}{p+1} + 1; \frac{-p}{p+1}; \frac{-p}{p+1} \right) \end{aligned} \quad (A.5)$$

where we have, without loss of generality, set $x_2 = 1$ and

$$I(a; b; c) = \int_0^{x_1} dx x^a (1-x)^b (x-x_1)^c$$

$I(a; b; c)$ may be expressed in terms of the hypergeometric function and gamma functions. Then (A.5) may be proved to be zero by use of the identities of contiguous hypergeometric functions. Thus the null state truncation at the first level in the trace over the identity sector may be proven.

APPENDIX B

We now show the computation of $S^* \phi^t(a) S$ for the (13, 13) block.

$$\begin{aligned} (S^* \phi^t(a) S)^{(13,13)} &= \sum_{r'=1}^{p-1} \sum_{s'=1}^{r'} \frac{4}{2p(p+1)} e^{-\frac{\pi i}{2} \alpha_-^2} (-1)^{r+r''+r'(s+s''+1)+s'(r+r'')} \\ &\times \sin \pi r r' \frac{(p+1)}{p} \sin \pi r'' r' \frac{(p+1)}{p} 2 \cos \pi s' \alpha_-^2 \frac{\sin \pi \alpha_-^2}{\sin 2\pi \alpha_-^2} \\ &\times \left[\frac{\sin \pi s'(s-1) \alpha_-^2}{\sin \pi (s-1) \alpha_-^2} - \frac{\sin \pi s'(s+1) \alpha_-^2}{\sin \pi (s+1) \alpha_-^2} \right] \\ &\times \left[\frac{\sin \pi s''(s'-1) \alpha_-^2}{\sin \pi (s'-1) \alpha_-^2} - \frac{\sin \pi s''(s'+1) \alpha_-^2}{\sin \pi (s'+1) \alpha_-^2} \right] \end{aligned} \quad (B.1)$$

Now we can express

$$\frac{\sin \pi s''(s' \pm 1)\alpha_-^2}{\sin \pi(s' \pm 1)\alpha_-^2} = \sum_{k=-(s''-1)}^{(s''-1)} \exp(i\pi k(s' \pm 1)\alpha_-^2) \quad (B.2)$$

Next we can increase the range of r' and s' to $(1, 2p)$ and $(1, 2(p+1))$ respectively, by inserting a factor 2^{-3} . Summing over r' yields the result $(-2)/(-4)(1 + (-1)^{s+s'+1})\delta_{r,r''}$. There are several terms in the s' sum. First let us consider the case with the -ve sign in (B.2) with $(2 \cos \pi s' \alpha_-^2)(\sin \pi s'(s-1)\alpha_-^2 / \sin \pi(s-1)\alpha_-^2)$ in (B.1). We can express sin and cos in terms of exponentials, and sum over s' : the result is:

$$2(p+1) \left[\frac{\sin \pi s \alpha_-^2}{\sin \pi(s-1)\alpha_-^2} H(s''-1-s) + \frac{\sin \pi(s-2)\alpha_-^2}{\sin \pi(s-1)\alpha_-^2} H(s''+1-s) \right] \quad (B.3)$$

where $H(x)$ is the Heaviside step function, $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$. Similarly we can take the term $(2 \cos \pi s' \alpha_-^2)(\sin \pi s'(s+1)\alpha_-^2 / \sin \pi(s+1)\alpha_-^2)$. The final result is:

$$\begin{aligned} & \frac{(1 + (-1)^{s''+s+1})}{2} e^{-i\frac{\pi}{2}\alpha_-^2} \frac{\sin \pi \alpha_-^2}{\sin 2\pi \alpha_-^2} \delta_{r,r'} \\ & \times \left[\left(\frac{\sin \pi s \alpha_-^2}{\sin \pi(s-1)\alpha_-^2} - \frac{\sin \pi s \alpha_-^2}{\sin \pi(s+1)\alpha_-^2} \right) H(s''-1-s) \right. \\ & \left. + \frac{\sin \pi(s-2)\alpha_-^2}{\sin \pi(s-1)\alpha_-^2} H(s''+1-s) - \frac{\sin \pi(s+2)\alpha_-^2}{\sin \pi(s+1)\alpha_-^2} H(s''-3-s) \right] \quad (B.4) \end{aligned}$$

The expression inside the bracket can be simplified. It is

$$\begin{aligned} & = 0 \quad \text{if } s'' < s-1 \\ & = 0 \quad \text{if } s'' > s+1 \end{aligned} \quad (B.5)$$

and

$$\begin{aligned} & = \frac{\sin \pi(s-2)\alpha_-^2}{\sin \pi(s-1)\alpha_-^2} \quad \text{if } s'' = s-1 \\ & = \frac{\sin \pi(s+2)\alpha_-^2}{\sin \pi(s+1)\alpha_-^2} \quad \text{if } s'' = s+1 \end{aligned} \quad (B.6)$$

and the whole expression vanishes if $s'' + s + 1$ is odd. The result for the +ve sign in (B.2) is identical. Combining together all the above factors one exactly obtains the last equation in (5.14).

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FIGURE CAPTIONS

Fig. 1 The contours C_1 and C_2 on the torus. Though not explicitly indicated, C_1 and C_2 appropriately wind around the point z_1 and the inner circle and z_2 and the outer circle respectively.

Fig. 2 The contours $\tilde{C}_1$ and $\tilde{C}_2$ on the torus.

Fig. 3 A conformal block of the correlation function in the $\phi(a)$ -diagonal basis. The cross indicates a trace in a particular representation $\phi_{r,s}$. The intermediate state $\phi_{r,s+1}$ corresponds to $\oint_{C_1}$ and $\phi_{r,s-1}$ corresponds to $\oint_{C_2}$.

Fig. 4 A conformal block of the correlation function in the "Verlinde" basis. The intermediate state ϕ_{11} corresponds to $\oint_{\tilde{C}_1}$ and ϕ_{13} corresponds to $\oint_{\tilde{C}_2}$.

Figures 5 to 10 are fully described in the text.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10(a)

Fig. 10(b)