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THE QUANTIZATION OF $N=3$ SUPER-YANG-MILLS OFF-SHELL IN HARMONIC SUPERSPACE

F. Delduc^{*}

CERN-Geneva

J. McCabe⁺

LAPP, BP.909, 74019 Annecy-le-Vieux, France

ABSTRACT

The supersymmetric extension of the transverse gauge is found for the harmonic superspace version of $N=3$ extended Super-Yang-Mills. Using this gauge, the $N=3$ supersymmetric Feynman rules are derived. The explicit form of the propagators is given along with the relevant $SU(3)/U(1) \times U(1)$ and $SU(3)/SU(2) \times U(1)$ distributions.

* On leave of absence from LPTHE, Jussieu, Paris (FRANCE)

+ Address after Oct. 1, 1988: Racah Institute, Hebrew University, Jerusalem 91905 (ISRAEL)

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1. Introduction

The off-shell formulation of supersymmetry, and in particular superspace techniques have proven to be extremely useful in allowing for an easy quantization of supersymmetric theories (simplifying the computation of quantum corrections and displaying very clearly the improved ultraviolet behavior of these theories). This is true not only for the $N=1$ case, but also for the extended supersymmetric theories. A complete off-shell formulation for $N=2$ and $N=3$ supersymmetric theories has been invented and developed since 1984 by the GIKOS group (Galperin, Ivanov, Kalitzin, Ogievetsky and Sokatchev).^[1,2] They have shown that the relevant superspace, which they called harmonic superspace, involves additional bosonic variables belonging to $SU(2)/U(1)$ ($SU(3)/U(1)\times U(1)$) in the $N=2$ ($N=3$) cases. For the $N=2$ supersymmetric theories, the GIKOS group derived all the necessary Feynman rules and analyzed the quantum corrections.^[3] In particular, an $N=2$ superfield contains an infinite number of ordinary fields. Thus, an $N=2$ supergraph corresponds to the summation of an infinite number of ordinary graphs. Such a sum may diverge, and the corresponding infinities would appear as coinciding singularities in the additional (harmonic) variables. The absence of these singularities was demonstrated in [4].

The $N=3$ Super-Yang-Mills action was derived in [2]. This action is in some sense simpler than in the $N=1$ and $N=2$ cases, since it is polynomial in the gauge superfields (in fact, it only involves trilinear couplings!). Contrary to $N=2$, it is also local in the harmonic variables. Since $N=3$ Super-Yang-Mills (SYM) has $N=4$ on-shell supersymmetry, this theory will be free of ultraviolet singularities.^[5] It is probable that in $N=3$ harmonic superspace, due to improved power counting, ultraviolet finiteness is realized supergraph by supergraph, and not as a result of graph combinations as in the $N=1$ approach.^[6] To study the quantum corrections in superspace, we first need a supersymmetric gauge-fixing and the set of Feynman rules for $N=3$ supergraphs. The present work answers these two problems, and we postpone the study of ultraviolet finiteness to a future

publication.

The paper is organized as follows: In part 2, we review the model introduced by the GIKOS group. In section 3, we discuss the gauge condition and propose a supersymmetric gauge-fixed Lagrangian. We study the gauge-fixed equations of motion and show that their zero-modes are also zero-modes of the d'Alembertian. This implies the existence of the propagators, which are given in section 4. We conclude in section 5, discussing in particular the difficulties which will appear in the study of quantum corrections.

In appendix A, we describe the way to expand a function of harmonic variables. In appendix B, details are given about distributions on the coset spaces $SU(3)/U(1)\times U(1)$ and $SU(3)/SU(2)\times U(1)$.

2. N=3 Super-Yang-Mills in harmonic superspace

All details about the construction of this theory may be found in [1,2]. only the notations and the results will be presented here.

2.1 N=3 HARMONIC SUPERSPACE

The ordinary superspace of N=3 supersymmetry contains variables $(x^{\alpha\dot{\alpha}}, \theta_i^\alpha, \bar{\theta}^{i\dot{\alpha}})$ $i=1,2,3$.^{*} Harmonic superspace is obtained by adding variables $u_i^a, \bar{u}_a^i$, $a,i=1,2,3$, inert under supersymmetry and taking values in the fundamental representation of $SU(3)$:

$$U = (u_i^1, u_i^2, u_i^3), \quad U^\dagger = \begin{pmatrix} \bar{u}_1^i \\ \bar{u}_2^i \\ \bar{u}_3^i \end{pmatrix}, \quad (2.1)$$

$$U^\dagger U = U U^\dagger = 1, \quad \det U = 1. \quad (2.2)$$

* For spinors and spinorial derivatives, we follow the conventions of "Superspace".[7]

It will be useful to express constraints (2.2) directly on the u_i^a , $\bar{u}_a^i$ variables:

$$\bar{u}_a \cdot u^b = \bar{u}_a^i u_i^b = \delta_a^b , \quad (2.3)$$

$$u_i^a \bar{u}_a^j = \delta_i^j , \quad (2.4)$$

$$\epsilon^{ijk} u_i^1 u_j^2 u_k^3 = 1 . \quad (2.5)$$

Eqs. (2.3, 2.5) can be used to express the $\bar{u}_a^i$ variables in terms of the u_i^a , for instance:

$$\bar{u}_3^i = \epsilon^{ijk} u_j^1 u_k^2 , \quad (2.6)$$

u_i^1 , u_i^2 and u_i^3 are triplets under the left action of SU(3). This SU(3) action is the SU(3) automorphism group of the N=3 supersymmetry algebra. The u^a variables may in fact be seen as vielbeins which will be used to transform SU(3)-tensors $T^{i_1 \dots i_n}$ into SU(3) singlets:

$$T^{a_1 \dots a_n} = u_{i_1}^{a_1} \dots u_{i_n}^{a_n} T^{i_1 \dots i_n} .$$

The right action of SU(3) on "u" is generated by the differential operators:

$$D_b^a = u_i^a \frac{\partial}{\partial u_i^b} - \bar{u}_b^i \frac{\partial}{\partial \bar{u}_a^i} , \quad a \neq b, \quad (2.7)$$

$$Q_1 = u_i^1 \frac{\partial}{\partial u_i^1} - u_i^2 \frac{\partial}{\partial u_i^2} - \bar{u}_1^i \frac{\partial}{\partial \bar{u}_1^i} + \bar{u}_2^i \frac{\partial}{\partial \bar{u}_2^i} , \quad (2.8)$$

$$Q_2 = u_i^1 \frac{\partial}{\partial u_i^1} + u_i^2 \frac{\partial}{\partial u_i^2} - 2u_i^3 \frac{\partial}{\partial u_i^3} - \bar{u}_1^i \frac{\partial}{\partial \bar{u}_1^i} - \bar{u}_2^i \frac{\partial}{\partial \bar{u}_2^i} + 2\bar{u}_3^i \frac{\partial}{\partial \bar{u}_3^i} . \quad (2.9)$$

Q_1 and Q_2 span the Cartan subalgebra, and the D_b^a , called the harmonic derivatives in the following, are the step operators. Notice that the finite SU(3) transformations $e^{i\pi Q_1}$ and $e^{i\pi Q_2}$ coincide, and thus the eigenvalues of Q_1 and Q_2 must have the same parity.

On the spinors, we make the invertible change of coordinates:

$$\theta_a^\alpha = \bar{u}_a^i \theta_i^\alpha, \quad \bar{\theta}^{a\dot{\alpha}} = u_i^a \bar{\theta}^{i\dot{\alpha}}. \quad (2.10)$$

The corresponding covariant derivatives D_α^a , $\bar{D}_{a\dot{\alpha}}$ satisfy the algebra:

$$\{D_\alpha^a, D_\beta^b\} = 0, \quad (2.11)$$

$$\{\bar{D}_{a\dot{\alpha}}, \bar{D}_{b\dot{\beta}}\} = 0, \quad (2.12)$$

$$\{D_\alpha^a, \bar{D}_{b\dot{\beta}}\} = i\partial_{\alpha\dot{\beta}} \delta_b^a. \quad (2.13)$$

The commutation relations of the harmonic and spinorial derivatives are easily obtained:

$$[D_b^a, D_\alpha^c] = \delta_b^c D_\alpha^a, \quad (2.14)$$

$$[D_b^a, D_{c\dot{\alpha}}] = -\delta_c^a D_{b\dot{\alpha}}. \quad (2.15)$$

Finally, all derivatives are eigenvectors of the charge operators Q_1 , Q_2 . The eigenvalues q_1 , q_2 are given in table 1 for harmonic derivatives, and table 2 for spinorial derivatives.

Following GIKOS, we define the $*$ -transformation on "u" as the right action of the $SU(3)$ transformation:

$$S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.16)$$

$$U^* = U \cdot S. \quad (2.17)$$

The essential property of the $*$ -transformation is to transform objects with definite q_1 , q_2 charge into new objects which still have definite q_1, q_2 charge

(in other words, S belongs to the normalizer of $U(1) \times U(1)$). Among the many possible definitions of S , (2.16) happens to be the most suited to our purposes. The combined action of $*$ -transformation and complex conjugation, denoted by $^{\bar{*}}$, on harmonic variables, harmonic derivatives and spinorial derivatives is given in tables 3, 4, and 5.

Harmonic superfields $\Phi^{(q_1, q_2)}(x, \theta, \bar{\theta}, u)$ are functions of all variables with definite Q_1, Q_2 charges. They are thus functions "up to a phase" on $SU(3)/U(1) \times U(1)$ (more rigorously, they are sections of some line-bundle on $SU(3)/U(1) \times U(1)$). The $^{\bar{*}}$ -transformation acts on harmonic superfields through:

$$\Phi^{\bar{*}(q_1, q_2)}(x, \theta_i, \bar{\theta}^i, u) = \bar{\Phi}^{(q_1, q_2)}(x, \theta_i, \bar{\theta}^i, u^*) . \quad (2.18)$$

Notice that it satisfies:

$$(\Phi^{\bar{*}(q_1, q_2)})^{\bar{*}} = (-1)^{q_2} \Phi^{(q_1, q_2)} . \quad (2.19)$$

Thus, only superfields with even second charge may be $^{\bar{*}}$ -real. Various subspaces of the whole harmonic superspace, stable under $N=3$ supersymmetry may be defined. The most important one for our purposes is the analytic subspace, where the variables θ_1^α and $\bar{\theta}_\alpha^2$ do not appear explicitly. A superfield depending only on the variables of the analytic subspace may be defined by the supersymmetric constraints:

$$D_\alpha^1 \Phi^{(q_1, q_2)} = 0 , \quad \bar{D}_{2\dot{\alpha}} \Phi^{(q_1, q_2)} = 0 . \quad (2.20)$$

Such a superfield will be called analytic. Notice that $^{\bar{*}}$ transforms an analytic superfield into an analytic superfield. Furthermore, only the three harmonic derivatives D_3^1 , D_2^3 and D_2^1 are analyticity preserving, and they will thus play a special role. Eqs. (2.20) are not the most stringent supersymmetric constraints which can be imposed on a superfield.^[2] In particular, we can define the chiral

analytic constraints by:

$$D_\alpha^1 \Phi^{(q_1, q_2)} = \bar{D}_{2\dot{\alpha}} \Phi^{(q_1, q_2)} = \bar{D}_{3\dot{\alpha}} \Phi^{(q_1, q_2)} = 0 . \quad (2.21)$$

Antichiral analytic superfields satisfy:

$$D_\alpha^1 \Phi^{(q_1, q_2)} = \bar{D}_{2\dot{\alpha}} \Phi^{(q_1, q_2)} = D_\alpha^3 \Phi^{(q_1, q_2)} = 0 . \quad (2.22)$$

Finally, linear analytic superfields satisfy:

$$D_\alpha^1 \Phi^{(q_1, q_2)} = \bar{D}_{2\dot{\alpha}} \Phi^{(q_1, q_2)} = (\bar{D}_3)^2 \Phi^{(q_1, q_2)} = (D^3)^2 \Phi^{(q_1, q_2)} = 0 . \quad (2.23)$$

Notice that $*$ transforms a *chiral* analytic into an *antichiral* analytic superfield. Such superfields cannot be $*$ -real.

Finally, we need to introduce the SU(3) invariant measure $[du]$ on SU(3)/U(1) $\times$ U(1). We simply define integration by the following two properties:

$$\int [du] 1 = 1 , \quad \int [du] D_b^a f(u) = 0. \quad (2.24)$$

The meaning of (2.24) is that integration extracts the SU(3)-right invariant part of the integrand, and also one can integrate by parts. We can write down the measure on the full harmonic superspace:

$$dV_T = d^4x \, d^6\theta \, d^6\bar{\theta} \, [du]$$

and on the analytic subspace:

$$dV_A^{(-4,0)} = d^4x_A \, d^2\theta_2 \, d^2\theta_3 \, d^2\bar{\theta}^1 \, d^2\bar{\theta}^3 \, [du]$$

The superscript $(-4,0)$ refers to the (Q_1, Q_2) charges of the analytic measure. Notice that this measure has dimension zero.

Having built the N=3 harmonic superspace, we now turn, following again GIKOS, to the construction of the action of N=3 SYM.

2.2 N=3 SUPER-YANG-MILLS IN HARMONIC SUPERSPACE

The prepotentials of N=3 SYM are the gauge connections of the analyticity-preserving harmonic derivatives:

$$\begin{aligned} \mathbf{D}_2^1 &= D_2^1 + iV_2^1 , \\ \mathbf{D}_3^1 &= D_3^1 + iV_3^1 , \\ \mathbf{D}_2^3 &= D_2^3 + iV_2^3 . \end{aligned} \quad (2.25)$$

where V_2^1 , V_3^1 , V_2^3 are analytic superfields taking value in the Lie algebra of the gauge group G. Gauge transformations read:

$$\mathbf{D}_b^a \Longrightarrow e^{+i\Lambda} \mathbf{D}_b^a e^{-i\Lambda} , \quad (2.26)$$

where the gauge parameter Λ is also an analytic superfield:

$$D_\alpha^1 \Lambda = \bar{D}_{2\dot{\alpha}} \Lambda = 0 . \quad (2.27)$$

Thus the flat D_α^1 and $\bar{D}_{2\dot{\alpha}}$ are covariant spinorial derivatives. All the superfields introduced up to now have definite $\pm$ -transformations:

$$V_2^{1\pm} = V_2^1 , \quad V_3^{1\pm} = V_3^1 , \quad \text{and} \quad V_2^{3\pm} = -V_2^3 , \quad (2.28)$$

$$\Lambda^\pm = \Lambda . \quad (2.29)$$

The field-strengths corresponding to the gauge connections (2.25), labelled by their (Q_1, Q_2) charges, are:

$$\begin{aligned} [\mathbf{D}_2^1, \mathbf{D}_3^1] &= F^{(3,3)} , \\ [\mathbf{D}_2^3, \mathbf{D}_2^1] &= F^{(3,-3)} , \\ [\mathbf{D}_3^1, \mathbf{D}_2^3] &= \mathbf{D}_2^1 + F^{(2,0)} . \end{aligned} \quad (2.30)$$

They satisfy the Bianchi identity:

$$\mathbf{D}_2^1 F^{(2,0)} + \mathbf{D}_3^1 F^{(3,-3)} + \mathbf{D}_2^3 F^{(3,3)} = 0 . \quad (2.31)$$

To write down the action, one introduces the interpolating connections tV_b^a ,

$0 \leq t \leq 1$, and the corresponding field strengths $F_t^{(q_1, q_2)}$, obtained by replacing everywhere in (2.30) V_b^a by tV_b^a . The gauge invariant action of N=3 SYM then reads:

$$S_{inv} = \int dV_A^{(-4,0)} \mathbf{L}^{(4,0)} , \quad (2.32)$$

where:

$$\mathbf{L}^{(4,0)} = -i \int_0^1 dt \text{Tr} \{ V_2^1 F_t^{(2,0)} + V_3^1 F_t^{(3,-3)} + V_2^3 F_t^{(3,3)} \} . \quad (2.33)$$

Notice that the Lagrangian density (2.33) is not gauge invariant. However, its variation under an infinitesimal gauge transformation may be shown to be a total (harmonic) derivative. The field equations corresponding to the action (2.32) are given by:

$$F^{(3,3)} = F^{(3,-3)} = F^{(2,0)} = 0 .$$

It has been shown in ref.[2] that these field equations lead back to the canonical on-shell constraints [8] of N=3 SYM.

This off-shell version of N=3 SYM is peculiar in many respects. First, as in N=2 harmonic superspace, the number of auxiliary and gauge fields is infinite (they are obtained by expanding the harmonic superfields with respect to the SU(3) variables "u"). This is necessary to overcome some no-go theorems.[9] Second, there is no equivalent of the N=2 Wess- Zumino gauge [3,7], where only the physical fields, plus a finite number of auxiliary fields, survive. In the N=3 case, there are, in fact, an infinite number of auxiliary (non-gauge) fields of any dimension between 0 and 4.

To check that there are indeed non-gauge fields of dimension 0, we look at the expansion of the $\theta = \bar{\theta} = 0$ components of V_3^1 and Λ using results from appendix A. It will be sufficient to study the SU(3) representations in this expansion

corresponding to the tableau:

$$(\bar{u}_{(a}\bar{u}_{b)})^{ij}(u^{(c}u^{d)})_{kl} . \quad (2.34)$$

Using the constraints coming from the charges, this representation appears in V_3^1 as follows:

$$V_3^1|_{\theta=\bar{\theta}=0} = \cdots + \begin{pmatrix} 4 & 2 & 0 \\ & 3 & 2 \\ & & 3 \end{pmatrix}_{kl}^{ij} F_{ij}^{kl} + \begin{pmatrix} 4 & 2 & 0 \\ & 4 & 1 \\ & & 3 \end{pmatrix}_{kl}^{ij} G_{ij}^{kl} + \cdots \quad (2.35)$$

Where F and G are both complex tensors. In the expansion of Λ , we have:

$$\Lambda|_{\theta=\bar{\theta}=0} = \cdots + \begin{pmatrix} 4 & 2 & 0 \\ & 4 & 0 \\ & & 2 \end{pmatrix}_{kl}^{ij} A_{ij}^{kl} + \begin{pmatrix} 4 & 2 & 0 \\ & 3 & 1 \\ & & 2 \end{pmatrix}_{kl}^{ij} B_{ij}^{kl} + \begin{pmatrix} 4 & 2 & 0 \\ & 2 & 2 \\ & & 2 \end{pmatrix}_{kl}^{ij} C_{ij}^{kl} + \cdots \quad (2.36)$$

However, the $\pm$ -reality of Λ gives some constraints on A,B and C:

$$\bar{A}_{ij}^{kl} = A_{kl}^{ij}, \quad \bar{B}_{ij}^{kl} = -B_{kl}^{ij}, \quad \text{and} \quad \bar{C}_{ij}^{kl} = C_{kl}^{ij} . \quad (2.37)$$

Thus, Λ contains only three real fields corresponding to the representation (2.34), and this is not enough to get rid of the two, zero-dimension, complex, tensor fields, F and G.

3. Gauge fixing

3.1 GAUGE CONDITION

The gauge condition that we shall use is very reminiscent of the one found in N=2 superspace.^[3] It reads:

$$D_2^1 V_2^1 = 0 \quad . \quad (3.1)$$

To check that (3.1) is a good gauge condition, let us first examine the residual gauge freedom. An infinitesimal gauge transformation with parameter Λ respects (3.1) if Λ satisfies the (linearized) equation:

$$(D_2^1)^2 \Lambda = 0 \quad . \quad (3.2)$$

Eq. (3.2) restricts the SU(2) irreducible representations (IR) contained in Λ , where SU(2) is the subgroup generated by D_2^1 , D_1^2 , and Q_1 . The highest weights of these irreducible representations can only have a Q_1 charge equal to 0 or 2. It is then clear that an equation equivalent to (3.2) is:

$$(D_1^2)^2 \Lambda = 0 \quad . \quad (3.3)$$

The LHS of eq. (3.3) is not analytic. We apply to it the analytic operator $(D^1)^2(\bar{D}_2)^2$, and use the analyticity of Λ to get:

$$(D^1)^2(\bar{D}_2)^2(D_1^2)^2 \Lambda = 2\partial^2 \Lambda = 0 \quad . \quad (3.4)$$

Thus, the residual gauge freedom satisfies the usual constraint that its parameter is annihilated by the space-time d'Alembertian.

Next, we have to show that the gauge (3.1) can be reached from arbitrary V_2^1 's. Thus, again at the linearized level, one must be able to find, for any

connection V_2^1 , an analytic gauge parameter Λ such that:

$$(D_2^1)^2 \Lambda = D_2^1 V_2^1 . \quad (3.5)$$

To invert eq. (3.5), we need an analytic distribution satisfying:

$$(D_2^1(2))^2 \mathbf{g}^{(00|00)}(1|2) = \delta_A^{(00|40)}(1|2) . \quad (3.6)$$

Where (1) stands for the set of coordinates $(x, \theta, \bar{\theta}, u)$ and (2) for $(x', \theta', \bar{\theta}', v)$; the RHS is the analytic delta function:

$$\delta_A^{(00|40)}(1|2) = (D^1(2))^2 (\bar{D}_2(2))^2 \delta^{16}(1|2) \delta^{(00|00)}(u, v) , \quad (3.7)$$

with

$$\delta^{16}(1|2) = \delta^4(x - x') \delta^6(\theta - \theta') \delta^6(\bar{\theta} - \bar{\theta}') . \quad (3.8)$$

The distribution $\mathbf{g}^{(00|00)}(1|2)$ is easily computed using the results of appendix B. It reads:

$$\mathbf{g}^{(00|00)}(1|2) = \frac{(D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{\partial^2(1)} \delta^{16}(1|2) \mu^{(-40|-40)}(u, v). \quad (3.9)$$

Where $\mu^{(-40|-40)}(u, v)$ satisfies:

$$(D_2^1(2))^2 \mu^{(-40|-40)}(u, v) = \frac{1}{2} (D_1^2(2))^2 \delta^{(-40|40)}(u, v) , \quad (3.10)$$

and can be written as:

$$\mu^{(-40|-40)}(u, v) = -\frac{(u^2 \bar{v}_1)}{(u^1 \bar{v}_2)^3} \Delta^{(2, -2)}(u, v) . \quad (3.11)$$

Using the distribution $\mathbf{g}^{(00|00)}$, we can solve eq. (3.5) to obtain:

$$\Lambda(1) = \int d[V(2)]_A^{(-4, 0)} \mathbf{g}^{(00|00)}(1|2) D_2^1(2) V_2^1(2) . \quad (3.12)$$

Finally, let us show that (3.1) reduces to the usual transversality condition on

the physical vector field. This field is given by:

$$A_{\alpha\dot{\alpha}}(x) = \int [du] [D_{\alpha}^2 \bar{D}_{1\dot{\alpha}} V_2^1]_{\theta=\bar{\theta}=0} . \quad (3.13)$$

(Remember that $[du]$ extracts the $SU(3)$ singlet part.). We shall get the condition satisfied by the vector fields $A_{\alpha\dot{\alpha}}(x)$ by taking appropriate spinor derivatives of the gauge condition (3.1):

$$(D^2)^2 (\bar{D}_1)^2 D_2^1 V_2^1 = D_2^1 ((D^2)^2 (\bar{D}_1)^2 V_2^1) - (D^2 \cdot D^1) (\bar{D}_1)^2 V_2^1 = 0 . \quad (3.14)$$

We use the analyticity of V_2^1 to obtain:

$$D_2^1 ((D^2)^2 (\bar{D}_1)^2 V_2^1) + i \partial^{\alpha\dot{\alpha}} (D_{\alpha}^2 \bar{D}_{1\dot{\alpha}} V_2^1) = 0 . \quad (3.15)$$

We then take $\theta = \bar{\theta} = 0$ and integrate over u to get:

$$\partial^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}} = 0 . \quad (3.16)$$

3.2 GAUGE FIXING ACTION

Let us now implement the gauge condition (3.1) in the action. We shall simply impose the gauge condition (3.1) by using an analytic, $\pm$ -real Lagrange multiplier $\lambda(x, \theta, \bar{\theta}, u)$. The action for the ghost fields $c, \bar{c}$, which are analytic, $\pm$ -real anticommuting superfields taking value in the Lie algebra of the gauge group G , is obtained by requiring the invariance of the gauge-fixing action with respect to BRST transformations ^[10]:

$$\begin{aligned} \delta V_b^a &= -[D_b^a c + i[V_b^a, c]] , \\ \delta \bar{c} &= \lambda , \quad \delta \lambda = 0 , \\ \delta c &= \frac{i}{2} [c, c] . \end{aligned} \quad (3.17)$$

We then obtain the gauge-fixing action:

$$S_{gf}^{(4,0)} = Tr \{ \lambda D_2^1 V_2^1 + \bar{c} D_2^1 [D_2^1 c + i[V_2^1, c]] \} . \quad (3.18)$$

3.3 GAUGE FIXED EQUATIONS OF MOTION

The computation of the propagators involves the inversion of the quadratic part of the Lagrangian density (2.33, 3.18). This will be possible if and only if the linearized gauge-fixed equations of motion have no zero modes other than those of the d'Alembertian. We already know from the discussion in Sec.(3.1) that the gauge freedom (2.26) is correctly fixed. However, it could happen that the quadratic invariant Lagrangian has more gauge symmetries, left unbroken by the gauge-fixing Lagrangian (3.18). We shall see that this is not the case.

The linearized field equations, plus gauge condition, read:

$$D_2^1 V_2^1 = 0 \quad . \quad (3.19)$$

$$V_2^1 - D_3^1 V_2^3 + D_2^3 V_3^1 = 0 \quad . \quad (3.20)$$

$$D_2^1 V_3^1 = D_3^1 V_2^1 \quad . \quad (3.21)$$

$$D_2^1 V_2^3 = D_2^3 V_2^1 \quad . \quad (3.22)$$

One can use (3.19) and the algebra of the harmonic operators to rewrite (3.21) and (3.22) as:

$$\begin{aligned} D_2^1 (V_3^1 - D_3^2 V_2^1) &= 0 \quad , \\ D_2^1 (V_2^3 + D_1^3 V_2^1) &= 0 \quad . \end{aligned} \quad (3.23)$$

Using the properties of SU(2) IR's, it is easy to show that, if $D_2^1 F^{(q_1, q_2)} = 0$ with $q_1 < 0$, then $F^{(q_1, q_2)} = 0$. However, the functions appearing in eqs. (3.23) have Q_1 -charge equal to +1. Therefore, we are going to use some spinorial derivatives to lower the Q_1 -charge. As a first step, it is convenient to define the non-analytic

superfields:

$$v_b^a = (D^3)^2(D^2)^2 V_b^a . \quad (3.24)$$

Using the analyticity of V_b^a , one gets the same set of equations for these new fields:

$$D_2^1 v_2^1 = 0 , \quad (3.25)$$

$$v_2^1 - D_3^1 v_2^3 + D_2^3 v_3^1 = 0 , \quad (3.26)$$

and:

$$D_2^1(v_3^1 - D_3^2 v_2^1) = 0 , \quad (3.27)$$

$$D_2^1(v_2^3 + D_1^3 v_2^1) = 0 .$$

However, the fields in (3.27) have Q_1 -charge equal to -1, and thus they vanish:

$$\begin{aligned} v_3^1 - D_3^2 v_2^1 &= 0 , \\ v_2^3 + D_1^3 v_2^1 &= 0 . \end{aligned} \quad (3.28)$$

We can then use (3.28) to rewrite eq. (3.26), which becomes:

$$(\mathbf{C} + \frac{2}{3})v_2^1 = 0 . \quad (3.29)$$

Here, $\mathbf{C}$ is the $SU(3)$ Casimir operator, defined in appendix B, (B.22). This operator has only positive eigenvalues, and thus eqs. (3.29) and (3.28) imply:

$$v_b^a = (D^3)^2(D^2)^2 V_b^a = 0 . \quad (3.30)$$

Applying the operator $(\bar{D}_2)^2$ to eq. (3.30) and using the analyticity of the harmonic connections, we obtain:

$$\partial^2(D^3)^2 V_b^a = 0 \quad (3.31)$$

and, taking the $\bar{\partial}$ of these equations:

$$\partial^2(\bar{D}_3)^2 V_b^a = 0 . \quad (3.32)$$

Eqs. (3.31) and (3.32) show the vanishing of the *chiral* and *antichiral* parts

of the fields $\partial^2 V_b^a$. A further effort is needed to gain information about the *linear* part. We use this time the spinorial superfields:

$$w_b^{a\dot{\alpha}} = (D^3)^2 \bar{D}_3^{\dot{\alpha}} (D^2)^2 V_b^a , \quad (3.33)$$

using (3.30),

$$w_b^{a\dot{\alpha}} = -i \partial^{\alpha\dot{\alpha}} D_\alpha^3 (D^2)^2 V_b^a$$

and repeating the same calculations as for v_b^a , we obtain:

$$\begin{aligned} w_3^{1\dot{\alpha}} &= D_3^2 w_2^{1\dot{\alpha}} , \\ w_2^{3\dot{\alpha}} &= -D_1^3 w_2^{1\dot{\alpha}} . \end{aligned} \quad (3.34)$$

and:

$$(\mathbf{C} + 1) w_2^{1\dot{\alpha}} = 0 . \quad (3.35)$$

The positivity of the Casimir operator again implies:

$$w_b^{a\dot{\alpha}} = (D^3)^2 \bar{D}_3^{\dot{\alpha}} (D^2)^2 V_b^a = 0 . \quad (3.36)$$

Acting with $(\bar{D}_2)^2$, we get:

$$(D^3)^2 \bar{D}_3^{\dot{\alpha}} \partial^2 V_b^a = 0 . \quad (3.37)$$

Eqs. (3.31), (3.32), and (3.37) can be combined using the completeness relation:

$$\partial^2 = (D^3)^2 (\bar{D}_3)^2 + (\bar{D}_3)^2 (D^3)^2 + \bar{D}_{3\dot{\alpha}} (D^3)^2 \bar{D}_3^{\dot{\alpha}} \quad (3.38)$$

to obtain finally:

$$(\partial^2)^2 V_b^a = 0 . \quad (3.39)$$

This terminates our proof that the gauge-fixed field equations fix uniquely all components of the V_b^a 's, up to on-shell fields ($\partial^2 \Phi(x) = 0$). This, in turn, implies that the vector propagators are non-singular in Euclidean space ($k^2 \neq 0$).

4. Propagators

From (2.33) and (3.18), the quadratic part of the full Lagrangian is:

$$\begin{aligned} \mathbf{L}_2^{(4,0)} = & Tr \{ \frac{1}{2} (V_2^1 (D_3^1 V_2^3 - D_2^3 V_3^1) + V_3^1 (D_2^3 V_2^1 - D_2^1 V_2^3) \\ & + V_2^3 (D_2^1 V_3^1 - D_3^1 V_2^1)) + \lambda D_2^1 V_2^1 + \bar{c} (D_2^1)^2 c \} . \end{aligned} \quad (4.1)$$

We chose a basis T^a of the algebra of the gauge group such that:

$$Tr T^a T^b = \delta^{ab} . \quad (4.2)$$

Then the propagators of any two fields ϕ_a^1, ϕ_b^2 may be factorized as:

$$\mathbf{G}_{ab}^{(\phi^1 \phi^2)}(1|2) = \delta_{ab} \mathbf{G}^{(\phi^1 \phi^2)}(1|2) . \quad (4.3)$$

We first compute the ghost propagator $\mathbf{g}^{(00|00)}(1|2) = \langle c(1) \bar{c}(2) \rangle_+$ which satisfies:

$$(D_2^1(1))^2 \mathbf{g}^{(00|00)}(1|2) = \delta_A^{(4,0|0,0)}(1|2) . \quad (4.4)$$

Such a distribution has already been met in eq. (3.6) and its expression was given in eqs. (3.9-11).

Next, the propagators of the gauge fields and Lagrange multiplier $(\lambda, V_2^1, V_2^3, V_3^1)$ satisfy the matrix equation:

$$\begin{aligned} & \begin{pmatrix} 0 & D_2^1 & 0 & 0 \\ -D_2^1 & -1 & D_3^1 & -D_2^3 \\ 0 & -D_3^1 & 0 & +D_2^1 \\ 0 & +D_2^3 & -D_2^1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{G}^{(00|00)} & \mathbf{G}^{(00|20)} & \mathbf{G}^{(00|1-3)} & \mathbf{G}^{(00|13)} \\ \mathbf{G}^{(20|00)} & \mathbf{G}^{(20|20)} & \mathbf{G}^{(20|1-3)} & \mathbf{G}^{(20|13)} \\ \mathbf{G}^{(1-3|00)} & \mathbf{G}^{(1-3|20)} & \mathbf{G}^{(1-3|1-3)} & \mathbf{G}^{(1-3|13)} \\ \mathbf{G}^{(13|00)} & \mathbf{G}^{(13|20)} & \mathbf{G}^{(13|1-3)} & \mathbf{G}^{(13|13)} \end{pmatrix} \\ & = \begin{pmatrix} \delta_A^{(40|00)} & 0 & 0 & 0 \\ 0 & \delta_A^{(20|20)} & 0 & 0 \\ 0 & 0 & \delta_A^{(33|1-3)} & 0 \\ 0 & 0 & 0 & \delta_A^{(3-3|13)} \end{pmatrix} . \end{aligned} \quad (4.5)$$

We only need half of the propagator matrix, thanks to the relation:

$$\mathbf{G}^{(q_1 q_2 | q'_1 q'_2)}(1|2) = \mathbf{G}^{(q'_1 q'_2 | q_1 q_2)}(2|1) . \quad (4.6)$$

The elements of the first column are related through the BRST symmetry to the ghost propagator, and are thus easily determined:

$$\begin{aligned} \mathbf{G}^{(00|00)}(1|2) &= 0 , \\ \mathbf{G}^{(20|00)}(1|2) &= D_2^1(1) \mathbf{g}^{(00|00)}(1|2) , \\ \mathbf{G}^{(1-3|00)}(1|2) &= D_2^3(1) \mathbf{g}^{(00|00)}(1|2) , \\ \mathbf{G}^{(13|00)}(1|2) &= D_3^1(1) \mathbf{g}^{(00|00)}(1|2) . \end{aligned} \quad (4.7)$$

The remaining elements are more difficult to compute. The method we use is identical to the one described in section (3.3) for the resolution of the gauge-fixed equations of motion. Again, the method involves consecutive steps which determine the *chiral*, *antichiral*, and *linear* analytic parts of the propagator. The results are expressed in terms of the distributions:

$$\begin{aligned} \mathbf{g}_{CA}^{(2,2|2,-2)}(1|2) &= \frac{(\bar{D}_3(1))^2 (D^3(1))^2 (D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{(v^1 \bar{u}_2)^2 (\partial^2)^2} \delta^{16}(1|2), \\ \mathbf{g}_{AC}^{(2,-2|2,2)}(1|2) &= \frac{(D^3(1))^2 (\bar{D}_3(1))^2 (D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{(u^1 \bar{v}_2)^2 (\partial^2)^2} \delta^{16}(1|2), \\ \mathbf{g}_{LL}^{(2,0|2,0)}(1|2) &= \frac{D^{3\alpha}(1) (\bar{D}_3(1))^2 D_\alpha^3(1) (D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{(v^1 \bar{u}_2)(u^1 \bar{v}_2)(\partial^2)^2} \delta^{16}(1|2) \end{aligned} \quad (4.8)$$

where the subscripts C,A, and L indicate that the distribution is *chiral*, *antichiral*, or *linear* analytic in the first or second set of variables. The corresponding constraints are defined in (2.21-23).

Furthermore, these distributions have non-singular limits as $u^1 \bar{v}_2 \longrightarrow 0$ or $v^1 \bar{u}_2 \longrightarrow 0$ and also satisfy the following properties:

$$\begin{aligned} \mathbf{g}_{AC}^{(2,-2|2,2)}(1|2) &= \mathbf{g}_{CA}^{(2,2|2,-2)}(2|1) = \mathbf{g}_{CA}^{(2,2|2,-2)\dagger}(1|2) \\ \mathbf{g}_{LL}^{(2,0|2,0)}(1|2) &= \mathbf{g}_{LL}^{(2,0|2,0)}(2|1) = \mathbf{g}_{LL}^{(2,0|2,0)\dagger}(1|2) . \end{aligned} \quad (4.9)$$

Finally, the distributions (4.8) are annihilated by the analyticity preserving har-

monic derivatives D_2^1, D_3^1 and D_2^3 .

$$D_b^a g_e(1|2) = 0, \quad (e) = (AC), (CA), (LL) \text{ and } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \quad (4.10)$$

To prove these properties, it is very often useful to transfer some spinorial derivatives from the (1) to the (2) variables, of the $\delta^{16}(1|2)$, or the converse. To do this, we use the completeness relation (2.4) and obtain:

$$D_\alpha^a(2)\delta^{16}(1|2) = -(v^a \bar{u}_1 D_\alpha^1(1) + v^a \bar{u}_2 D_\alpha^2(1) + v^a \bar{u}_3 D_\alpha^3(1))\delta^{16}(1|2). \quad (4.11)$$

We apply (4.11) to obtain a form of $\mathbf{g}_{CA}^{(2,2|2,-2)}(1|2)$ that is manifestly non-singular as $v^1 \bar{u}_2 \rightarrow 0$:

$$\begin{aligned} \mathbf{g}_{CA}^{(2,2|2,-2)}(1|2) &= \frac{-(\bar{D}_3(1))^2(\bar{D}_2(1))^2(D^3(1))^2(D^1(1))^2(\bar{D}_2(2))^2 D^{1\alpha}(2)}{2(\partial^2)^2} x \\ &\quad x(v^1 \bar{u}_1 D_\alpha^1(1) + v^1 \bar{u}_2 D_\alpha^2(1) + v^1 \bar{u}_3 D_\alpha^3(1)) \frac{\delta^{16}(1|2)}{(v^1 \bar{u}_2)^2}. \end{aligned}$$

The first and the third terms between the parentheses vanish because there are too many spinor derivatives with the same index. Applying again (4.11), we obtain:

$$\begin{aligned} \mathbf{g}_{CA}^{(2,2|2,-2)}(1|2) &= \\ &= \frac{1}{(\partial^2)^2} (\bar{D}_3(1))^2 (\bar{D}_2(1))^2 (D^3(1))^2 (D^2(1))^2 (D^1(1))^2 (\bar{D}_2(2))^2 \delta^{16}(1|2). \end{aligned}$$

We introduce the notations:

$$D^6 = (D^1)^2 (D^2)^2 (D^3)^2 \text{ and } \bar{D}^6 = (\bar{D}_1)^2 (\bar{D}_2)^2 (\bar{D}_3)^2.$$

Notice that these two operators are SU(3) singlets (i.e. they are independent of

the harmonic variables u^a). We can then rewrite $\mathbf{g}_{CA}^{(2,2|2,-2)}(1|2)$ as:

$$\begin{aligned}\mathbf{g}_{CA}^{(2,2|2,-2)}(1|2) &= \frac{(\bar{D}_3(1))^2(\bar{D}_2(1))^2 D^6(1)(\bar{D}_2(2))^2}{(\partial^2)^2} \delta^{16}(1|2) \\ &= \frac{(D^1(1))^2 \bar{D}^6(1)(\bar{D}_2(2))^2 D^6(2)}{(\partial^2)^3} \delta^{16}(1|2) .\end{aligned}\quad (4.12)$$

Obviously, these last two expressions are non-singular as $v^1 \bar{u}_2 \rightarrow 0$. Similar forms may be obtained for $\mathbf{g}_{AC}^{(2,-2|2,2)}(1|2)$ and $\mathbf{g}_{LL}^{(2,0|2,0)}(1|2)$:

$$\begin{aligned}\mathbf{g}_{AC}^{(2,-2|2,2)}(1|2) &= \frac{1}{(\partial^2)^3} (\bar{D}_2(1))^2 D^6(1) (D^1(2))^2 \bar{D}^6(2) \delta^{16}(1|2) , \\ \mathbf{g}_{LL}^{(2,0|2,0)}(1|2) &= \frac{i \partial^{\alpha\dot{\alpha}}(1)}{(\partial^2)^4} \bar{D}_{3\dot{\alpha}}(1) (\bar{D}_2(1))^2 D^6(1) D_\alpha^3(2) (D^1(2))^2 \bar{D}^6(2) \delta^{16}(1|2) .\end{aligned}\quad (4.13)$$

Using these expressions, it is easy to check all properties of distributions (4.8). One can also demonstrate these properties directly, using eqs. (B.31) and (B.32) of appendix B.

Next, we introduce the notation:

$$\begin{aligned}\mathbf{H}(C_A, C_B) &= -\mathbf{g}_{AC}^{(2,-2|2,2)} C_A(1) C_B(2) \{v^3 \bar{u}_3 \Theta(u, v)\} \\ &\quad -\mathbf{g}_{CA}^{(2,2|2,-2)} C_A(1) C_B(2) \{u^3 \bar{v}_3 \Theta(u, v)\} \\ &\quad -\mathbf{g}_{LL}^{(2,0|2,0)} C_A(1) C_B(2) \{\Theta(u, v)\}\end{aligned}\quad (4.14)$$

where C_A and C_B are harmonic operators, and $\Theta(u, v)$ is the "inverse Casimir" defined in appendix B, eq. (B.25):

$$\Theta(u, v) = \frac{1}{2(1 - |u^3 v_3|^2)} \quad (4.15)$$

We shall also need the distribution:

$$\pi^{(13|1-3)}(1|2) = \frac{(D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{\partial^2} \delta^{16}(1|2) \frac{\Delta^{(0,0)}(u, v)}{(v^1 \bar{u}_2)^3} , \quad (4.16)$$

satisfying the eq. $D_2^1 \pi^{(13|1-3)}(1|2) = \delta_A^{(33|1-3)}(1|2)$. $\Delta^{(0,0)}(u, v)$ is the delta-function on $SU(3)/SU(2) \times U(1)$ given in appendix B, eq. (B.7). We are now in a

position to write down all elements of the propagator matrix:

$$\begin{aligned}
\mathbf{G}^{(20|20)}(1|2) &= \mathbf{H}(1,1) \ , \\
\mathbf{G}^{(1-3|20)}(1|2) &= -\mathbf{H}(D_1^3, 1) \ , \\
\mathbf{G}^{(13|20)}(1|2) &= \mathbf{H}(D_3^2, 1) \ , \\
\mathbf{G}^{(1-3|1-3)}(1|2) &= \mathbf{H}(D_1^3, D_1^3) \ , \\
\mathbf{G}^{(13|1-3)}(1|2) &= \pi^{(13|1-3)}(1|2) - \mathbf{H}(D_3^2, D_1^3) \ , \\
\mathbf{G}^{(13|13)}(1|2) &= \mathbf{H}(D_3^2, D_3^2) \ .
\end{aligned} \tag{4.17}$$

Eqs.(4.17) constitute the main result of this chapter. In the remaining, we are only going to check that (4.17) indeed solves the equations for the second column of the propagator matrix. These equations read:

$$D_2^1 \mathbf{G}^{(20|20)}(1|2) = 0 \ , \tag{4.18}$$

$$\mathbf{G}^{(20|20)}(1|2) - D_3^1 \mathbf{G}^{(1-3|20)}(1|2) + D_2^3 \mathbf{G}^{(13|20)}(1|2) = \pi^{(20|20)}(1|2) \ , \tag{4.19}$$

$$D_2^1 \mathbf{G}^{(1-3|20)}(1|2) = D_2^3 \mathbf{G}^{(20|20)}(1|2) \ , \tag{4.20}$$

$$D_2^1 \mathbf{G}^{(13|20)}(1|2) = D_3^1 \mathbf{G}^{(20|20)}(1|2) \ , \tag{4.21}$$

where:

$$\begin{aligned}
\pi^{(20|20)}(1|2) &= -\delta_A^{(20|20)}(1|2) - D_2^1(1) D_2^1(2) \mathbf{g}^{(00|00)}(1|2) \\
&= \frac{-(D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{\partial^2} \delta^{16}(1|2) \frac{\Delta^{(2,-2)}(u,v)}{(u^1 \bar{v}_2)^2}
\end{aligned} \tag{4.22}$$

Using the fact that the derivatives D_2^1 , D_3^1 and D_2^3 annihilate the distributions

(4.8), and also:

$$D_2^1 \Theta(u, v) = 0 \quad , \quad (4.23)$$

eq. (4.18) is easily shown to be satisfied. Eq. (4.20) becomes:

$$\begin{aligned} D_2^1 \mathbf{G}^{(1-3|20)} &= -D_2^1 \mathbf{H}(D_1^3, 1) = -\mathbf{H}(D_2^1 D_1^3, 1) \\ &= +\mathbf{H}(D_2^3, 1) = D_2^3 \mathbf{H}(1, 1) \\ &= D_2^3 \mathbf{G}^{(20|20)} \end{aligned} \quad (4.24)$$

and analogously for eq. (4.21). Finally, from (4.17) the left-hand side of (4.19) may be rewritten as:

$$\begin{aligned} LHS &= \mathbf{G}^{(20|20)}(1|2) - D_3^1 \mathbf{G}^{(1-3|20)}(1|2) + D_2^3 \mathbf{G}^{(13|20)}(1|2) \\ &= \mathbf{H}(1 + D_3^1 D_1^3 + D_2^3 D_3^2, 1) \quad . \end{aligned}$$

Using eqs. (B.24), (B.30) of appendix B, and (4.23):

$$LHS = -\{\mathbf{g}_{AC}^{(2-2|22)}(v^3 \bar{u}_3) + \mathbf{g}_{CA}^{(22|2-2)}(u^3 \bar{v}_3) + \mathbf{g}_{LL}^{(20|20)}\} \Delta^{(0,0)}(u, v) \quad .$$

We can then use the completeness relation (3.38), and also (B.17) to obtain:

$$LHS = \frac{-(D^1(1))^2 (\bar{D}_2(1))^2 (D^1(2))^2 (\bar{D}_2(2))^2}{\partial^2} \delta^{16}(1|2) \frac{(v^3 \bar{u}_3)}{(u^1 \bar{v}_2)^2} \Delta^{(2,-2)}(u, v) \quad (4.25)$$

which is identical to the right-hand side of eq. (4.19) as defined in (4.22). The other equations may be verified in a similar way.

Finally, the three-point vertices are immediately deduced from the interaction Lagrangian, which is (eqs. (2.33) and (3.18)):

$$\mathbf{L}_{INT} = -i Tr \{ V_2^1 [V_2^3, V_3^1] + D_2^1 \bar{c} [V_2^1, c] \} \quad (4.26)$$

5. Conclusion

The Feynman rules for off-shell $N=3$ super-Yang-Mills are summarized by the ghost and vector propagators (4.4,17) and the interaction action of eq. (4.26). These results, together with the explicit form of the harmonic distributions given in appendix B, suffice for all graph calculations.

A surprising fact in this computation is the necessity to split the vector propagators into parts which satisfy stronger supersymmetric constraints (denoted by *chiral*, *antichiral* and *linear analytic*) than the original fields.

Let us note also that we have stayed throughout this work in a particular gauge (the Landau gauge). Covariant gauges are now under investigation.

The major advantage of a superspace quantization of $N=3$ super-Yang-Mills is that the proof of the ultraviolet finiteness of the theory should be fairly easy. In fact, it was remarked by the authors of ref. [3] that the momentum integrals will be ultraviolet finite if one can do two things: use spinor derivatives to reconstruct integrals over the full harmonic superspace at each vertex $(dV_A^{(-4,0)}) \rightarrow (dV_T)$, and write the propagators using spinor derivatives acting on $\delta^{16}(1|2)$ without any other θ -dependence. These conditions are met by the propagators in section 4. Repeating non-renormalization arguments, like in $N=1$ supersymmetric theories^[6], by first doing integrations over all θ 's except one, we can reduce any graph to a sum of terms:

$$\prod_{\text{all vertices}} du_v \int d^{12}\theta \prod_{e=1}^N \frac{d^4 p_e}{(2\pi^4)} \left[(D_\alpha^A \dots) V_b^a(p_1, u_1, \theta) \dots (D_\beta^B \dots) V_d^c(p_N, u_N, \theta) \right]$$

$$\cdot \delta^4\left(\sum_{l=1}^N p_e\right) \prod_{l=1}^L \int \frac{d^4 k_l}{(2\pi^4)} F(k_l, p_e, u_v).$$

where the V_b^a are external sources of dimension zero, on which some spinorial derivatives may act. N is the number of external vertices and L is the number of loops. The corresponding momenta are denoted respectively by p_e and k_l . Since the total graph is of dimension zero, it is clear that the loop integrations are superficially convergent. One should notice that there may still be divergences due to coinciding singularities in the " u " harmonic distributions. Preliminary investigations indicate, as in the $N=2$ case^[4], that these divergences do not occur, graph by graph, after integration on the Grassman variables.

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APPENDIX A

Expansion of harmonic functions

Finding a basis for the expansion of functions on SU(3) is more complicated than in the SU(2) case, where a basis is given by the completely symmetrized tensors (see [1] for notations)

$$u_{i_1}^+ \dots u_{i_m}^+ u_{i_{m+1}}^- \dots u_{i_n}^- \quad (A.1)$$

In the case of SU(3), we would like to use the fact that a complete set of irreducible representations (IR) is given by the tableaux with 2 covariant lines, or with 1 covariant and 1 contravariant lines

$$\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \quad \equiv \quad \begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & & & \\ \hline \end{array} \quad (A.2)$$

The corresponding set of tensors is given by:

$$\bar{u}_1^{(i_1} \dots \bar{u}_1^{i_p} \bar{u}_2^{i_{p+1}} \dots \bar{u}_2^{i_q} \bar{u}_3^{i_{q+1}} \dots \bar{u}_3^{i_r}) u_{j_1}^1 \dots u_{j_m}^1 u_{j_{m+1}}^2 \dots u_{j_n}^2 u_{j_{n+1}}^3 \dots u_{j_l}^3 - traces. \quad (A.3)$$

From the completeness relation (2.4), we see that the tensors in the set (A.3) are not all independent. One could obtain a basis by restricting the tensors to contain either u_i^3 , or $\bar{u}_3^i$, but not both. However, such a basis is not orthogonal. For instance:

$$\int [du] (\bar{u}_1^i u_j^1 - \frac{1}{3} \delta_j^i) (\bar{u}_2^k u_l^2 - \frac{1}{3} \delta_l^k) = -\frac{1}{18} (\delta_l^i \delta_j^k - \frac{1}{3} \delta_j^i \delta_l^k).$$

Notice that the tensor basis just introduced decomposes into irreducible representations of the SU(3) group generated by the differential operators Q_1 , Q_2 and D_b^a , eqs. (2.7-9). A way to construct an orthonormal basis for any irreducible representation of U(3) has been introduced by Gel'fand and Zetlin^[11]. It may be described as follows

$$\left(\begin{array}{c} \}^{U(1)} \\ \}^{U(2)} \end{array} \right) \}^{U(3)}. \quad (A.4)$$

We choose a U(2) subgroup inside U(3) and decompose any U(3) IR into U(2) IR's. Two vectors corresponding to different U(2) IR's will automatically be orthogonal. Inside a given U(2) IR, we take a basis of U(1) eigenvectors. Since the U(1) eigenvalues inside a

U(2) IR are non-degenerate, this basis will be orthogonal. A vector in this basis is labelled by a set of integers:

$$\begin{pmatrix} m_1 & m_2 & 0 \\ & k_1 & k_2 \\ & & s \end{pmatrix} \begin{array}{ll} SU(3) & IR, \\ U(2) & IR, \\ U(1) & eigenvalue. \end{array} \quad (A.5)$$

m_1 and m_2 may be seen to be the lengths of the first and second lines in the tableau (A.2). Restricting ourselves to SU(3) representations, we can take the third entry of the first line to be zero. From the laws governing the decomposition of a SU(3) IR into U(2) IR's, we get the following inequalities:

$$\begin{aligned} m_1 &\geq k_1 \geq m_2 \geq k_2 \geq 0, \\ k_1 &\geq s \geq k_2. \end{aligned} \quad (A.6)$$

The explicit construction of the vectors (A.5) can be found in Zelobenko^[12]. We apply this technique to construct an orthonormal basis of SU(3) tensors. We first introduce a set of modified lowering operators

$$\Delta_1^2 = D_1^2, \quad \Delta_2^3 = D_2^3, \quad \Delta_1^3 = (Q_1 + 1)D_1^3 + D_2^3 D_1^2. \quad (A.7)$$

We then start from the highest weight tensor:

$$\begin{pmatrix} m_1 & m_2 & 0 \\ & m_1 & m_2 \\ & & m_1 \end{pmatrix}^{i_1 \dots i_{m_2}}_{j_1 \dots j_{m_1 - m_2}} = \sqrt{d_{m_1, m_2}} \bar{u}_3^{(i_1} \dots \bar{u}_3^{i_{m_2})} u_{(j_1}^1 \dots u_{j_{m_1 - m_2}}^1). \quad (A.8)$$

Where symmetrization over m indices includes a $\frac{1}{m!}$ and d_{m_1, m_2} is the dimension of the (m_1, m_2) representation of SU(3):

$$d_{m_1, m_2} = \frac{1}{2}(m_1 + 2)(m_2 + 1)(m_1 - m_2 + 1).$$

We then write the generic tensor of our basis as:

$$\begin{pmatrix} m_1 & m_2 & 0 \\ & k_1 & k_2 \\ & & s \end{pmatrix}^{i_1 \dots i_{m_2}}_{j_1 \dots j_{m_1 - m_2}} =$$

$$= \frac{1}{\sqrt{N}} (\Delta_1^2)^{k_1-s} (\Delta_1^3)^{m_1-k_1} (\Delta_2^3)^{m_2-k_2} \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ s \end{pmatrix}_{j_1 \dots j_{m_1-m_2}}^{i_1 \dots i_{m_2}} \quad (A.9)$$

with:

$$N = \frac{(k_1 - k_2)!(k_1 - s)!(m_1 - m_2)!(m_1 + 1)!m_2!}{(s - k_2)!(k_1 - m_2)!(k_1 + 1)!k_2!} (m_1 - k_1)! \cdot \frac{(m_1 - k_2 + 1)!}{(k_1 - k_2 + 1)!} (m_2 - k_2)!. \quad (A.10)$$

The rules for bar- or star-conjugation of a basis tensor are given by

$$\overline{\begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ s \end{pmatrix}_{j_1 \dots j_{m_1-m_2}}^{i_1 \dots i_{m_2}}} = (-1)^{k_1-s+m_2-k_2} \begin{pmatrix} m_1 & m_1-m_2 & 0 \\ m_1-k_2 & m_1-k_1 \\ m_1-s \end{pmatrix}_{i_1 \dots i_{m_2}}^{j_1 \dots j_{m_1-m_2}},$$

$$\left[\begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ s \end{pmatrix}_{j_1 \dots j_{m_1-m_2}}^{i_1 \dots i_{m_2}} \right]^* = (-1)^{s+k_2} \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ k_1+k_2-s \end{pmatrix}_{j_1 \dots j_{m_1-m_2}}^{i_1 \dots i_{m_2}}.$$

This basis is orthogonal in the following sense: if $F_{j_1 \dots j_{m_1-m_2}}^{i_1 \dots i_{m_2}}$ and $G_{j_1' \dots j_{m_1'-m_2}'}^{i_1' \dots i_{m_2}'}$ are two irreducible (symmetric traceless) SU(3) tensors, one has in obvious notations:

$$\int [du] \left[\begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ s \end{pmatrix} \cdot F \right] \overline{\left[\begin{pmatrix} m_1' & m_2' & 0 \\ k_1' & k_2' \\ s' \end{pmatrix} \cdot G \right]} =$$

$$= \delta_{m_1 m_1'} \delta_{m_2 m_2'} \delta_{k_1 k_1'} \delta_{k_2 k_2'} \delta_{s s'} F \bar{G}. \quad (A.11)$$

Using such a basis, it is very simple to write down the delta functions in a given charge sector. Noticing that the charges are given by:

$$Q_1 \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ s \end{pmatrix} = (2s - k_1 - k_2) \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 \\ s \end{pmatrix},$$

$$Q_2 \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 & \\ s & & \end{pmatrix} = (3(k_1 + k_2) - 2(m_1 + m_2)) \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 & \\ s & & \end{pmatrix}. \quad (\text{A.12})$$

The distribution:

$$\delta^{(q_1, q_2 | -q_1, -q_2)}(u, v) = \sum_{\substack{2s - k_1 - k_2 = q_1 \\ 3(k_1 + k_2) - 2(m_1 + m_2) = q_2}} \begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 & \\ s & & \end{pmatrix}_u \cdot \overline{\begin{pmatrix} m_1 & m_2 & 0 \\ k_1 & k_2 & \\ s & & \end{pmatrix}_v} \quad (\text{A.13})$$

will satisfy:

$$\int [dv] \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v) F^{(q_1, q_2)}(v) = F^{(q_1, q_2)}(u). \quad (\text{A.14})$$

Finally, we will also have to deal with functions on the smaller space $SU(3)/SU(2) \times U(1)$ included in $SU(3)/U(1) \times U(1)$. They may be characterized by the constraints:

$$D_2^1 F^{(0, q_2)} = D_1^2 F^{(0, q_2)} = 0. \quad (\text{A.15})$$

Eqs. (A.15) mean that $F^{(0, q_2)}$ is a singlet of the $SU(2)$ group generated by (D_2^1, D_1^2, Q_1) . A basis for the expansion of these functions may be obtained from (A.9) by restricting to tensors which are $SU(2)$ singlets, i.e. $k_1 = k_2$. An explicit form for these tensors is:

$$\begin{pmatrix} p+q & q & 0 \\ q & q & \\ q & & \end{pmatrix}_{j_1 \dots j_p}^{i_1 \dots i_q} = \sum_{k=0}^{inf(p, q)} \alpha_k \delta_{(j_1}^{i_1} \dots \delta_{j_k}^{i_k} u_{j_{k+1}}^3 \dots u_{j_p}^3 \bar{u}_3^{i_{k+1}} \dots \bar{u}_3^{i_q}) \quad (\text{A.16})$$

where

$$\alpha_k = \sqrt{\frac{(p+q+2)!}{2p!q!}} (-1)^k \frac{p!q!}{(p-k)!(q-k)!k!} \frac{1}{(p+q+1)!}. \quad (\text{A.17})$$

Again, using this basis, the delta function on $SU(3)/SU(2) \times U(1)$ in a given charge sector is easily obtained:

$$\Delta^{(q_2 | -q_2)}(u, v) = \sum_{\substack{q, p \\ 2(q-p) = q_2}} \begin{pmatrix} p+q & q & 0 \\ q & q & \\ q & & \end{pmatrix}_u \cdot \overline{\begin{pmatrix} p+q & q & 0 \\ q & q & \\ q & & \end{pmatrix}_v}. \quad (\text{A.18})$$

For a function $F^{(0, q_2)}$ satisfying (A.15), we then have:

$$\int [dv] \Delta^{(q_2 | -q_2)}(u, v) F^{(0, q_2)}(v) = F^{(0, q_2)}(u)$$

We shall give in appendix B an alternative (and simpler) version of the delta-functions

(A.13), (A.18). Many other useful formulae, including the action of the step operators on the basis tensors, may be found in Zelobenko^[12].

APPENDIX B

Harmonic distributions

The harmonic distributions we need, to compute the propagators, are of two kinds. The first kind is closely related to the distributions appearing in the N=2 case^[3]. One has to invert a step operator, D_2^1 in our case, on a delta-function, or some derivative of a delta-function. The relation with the N=2 case is made explicit in subsection B.1, and the distributions are computed in subsections B.2 and B.3. The second kind is a distribution on $SU(3)/SU(2) \times U(1)$, which inverts the operator $\mathbf{C} + 1$, where $\mathbf{C}$ is the quadratic Casimir operator of $SU(3)$. It is studied in subsection B.4. Finally, some formulae which are useful in the derivation of the propagators are given in subsection B.5.

B.1 A special set of coordinates

We use the set of coordinates on $SU(3)$ coming from the following decomposition of an $SU(3)$ matrix:

$$U = \begin{pmatrix} u^1 & u^2 & u^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+\varphi\varphi^\dagger}} & \frac{\varphi}{\sqrt{1+\varphi\varphi^\dagger}} \\ \frac{-\varphi^\dagger}{\sqrt{1+\varphi\varphi^\dagger}} & \frac{1}{\sqrt{1+\varphi\varphi^\dagger}} \end{pmatrix} \begin{pmatrix} e^{\frac{i\psi}{2}} \begin{pmatrix} u^+ & u^- \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & 0 & e^{-i\psi} \end{pmatrix}. \quad (B.1)$$

Where $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}$ is a two-dimensional complex vector parametrizing the coset space $SU(3)/SU(2) \times U(1)$, $\begin{pmatrix} u^+ & u^- \end{pmatrix}$ is a $SU(2)$ matrix, and ψ is a $U(1)$ phase. Notice that not any $SU(3)$ matrix may be decomposed this way, the necessary and sufficient condition being $u_3^3 \neq 0$. However, the set of unitary matrices satisfying $u_3^3 = 0$ is of lower dimension, and thus of zero measure. Therefore, one can integrate using variables (B.1). The main advantage of coordinates (B.1) is that a subset of the harmonic derivatives is greatly simplified:

$$D_2^1 = u_i^+ \frac{\partial}{\partial u_i^-}, \quad D_1^2 = u_i^- \frac{\partial}{\partial u_i^+}. \quad (B.2)$$

The charge operators also are simple:

$$Q_1 = [D_2^1, D_1^2] = u_i^+ \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^-}, \quad (B.3)$$

$$Q_2 = \frac{2}{i} \frac{\partial}{\partial \psi}. \quad (B.4)$$

The invariant $SU(3)$ measure may be written as:

$$[du] = \frac{1}{\pi^3} \frac{d^2\varphi d^2\bar{\varphi}}{(1 + \varphi^\dagger\varphi)^3} d\psi [du^\mp]. \quad (B.5)$$

Where $[du^\mp]$ is the $SU(2)$ measure [1,3].

B.2 Distributions in special coordinates

Introducing a second set of $SU(3)$ variables, $V = (v^1 \ v^2 \ v^3) \leftrightarrow (\varphi', \psi', v^\mp)$, the delta-functions introduced in appendix A are easily written as:

$$\delta^{(q_1, q_2 | -q_1, -q_2)}(u, v) = \Delta^{(q_2 | -q_2)}(u, v) \delta^{(q_1 | -q_1)}(u, v) \quad (B.6)$$

with

$$\Delta^{(q_2 | -q_2)}(u, v) = \frac{\pi^2}{2} (1 + \varphi^\dagger\varphi)^3 \delta^2(\varphi - \varphi') \delta^2(\bar{\varphi} - \bar{\varphi}') e^{iq_2 \frac{\psi - \psi'}{2}}. \quad (B.7)$$

And a convenient formula for $SU(2)$ delta-functions is obtained from ref.[3]:

$$\delta^{(q_1 | -q_1)}(u, v) = \pi (u^+ v^-)^{q_1} \delta(u^+ v^+) \delta(u^- v^-). \quad (B.8)$$

From the work of GIKOS [3], we can construct the distributions $\Omega^{(-q_1, q_2 | -q_1, -q_2)}$, $q_1 > 0$, satisfying:

$$D_2^1(u) \Omega^{(-q_1, q_2 | -q_1, -q_2)}(u, v) = (D_1^2(u))^{q_1-1} \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v). \quad (B.9)$$

They read

$$\Omega^{(-q_1, q_2 | -q_1, -q_2)}(u, v) = \Delta^{(q_2 | -q_2)}(u, v) \frac{(-1)^{q_1} (q_1 - 1)!}{(u^+ v^+)^{q_1}}. \quad (B.10)$$

Notice that the existence of distributions (B.9) can be discussed using the properties of the representations of the $SU(2)$ group generated by D_2^1 , D_1^2 and Q_1 . Indeed, the distribution $\delta^{(q_1, q_2 | -q_1, -q_2)}(u, v)$ can be expanded in tensors belonging to irreducible representations of $SU(2)$ (see formula (A.13)). The highest weight of any such representation will have a Q_1 -charge bigger or equal to q_1 , or equivalently the lowest weight will have a Q_1 -charge smaller or equal to $-q_1$. Consequently, $(D_1^2(u))^{q_1-1} \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v)$, which has a Q_1 -charge $-q_1 + 2$, does not contain any $SU(2)$ lowest weight, and thus D_2^1 can be inverted on it.

B.3 Distributions in covariant notations

An easy change of variables leads to the following useful form of the charge zero delta-function:

$$\delta^{(0,0|0,0)}(u, v) = \frac{\pi^3}{2} \delta^2(u^1 \bar{v}_3) \delta^2(u^2 \bar{v}_3) \delta^2(u^1 \bar{v}_2) \quad (B.11)$$

with

$$\delta^2(z) = \delta(z) \delta(\bar{z}).$$

Clearly, many choices are possible for the arguments of the delta-function in (B.11). They can be obtained from (B.11) by applying any permutation to the indices of u and $\bar{v}$, and by exchanging the variables u and v . To obtain the delta-functions in a given charge sector, we multiply (B.9) by appropriate powers of $u^a \bar{v}_a$, $v^a \bar{u}_a$. For instance:

$$(u^1 \bar{v}_1) \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v) = \delta^{(q_1+1, q_2+1 | -q_1-1, -q_2-1)}(u, v). \quad (B.12)$$

Notice also the identities:

$$(u^a \bar{v}_b) \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v) = 0, \quad a \neq b, \quad (B.13)$$

$$(u^1 \bar{v}_1)(u^2 \bar{v}_2)(u^3 \bar{v}_3) \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v) = \delta^{(q_1, q_2 | -q_1, -q_2)}(u, v). \quad (B.14)$$

The delta-function on $SU(3)/SU(2) \times U(1)$ may be written in the same notations as:

$$\Delta^{(0|0)}(u, v) = \frac{\pi^2}{2} \delta^2(u^1 \bar{v}_3) \delta^2(u^2 \bar{v}_3). \quad (B.15)$$

To obtain the delta-function in a given charge sector, we multiply (B.15) by appropriate powers of $u^3 \bar{v}_3$, $v^3 \bar{u}_3$. Using formula (2.6) for u and v , we get the relation:

$$u^1 \bar{v}_2 = (v^3 \bar{u}_2)(v^1 \bar{u}_3) - (v^3 \bar{u}_3)(v^1 \bar{u}_2). \quad (B.16)$$

Then, from eqs. (B.15), (B.16) we obtain:

$$(u^1 \bar{v}_2) \Delta^{(0|0)}(u, v) = -(v^3 \bar{u}_3)(v^1 \bar{u}_2) \Delta^{(0|0)}(u, v). \quad (B.17)$$

We can also rewrite distributions (B.10) as:

$$\Omega^{(-q_1, q_2 | -q_1, -q_2)}(u, v) = (-1)^{q_1-a} (q_1-1)! \frac{(v^3 \bar{u}_3)^{\frac{q_1+q_2}{2}-a}}{(u^1 \bar{v}_2)^{q_1-a} (v^1 \bar{u}_2)^a} \Delta^{(0|0)}(u, v). \quad (B.18)$$

where a is an integer which may be chosen freely between 0 and q_1 (this freedom is directly related to eq. (B.17)). Formulae (B.18) may be verified directly [3] using the properties of

the integration, eqs. (2.24). For instance, we want to check:

$$(u^1 \bar{v}_2) \Omega^{(-1,1|-1,-1)}(u, v) = -\Delta^{(2|-2)}(u, v). \quad (B.19)$$

Any test function $F^{(0,2)}(u)$ may be decomposed as:

$$F^{(0,-2)}(u) = F_S^{(0,-2)}(u) + D_2^1 H^{(-2,-2)}(u). \quad (B.20)$$

Where $F_S^{(0,-2)}(u)$ is the SU(2)-singlet part of $F^{(0,-2)}(u)$:

$$D_2^1 F_S^{(0,-2)}(u) = 0.$$

We then have, using integration by parts and (B.13):

$$\begin{aligned} \int [du] D_2^1 H^{(-2,-2)}(u) (u^1 \bar{v}_2) \Omega^{(-1,1|-1,-1)}(u, v) = \\ = \int [du] H^{(-2,-2)}(u^1 \bar{v}_2) \delta^{(1,1|-1,-1)}(u, v) = 0. \end{aligned}$$

Finally, we use $u^1 \bar{v}_2 = D_2^1(u^2 \bar{v}_2)$ to get:

$$\int [du] F^{(0,-2)}(u) \left[(u^1 \bar{v}_2) \Omega^{(-1,1|-1,-1)}(u, v) \right] = -F_S^{(0,-2)}(v) \quad (B.21)$$

which is identical to (B.19).

B.4 Inverse Casimir

The Casimir operator on SU(3) is quadratic in the generators of SU(3) right and commutes with all transformations of SU(3) right. It is given by:

$$\mathbf{C} = \frac{1}{2}(D_2^1 D_1^2 + D_1^2 D_2^1) + \frac{1}{2}(D_3^1 D_1^3 + D_1^3 D_3^1) + \frac{1}{2}(D_3^2 D_2^3 + D_2^3 D_3^2) + \frac{1}{4}(Q_1)^2 + \frac{1}{12}(Q_2)^2. \quad (B.22)$$

The value of $\mathbf{C}$ on the (m_1, m_2) IR of SU(3) is

$$C(m_1, m_2) = m_1 + \frac{1}{4}(m_1)^2 + \frac{1}{3}(m_2 - \frac{1}{2}m_1)^2 \geq 0. \quad (B.23)$$

We shall need the distribution $\Theta(u, v)$ on SU(3)/SU(2)×U(1) satisfying:

$$(\mathbf{C} + 1)\Theta(u, v) = \Delta^{(0|0)}(u, v). \quad (B.24)$$

The existence of $\Theta(u, v)$ is obvious, the operator $\mathbf{C} + 1$ having no zero-modes. We shall

show that:

$$\Theta(u, v) = \frac{1}{2(1 - |u^3 \bar{v}_3|^2)} = \frac{1}{2(|u^1 \bar{v}_3|^2 + |u^2 \bar{v}_3|^2)}. \quad (B.25)$$

Notice that eq. (B.25) is very similar to the inverse Laplacian in 4-dimensional Euclidian space. In particular, these two distributions have the same behaviour at the origin $|u^3 \bar{v}_3| = 1$. To verify that (B.25) satisfies (B.24), we first define $\Theta(u, v)$ as the limit as $\alpha \rightarrow 0^+$ of a function $\Theta_\alpha(u, v)$, non-singular at $|u^3 \bar{v}_3| = 1$

$$\Theta_\alpha(u, v) = \frac{1}{2(|u^1 \bar{v}_3|^2 + |u^2 \bar{v}_3|^2 + \alpha |u^3 \bar{v}_3|^2)}. \quad (B.26)$$

We then find:

$$(C + 1)\Theta_\alpha(u, v) = \frac{\alpha(1 + (1 + \alpha)|u^3 \bar{v}_3|^2)}{2(1 - (1 - \alpha)|u^3 \bar{v}_3|^2)^3} \quad (B.27)$$

Introducing a test function $F^{(0,0)}(u)$ on $SU(3)/SU(2) \times U(1)$, we need to compute

$$I = \lim_{\alpha \rightarrow 0} \int [du] F^{(0,0)}(u) (C + 1)\Theta_\alpha(u, v). \quad (B.28)$$

We make the change of variables

$$U' = UV^\dagger$$

(B.28) then writes:

$$I = \lim_{\alpha \rightarrow 0} \int [du'] F^{(0,0)}(u) \frac{\alpha}{2} \frac{(1 + (1 + \alpha)|u_3'^3|^2)}{(1 - (1 - \alpha)|u_3'^3|^2)^3}. \quad (B.29)$$

Where $F^{(0,0)}(u)$ only depends on:

$$u_i^3 = u_1'^3 v_i^1 + u_2'^3 v_i^2 + u_3'^3 v_i^3$$

We can now use the special coordinates (B.1) to obtain:

$$I = \frac{1}{\pi^2} \lim_{\alpha \rightarrow 0} \int \frac{d^2 \varphi d^2 \bar{\varphi}}{1 + \varphi^\dagger \varphi} F^{(0,0)}(u) \frac{\alpha(2 + \varphi^\dagger \varphi + \alpha)}{(\varphi^\dagger \varphi + \alpha)^3}$$

Since $F^{(0,0)}(u(\varphi))$ is a bounded function of φ and $\bar{\varphi}$, the limit $\alpha \rightarrow 0$ in eq. (B.29) only

receives a contribution from the region $\varphi^\dagger \varphi < 1$. Therefore, (B.29) can be simplified to:

$$I = \frac{2}{\pi^2} \lim_{\alpha \rightarrow 0} \int_{\varphi^\dagger \varphi < 1} d^2 \varphi d^2 \bar{\varphi} F^{(0,0)}(u) \frac{\alpha}{(\alpha + \varphi^\dagger \varphi)^3}.$$

Finally, we make the change of variables $\varphi = \sqrt{\alpha} w$ and take the limit $\alpha \rightarrow 0$ to get

$$I = \frac{2}{\pi^2} \int d^2 w d^2 \bar{w} F^{(0,0)}(v) \frac{1}{(1 + w^\dagger w)^3} = F^{(0,0)}(v)$$

which demonstrates (B.24). Notice also the important relation:

$$\left(\mathbf{C} + \frac{2}{3}\right) [(u^3 \bar{v}_3) \Theta(u, v)] = (u^3 \bar{v}_3) \Delta^{(0|0)}(u, v). \quad (B.30)$$

B.5 Some more formulae

In the determination of the propagators, some more equations will be useful, which will be stated without proof:

$$\begin{aligned} D_2^1 \frac{1}{(u^1 \bar{v}_2)^q} &= \frac{(-1)^q}{(q-1)!} \pi(v^2 \bar{u}_2) \delta^{[q-1]}(u^1 \bar{v}_2) \delta(v^2 \bar{u}_1), \\ D_3^1 \frac{1}{(u^1 \bar{v}_2)^q} &= \frac{(-1)^q}{(q-1)!} \pi(v^2 \bar{u}_3) \delta^{[q-1]}(u^1 \bar{v}_2) \delta(v^2 \bar{u}_1), \\ D_2^3 \frac{1}{(u^1 \bar{v}_2)^q} &= 0. \end{aligned} \quad (B.31)$$

Analogously:

$$\begin{aligned} D_2^1 \frac{1}{(v^1 \bar{u}_2)^q} &= \frac{(-1)^q}{(q-1)!} \pi(u^1 \bar{v}_1) \delta^{[q-1]}(v^1 \bar{u}_2) \delta(u^2 \bar{v}_1), \\ D_2^3 \frac{1}{(v^1 \bar{u}_2)^q} &= \frac{(-1)^q}{(q-1)!} \pi(u^3 \bar{v}_1) \delta^{[q-1]}(v^1 \bar{u}_2) \delta(u^2 \bar{v}_1), \\ D_3^1 \frac{1}{(v^1 \bar{u}_2)^q} &= 0. \end{aligned} \quad (B.32)$$

And $\delta^{[p]}(z)$ is the p^{th} derivative of the delta-function.

	D_2^1	D_3^1	D_2^3	D_1^2	D_1^3	D_3^2
q_1	2	1	1	-2	-1	-1
q_2	0	3	-3	0	-3	3

Table 1

	D_α^1	D_α^2	D_α^3	$\bar{D}_{1\dot{\alpha}}$	$\bar{D}_{2\dot{\alpha}}$	$\bar{D}_{3\dot{\alpha}}$
q_1	1	-1	0	-1	1	0
q_2	1	1	-2	-1	-1	2

Table 2

	u^1	u^2	u^3	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$
$\underline{*}$	$-\bar{u}_2$	$\bar{u}_1$	$\bar{u}_3$	$-u^2$	u^1	u^3

Table 3

	D_2^1	D_3^1	D_2^3	D_1^2	D_1^3	D_3^2	Q_1	Q_2
$\underline{*}$	D_2^1	D_2^3	$-D_3^1$	D_1^2	D_3^2	$-D_1^3$	Q_1	$-Q_2$

Table 4

	D_α^1	D_α^2	D_α^3	$\bar{D}_{1\dot{\alpha}}$	$\bar{D}_{2\dot{\alpha}}$	$\bar{D}_{3\dot{\alpha}}$
$\underline{*}$	$-\bar{D}_{2\dot{\alpha}}$	$\bar{D}_{1\dot{\alpha}}$	$\bar{D}_{3\dot{\alpha}}$	$-D_\alpha^2$	D_α^1	D_α^3

Table 5

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