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Space-time supersymmetries, world-sheet symmetries and exceptional groups*

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ABSTRACT

We review the connection between space-time supersymmetries and world-sheet symmetries in four-dimensional heterotic string models. The relevance of the exceptional groups, which unify these symmetries, is stressed and used in the construction of partition functions.

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Space-time supersymmetries, world-sheet symmetries and exceptional groups

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The original hope that string theory would provide a *unique* “theory of everything” has suffered serious drawbacks with the discovery [1] of an enormous number of models satisfying all the known consistency requirements. In general, any unitary conformal field theory with central charges for the holomorphic and antiholomorphic sectors $(c, \bar{c}) = (15 - \frac{3}{2}d, 26 - d)$, where d is the number of (uncompactified) space-time dimensions, provides, when combined with the space-time part in a consistent (modular invariant, etc.) way, a valid heterotic string model. Of course, there is the hope that some non-perturbative mechanism will be able to interpolate between the many different string vacua. At the moment there does not exist a formalism that allows to solve or even attack this problem. In view of this it seems to be desirable to make as many model independent statements of string theories as possible. It has been known that requiring space-time supersymmetry imposes severe restrictions on the models [2, 3]. We will show how it determines the world-sheet symmetries and how the properties under space-time transformations, the superconformal ghost charge and the transformation under the global world-sheet symmetries of all allowed string states are unified by an underlying exceptional group. It also contains the information of extra, left-moving gauge bosons which extend the rank of the gauge group beyond $26 - d$. The relation between the exceptional groups and space-time supersymmetry was first observed in the context of covariant lattices [4]; it is however a general feature of supersymmetric string theories (c.f. below). We will restrict our attention to the phenomenologically interesting case of four-dimensional heterotic string theories for which the table below provides an exhaustive list of the connection between the various symmetries.

space – time supersymmetry	world – sheet SCA	right – moving gauge bosons	exceptional group
$N = 1$	$N = 2, c = 9$	–	E_6
$N = 2$	$N = 2, c = 3$ $N = 4, c = 6$	$[U(1)]^2$	E_7
$N = 4$	$3 \times (N = 2, c = 3)$	$[U(1)]^6$	E_8

Note that the requirement of space-time supersymmetry imposes restrictions on the internal left-moving (or, equivalently, holomorphic) part only. The right-moving part is still arbitrary (up to the general consistency requirements). In the following we want to explain how space-time supersymmetry implies the world-sheet symmetries and also how it determines the allowed conformal fields. We then show how the exceptional groups enter and lead to the same restrictions on the spectrum. The characters of the exceptional groups can be used to write down string partition functions exhibiting their model independent parts. Due to lack of space we cannot present many details and refer for a detailed discussion to refs. [3] where also a more complete list of references is given.

The energy-momentum tensor T and the super-current T_F of the internal left-moving part of any four-dimensional heterotic string theory generate a $N = 1$, $c = 9$ superconformal algebra (SCA); however, space-time supersymmetry always implies an extension of the $N = 1$ SCA. (The converse conclusion is in general not correct (cf. below).)

The N -extended space-time supersymmetry algebra in $d = 4$ is ($A = 1, \dots, N$)

$$\begin{aligned}\{Q_\alpha^A, Q_{\dot{\beta}B}\} &= 2i\delta_B^A \gamma_{\alpha\dot{\beta}}^\mu P_\mu \\ \{Q_\alpha^A, Q_\alpha^B\} &= 2C_{\alpha\beta} Z^{AB}\end{aligned}$$

where $Q_\alpha^A = (Q_{\dot{\alpha}A})^\dagger$ are the supersymmetry generators and $Z^{AB} = -Z^{BA}$ are the central charges (obviously there are none for $N = 1$). The supersymmetry charges are given by the contour integral of the holomorphic part of the gravitino vertex operators at zero momentum, i.e.

$$Q_{\alpha(q)}^A = \oint \frac{dz}{2\pi i} V_{\alpha(q)}^A(z)$$

where q is the superconformal ghost charge. In the canonical ghost picture ($q = -1/2$ for fermions and $q = -1$ for bosons) we have

$$V_{\alpha(-1/2)}^A = S_\alpha e^{-\phi/2} \Sigma^A$$

where S_α is a $SO(4)$ spin field, ϕ is the bosonized superconformal ghost and Σ^A has conformal dimension $3/8$. The supersymmetry algebra translates immediately to the following operator products:

$$\begin{aligned}\Sigma^A(z) \Sigma_B(w) &\sim (z-w)^{-3/4} \delta_B^A + (z-w)^{+1/4} J_B^A(w) \\ \Sigma^A(z) \Sigma^B(w) &\sim (z-w)^{-1/4} \psi^{AB}\end{aligned}$$

where J_B^A is of conformal dimension one and the fields $\psi^{AB} = -\psi^{BA}$ with conformal dimension $1/2$ are related to the central charges in the $q = -1$ picture through $Z_{(-1)}^{AB} = \oint \frac{dz}{2\pi i} e^{-\phi} \psi^{AB}$. The operator product expansions above are sufficient to evaluate various three- and four-point correlation functions from which we can deduce the whole operator algebra of the fields Σ^A , ψ^{AB} and J_B^A . One finds the following results:

(i) $N = 1$: J is the current of a level six $U(1)$ Kac-Moody algebra and can be expressed in terms of a free boson H as $J = i\sqrt{3}\partial H$. The fields $\Sigma^\pm = e^{\pm i\frac{\sqrt{3}}{2}H}$ have $U(1)$ charge $\pm\frac{3}{2}$. The supercurrent T_F splits into two parts $T_F = T_F^+ + T_F^-$ with $U(1)$ charges ± 1 respectively. T , $T_F^\pm$ and J generate a $N = 2$, $c = 9$ SCA. It is important to note that the operator product of the gravitino with itself contains two fields whose holomorphic parts are $G^\pm = \frac{1}{2\sqrt{3}}e^{\pm i\sqrt{3}H}$. Together with $J' = \frac{i}{\sqrt{3}}\partial H$ and $T^{c=1} = -\frac{1}{2}(\partial H)^2$ they generate a $N = 2$, $c = 1$ SCA.

(ii) $N = 2$: Here J_B^A splits into a $SU(2)$ and a $U(1)$ part generating a level one $SU(2)$ and a level two $U(1)$ Kac-Moody algebra respectively. The fields Σ^A (Σ_A) transform as a $SU(2)$ doublet with $U(1)$ charge $+\frac{1}{2}(-\frac{1}{2})$. BRST invariance of the gravitino vertex operator tells us that the supercurrent splits into two parts: $T_F =$

$T_F^{U(1)} + T_F^{SU(2)}$. $T_F^{U(1)}$, which carries no $SU(2)$ charge can be written as a sum of two terms with $U(1)$ charge ± 1 : $T_F^{U(1)} = T_F^+ + T_F^-$. ψ^+ , which enters the central charge in the $q = -1$ picture is a free fermion with $U(1)$ charge -1 . It combines with a free boson Y^+ to a free superfield $\mathcal{D}Y^+ = \psi^+ + \theta\partial Y^+$. ∂Y^+ is the central charge in the $q = 0$ picture. (We have a similar expression for the hermitian conjugate fields combined in $\mathcal{D}Y^-$.) In terms of $Y^\pm$ and $\psi^\pm$ the supercurrent is $T_F^{U(1)} = -\frac{1}{2}(\psi^+\partial Y^- + \psi^-\partial Y^+)$. $T_F^\pm$, $T^{c=3}$ and $J^{U(1)}$ generate a $N = 2$, $c = 3$ SCA. Here $T^{c=3}$ is the energy-momentum tensor for two free superfields. Expressed in terms of free superfields it is clear that the central charges are just internal momenta; i.e. $N = 2$ space-time supersymmetric theories in $d = 4$ are, as far as the holomorphic part is concerned, torus compactifications of $N = 1$ theories in $d = 6$. The compactified momenta appear as two $U(1)$ gauge bosons; the holomorphic part of their vertex operators at zero momentum are just the central charges. The $U(1)$ -invariant part of T_F also splits into two parts: $T_F^{SU(2)} = T_F^1 + T_F^{1\dagger} = T^1 O + T^2 O^\dagger$ where $T^{1,2}$ are the two components of a $SU(2)$ doublet whereas O is invariant. Obviously, since $J_{SU(2)}^\pm$ interpolate between T^1 and T^2 , the field $\tilde{T}_F^{SU(2)} = T_F^2 + T_F^{2\dagger} = T^2 O + T^1 O^\dagger$ must also be present in any $N = 2$ supersymmetric four-dimensional string theory. In fact, it can be used as the $SU(2)$ part of the supercurrent instead of $T_F^{SU(2)}$. Their sum, however, is not a valid supercurrent. $T_F^{1,2}$ and their hermitian conjugates, together with $J_i^{SU(2)}$ and $T^{c=6} = T - T^{c=3}$ generate a $N = 4$, $c = 6$ SCA.

(iii) $N = 4$: Here the fields Σ^α can be shown to be spin fields of $SO(6)$. The J^A_B can be expressed as $J^A_B = \frac{1}{4}(\Gamma_{MN})^A_B J^{MN}$ where Γ_{MN} are $SO(6)$ generators and the J^{MN} satisfy a level one $SO(6)$ Kac-Moody algebra. The central charges can be expressed in terms of six free real fermions transforming as a vector of $SO(6)$: $\psi^{AB} = \frac{1}{\sqrt{2}}\Gamma_M^{AB}\psi^M$. They can be combined with six $U(1)$ currents to six free superfields. The situation is as in the $N = 2$ case only that now we have three pairs of free complex conjugate superfields, each, together with one of the Cartan-subalgebra currents of $SO(6)$, generating a $N = 2$, $c = 3$ SCA. Again, the central charges are just internal momenta, saying that $N = 4$ theories in $d = 4$ are left-moving torus compactifications of an $N = 1$ theory in $d = 10$. The six compactified momenta appear as $U(1)$ gauge fields.

Now, the possible conformal fields in the theory are restricted by requiring locality of their operator products with the gravitino. The holomorphic part of a general field can be written as

$$\Phi_{(q)}(z) = \Phi^{\text{space-time}} e^{-q\phi} \Phi^{\text{Kac-Moody}} \Phi^{\text{rest}}$$

where $\Phi^{\text{Kac-Moody}}$ transforms under some representation of $U(1)$, $U(1) \times SU(2)$ or $SO(6)$ for the three cases $N = 1, 2$ or 4 respectively. Locality then restricts the allowed conjugacy classes of $(\text{space-time}) \times (\text{Kac-Moody})$. For instance, in the $N = 1$ case we get 12 conjugacy classes of $D_1 \times U(1)$ (D_1 is the transverse Lorentz group):

$$\begin{aligned} \text{space-time bosons} : & \quad (0, \sqrt{3}), (0, \pm \frac{1}{\sqrt{3}}), (v, 0); (v, \pm \frac{2}{\sqrt{3}}) \\ \text{space-time fermions} : & \quad (s, \frac{\sqrt{3}}{2}), (c, -\frac{\sqrt{3}}{2}), (s, -\frac{5}{6}\sqrt{3}), (c, \frac{5}{6}\sqrt{3}), (s, -\frac{\sqrt{3}}{6}), (c, \frac{\sqrt{3}}{6}) \end{aligned}$$

The $U(1)$ charges within each conjugacy class are modulo $2\sqrt{3}$, which follows from the existence of the fields $G^\pm$. In the $N = 2$ case we find 8 conjugacy classes of $D_1 \times A_1 \times U(1)$ and for $N = 4$ there are four conjugacy classes of $D_1 \times D_3$.

It is now clear that the mere existence of for instance a $N = 2$, $c = 9$ SCA is not sufficient for $N = 1$ space-time supersymmetry. Only if all fields belong to one of the 12 conjugacy classes consistent with the existence of the gravitino do we have space-time supersymmetry. So do all covariant lattice models have a $N = 2$ SCA but not all of them have space-time supersymmetry.

Let us now make the connection to the exceptional groups. We recall that the space-time and superconformal ghost degrees of freedom of any physical state can be described by a vector $(\vec{\lambda}_{D_2}, q)$ of the Lorentzian lattice $D_{2,1}$ where for bosons $\vec{\lambda}_{D_2}$ is in the 0 or v conjugacy class of D_2 and $q \in \mathbf{Z}$ and for space-time fermions $\vec{\lambda}_{D_2} \in s, c$ and $q \in \mathbf{Z} + \frac{1}{2}$. As explained in ref. [5], we can map the lattice $D_{2,1}$ to a Euclidean lattice D_5 without changing conjugacy classes and mutual scalar products (mod 1) of the lattice vectors: $D_{2,1} \rightarrow D_5 \subset D_1 \times D_4$, where the D_1 factor now contains the transverse (light-cone) and D_4 the longitudinal and superconformal ghost degrees of freedom. Physical states are then characterized by D_5 vectors $\vec{\lambda}_{D_5} = (\lambda_{D_1}, \vec{\lambda}_{D_4})$ with fixed $\vec{\lambda}_{D_4}$ vector (to avoid overcounting):

$$\begin{aligned} \text{bosons : } & \lambda_{D_1} \in \mathbf{Z} \quad , \quad \vec{\lambda}_{D_4} = (0, 0, 0, -1) \\ \text{fermions : } & \lambda_{D_1} \in \mathbf{Z} + \frac{1}{2}, \quad \vec{\lambda}_{D_4} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right). \end{aligned}$$

In order to have N supersymmetry charges we need N gravitinos, i.e. N massless spin $3/2$ fields. Within the covariant lattice construction their holomorphic part should be described by a vector $\vec{\lambda} = (\vec{\alpha}, \vec{\lambda}_{D_5})$, where $\vec{\lambda}_{D_5}$ is a vector of $(\text{length})^2 = 5/4$ in one of the two spinor conjugacy classes of D_5 and $\vec{\alpha}$ has $(\text{length})^2 = 3/4$. Since $\vec{\lambda}$ must be the weight vector of a simply laced group, one finds [4] that D_5 must be the regular subalgebra of some larger algebra whose adjoint contains N D_5 spinors. In $d = 4$ this leads, for the case of $N = 2^i$ gravitinos, to the embeddings $D_5 \subset E_{6+i}$, $i = 0, 1, 2$. Since the triality rotation of the D_4 factor becomes an inner automorphism after the embedding in the exceptional groups and since physical bosons (fermions) are characterized by a vector (spinor) weight of D_4 , it follows [6] that we always have an equal number of bosonic and fermionic degrees of freedom as required for a supersymmetric spectrum.

The connection with our previous discussion of the world-sheet symmetries following from space-time supersymmetry is made by noting that the centralizers of D_5 in E_{6+i} are exactly the world-sheet symmetry groups $U(1)$, $SU(2) \times U(1)$ and $SO(6)$ for $i = 0, 1$ and 2 respectively. If we now decompose the conjugacy classes of

E_{6+i} to $D_5 \times \begin{pmatrix} U(1) \\ U(1) \times A_1 \\ D_3 \end{pmatrix}$ and make the truncation onto physical states $D_5 \rightarrow D_1$,

which in terms of conjugacy classes means $s \leftrightarrow c$ and $v \leftrightarrow 0$, we get exactly the conjugacy classes which resulted from locality in our discussion above. Since that discussion was completely model independent (relying only on the space-time supersymmetries), so is the appearance of the exceptional groups; it is not restricted to the covariant lattice models.

As was already indicated above, the representations of the exceptional groups contain the on-shell states of supersymmetry multiplets. Let us demonstrate this on one example. The $\underline{133}$ (adjoint) of E_7 decomposes into representations of $D_5 \times A_1 \times U(1)$ as follows:

$$\begin{aligned} \underline{133} = & \underbrace{(\underline{45}, \underline{1}, 0)}_{\substack{\text{graviton+dilaton} \\ + \text{antisymmetric tensor}}} + \underbrace{(\underline{16}, \underline{2}, -\frac{1}{2}) + (\overline{\underline{16}}, \underline{2}, +\frac{1}{2})}_{2 \text{ gravitinos} + 2 \text{ dilatinos}} \\ & + \underbrace{(\underline{10}, \underline{1}, \pm 1)}_{\substack{2 \text{ physical} \\ \text{vectors}}} + \underbrace{(\underline{1}, \underline{3}, 0) + (\underline{1}, \underline{1}, 0)}_{3+1 \text{ auxiliary vectors}} \end{aligned}$$

The interpretation of the different terms follows after multiplying the holomorphic part of the vertex operators with $\partial \bar{X}^\mu(\bar{z})$. We easily see that we have an equal number of physical bosonic (8) and fermionic (8) degrees of freedom. The two physical vectors are the two $U(1)$ gauge bosons arising from the torus compactification of a $N = 1$ theory in $d = 6$. The auxiliary vectors correspond to the (ungauged) $SU(2) \times U(1)$ world-sheet symmetry.

Since, as we have demonstrated, the appearance of the exceptional groups is a common feature of all supersymmetric string theories, we can utilise this and make some model independent statements about their partition functions. To be specific we will discuss the phenomenologically most interesting case $N = 1$. We denote the E_6 characters by $P_{(E_6)i}^{c=6}$ where the subscript i refers to the three E_6 conjugacy classes 0, 1 and $\bar{1}$ whose lowest dimensional representations are the $\underline{1}$, $\underline{27}$ and $\overline{\underline{27}}$. The superscript refers to the fact that the E_6 weight lattice is six-dimensional. The E_6 characters can be expressed in terms of Jacobi θ -functions (lattice contribution) and Dedekind's η -function (oscillator contribution). In terms of these the truncation $D_5 \rightarrow D_1$ is ($i = 2, 3, 4$):

$$\left[\frac{\theta_i(0|\tau)}{\eta(\tau)} \right]^5 \rightarrow -(-1)^i \frac{\theta_i(0|\tau)}{\eta(\tau)}$$

where the minus sign for $i = 2$ takes into account Fermi statistics. Making the truncation on the E_6 characters we arrive at the light-cone characters denoted by $P_{(E_6)i}^{c=2}$. It encompasses the $U(1)$ and the transverse space-time part. It can be shown that $P_{(E_6)i}^{c=2} = 0$ which is the reflection of a space-time supersymmetric spectrum. (Note that the original E_6 characters $P_{(E_6)i}^{c=6}$ do not vanish.) Through some simple manipulations the characters $P_{(E_6)i}^{c=2}$ can be written as

$$P_{(E_6)i}^{c=2} = \sum \frac{\left(\begin{smallmatrix} U(1) \text{ theta function} \\ \text{at level 6} \end{smallmatrix} \right)}{\eta(\tau)} \times \frac{\left(\begin{smallmatrix} D_1 \text{ theta function} \\ \text{at level 1} \end{smallmatrix} \right)}{\eta(\tau)}$$

which, when done explicitly, corresponds exactly to the decomposition of E_6 conjugacy classes into conjugacy classes of $D_1 \times U(1)$. The $U(1)$ theta functions are related to the characters of the six conformal families generated by the $N = 2$, $c = 1$ algebra with generators $T^{c=1}$, $G^\pm$ and J' (c.f. above). Those however do not transform irreducibly under modular transformations. Only after projecting onto states

with definite world-sheet fermion number do we get a representation of the modular group, namely the 12 $U(1)$ θ -functions at level six. The same discussion that was presented for the $N = 1$ case in terms of E_6 characters can now be repeated for $N = 2$ and $N = 4$ theories in terms of E_7 and E_8 characters.

Now, the general form of the partition function of any $N = 1$ supersymmetric heterotic string theory in $d = 4$ is of the form

$$\chi(\tau, \bar{\tau}) \sim \frac{1}{\text{Im}\tau} \frac{1}{|\eta(\tau)|^4} \sum_{ijk} P_{(E_6)_i}^{c=2}(\tau) P_{(N=2)_j}^{c=8}(\tau) P_k^{\bar{c}=22}(\bar{\tau}) a_{ijk}$$

where the a_{ijk} are integers constrained by modular invariance. $P_{(N=2)_j}^{c=8}$ are the characters of a $c = 8$ conformal field theory corresponding to Φ^{rat} . If we combine $\sum_{jk} a_{ijk} P_{(N=2)_j}^{c=8}(\tau) P_k^{\bar{c}=22}(\bar{\tau})$ to conformal families $P_i^{c=8, \bar{c}=22}(\tau, \bar{\tau})$, $i = 0, 1, \bar{1}$,

then modular invariance is preserved if $P_i^{c=8, \bar{c}=22}(\tau, \bar{\tau})$ transforms exactly like the characters $P_{(E_6)_i}^{c=2}(\tau)$. This is for instance the case in all left-right symmetric (2,2) compactifications where the partition function has the following structure:

$$\chi(\tau, \bar{\tau}) \sim \frac{1}{\text{Im}\tau} \frac{1}{|\eta(\tau)|^4} P_{(E_6)_i}^{c=2}(\tau) P_{(N=2)_j}^{c=8}(\tau) P_{(N=2)_{j'}}^{\bar{c}=8}(\bar{\tau}) P_{(E_6)_{i'}}^{\bar{c}=6}(\bar{\tau}) P_{(E_8)}^{\bar{c}=8}(\bar{\tau}) a_{ii'jj'}$$

where the $c = 8$ conformal theories are chosen to be the same.

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