CERN - TH.5203/88

BRST QUANTIZATION, $IOSp(D,2|2)$ INVARIANCE AND THE PCT THEOREM

E.GOZZI AND M.REUTER
CERN, Geneva

ABSTRACT

In this paper we formulate the BRST quantization of the spinless relativistic point particle without restricting the parameter space to one fundamental domain of the modular group. Modular invariance is instead enforced by a constraint on the physical subspace of the Hilbert space. The second-quantized theory has then a manifest $IOSp(D,2|2)$ -invariance. Studying this symmetry we find that a special element of the $SO(D,2)$ subgroup of $IOSp(D,2|2)$ can be identified which induces the PCT-transformation on the physical subspace. Using this, the PCT-theorem is derived as a simple consequence of the $IOSp(D,2|2)$ invariance, thus confirming the relevant role of this graded symmetry. The subgroup $SO(D,2)$ seems to play a central role also in another respect: it is by requiring covariance of the BRST quantization with respect to it, and not just invariance with respect to the Lorentz group $SO(D-1,1)$, that the overall $IOSp(D,2|2)$ symmetry is generated.

CERN-TH 5203/88

October 1988

1. INTRODUCTION

Recently the covariant quantization of reparametrization-invariant systems, such as (super)strings, has attracted much attention. The simplest system of this kind is the relativistic point particle, which has been studied in a variety of ways [1 – 14]. It turned out that BRST methods, in the frame-work of the Batalin- Fradkin-Vilkovisky [BVF] path-integral quantization [15, 16] are a powerful tool to handle such systems.

One of the most intriguing results, obtained in this context, was the discovery of Neveu and West [6] that the BRST and anti-BRST operator of the relativistic point particle in D-dimensional Minkowski space can be unified with the Poincaré group $IO(D-1,1)$, in an inhomogeneous orthosymplectic supergroup $IOSp(D,2|2)$ which, in addition, contains the generators of the Parisi-Sourlas-type supersymmetries [17]. This enlarged group acts on a D+4 dimensional space consisting of the D-dimensional Minkowski space with two additional bosonic coordinates (the evolution parameter and the Lagrange multiplier of the constraint) and two fermionic coordinates (the BFV ghosts). Parisi-Sourlas dimensional reduction then guarantees that the physical states of the theory “live” in D dimensions only. This result, first obtained for a scalar particle, has been generalized in various ways. Here we only mention the massive spinning particle [7][8][9], the massless particle and the associated orthosymplectic extension of the conformal group [10], the non-relativistic particle [11], and, most importantly, the application of the same techniques to string field theory [18 – 23] which was the original motivation.

With this paper we want to further clarify the *physical* meaning of the, still somewhat mysterious, $IOSp(D,2|2)$ -invariance and of some of its subgroups. It would be nice to do all this for strings, but we limit ourselves, in this paper, to the relativistic spinless particle.

Before going into details, let us first review some of the basic facts [24]. We consider world-lines $X^\mu(\tau)$, $\tau \in [\tau_1, \tau_2]$, $\mu=0, \dots, D-1$, of particles propagating in D-dimensional Minkowski space. The classical action is given by

$$S_{cl} = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{X}_\mu^2(\tau)} \quad (1.1)$$

and the canonical momenta are $P^\mu = m\dot{X}^\mu(\dot{X}_\mu^2)^{-\frac{1}{2}}$. They give rise to the constraint

$$\phi(\tau) \equiv P^2(\tau) - m^2 = 0 \quad (1.2)$$

Since the classical Hamiltonian vanishes identically, the corresponding first order action

reads

$$S_1 = - \int_{\tau_1}^{\tau_2} d\tau \{ \dot{X}^\mu P_\mu - \lambda(\tau)(P^2 - m^2) \} \quad (1.3)$$

where $\lambda(\tau)$ is the Lagrange multiplier enforcing the constraint $\phi(\tau) = 0$. The action (1.3) is invariant under the infinitesimal world line diffeomorphism generated by ϕ ,

$$\begin{aligned} \delta X^\mu(\tau) &= 2\varepsilon(\tau)P^\mu(\tau) \\ \delta P_\mu(\tau) &= 0 \\ \delta \lambda(\tau) &= \dot{\varepsilon}(\tau) \end{aligned} \quad (1.4)$$

provided the parameter $\varepsilon(\tau)$ vanishes at the end points: $\varepsilon(\tau_1) = \varepsilon(\tau_2) = 0$. The latter condition implies that canonical gauge choices [3] are not permissible. The Fradkin-Vilkovisky quantization of (1.3) consists of extending the phase-space by promoting λ and its conjugate momentum π to dynamical degrees of freedom. One also associates two ghosts η^i , $i=1,2$ and two antighosts Q_i , $i=1,2$ with the constraints $\phi = 0$ and $\pi = 0$, the latter being a consequence of treating λ as a dynamical variable. The Fradkin-Vilkovisky theorem states that, indicating with Ω the BRS charge, and for suitable boundary conditions,

$$Z_\chi \equiv \int [dX^\mu][dP_\mu][d\lambda][d\pi][d\eta^i][dQ_i] \exp \left\{ iS_1 + \int_{\tau_1}^{\tau_2} d\tau [\dot{\lambda}\pi + \dot{\eta}^i Q_i - \{\chi, \Omega\}] \right\} \quad (1.5)$$

is independent of the gauge-fixing functional χ and is identical to the standard path-integral for constrained systems [25]. This, however, assumes that χ is an admissible gauge fixing in the following sense [24]: Let us consider the action of the gauge group (1.4) in the space of functions $\lambda(\tau)$ and let us define (in analogy with Riemann surface theory) Teichmüller space to be the space of orbits of the connected part of the gauge group. The modular group is defined as the quotient of the complete gauge group by its connected component. Finally, the quotient of Teichmüller space by the modular group, is called modular space: the gauge inequivalent configurations of the system are characterized by the points in the modular space. For each point in modular space, an *admissible* gauge fixing selects one and only one representative from the orbits in the equivalence class of orbits belonging to the same point in modular space. Since the BFV quantization incorporates only gauge transformations continuously connected to the identity, there will be no such gauge fixing in general. The best one can achieve is to find a “good” gauge fixing χ such that the path-integral reduces to an integral over Teichmüller space. This integral then has to

be restricted “by hand” to a fundamental domain of the modular group in Teichmüller space. (For a thorough discussion of this point, we refer to Govaerts[24]). The above remarks apply to any reparametrization-invariant system. For the point particle they have the following concrete meaning: the orbits of the gauge group are characterized by the Teichmüller parameter

$$C \equiv \int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) \quad (1.6)$$

Because of the condition $\varepsilon(\tau_{1,2}) = 0$, C cannot be changed by the transformation (1.4). Hence Teichmüller space consists of the entire real line $\mathbb{R}$. The full symmetry group of (1.1) is doubly connected: besides the transformations connected to the identity, it contains orientation-reversing diffeomorphisms which exchange the endpoints $X^\mu(\tau_1)$ and $X^\mu(\tau_2)$, so the modular group is $\mathbf{Z}_2$. It is generated by $\tau \rightarrow -\tau$. Invariance of (1.3) requires that $\lambda(\tau)$ transforms as $\lambda(\tau) \rightarrow -\lambda(-\tau)$ and so the Teichmüller parameter changes its sign under $\mathbf{Z}_2$: $C \rightarrow -C$. Consequently modular space is the real half-line $\frac{\mathbb{R}}{\mathbf{Z}_2} = \mathbb{R}^+$. It has been shown[5][24] that the condition

$$\dot{\lambda}(\tau) = 0 \quad (1.7)$$

i.e., $\lambda(\tau) = \lambda_0 = \text{const}$ is a good gauge fixing for the point particle. The corresponding Teichmüller parameter is simply

$$C = \lambda_0(\tau_2 - \tau_1) \quad (1.8)$$

The transition amplitude from an initial point X_1^μ to a final point X_2^μ , obtained by evaluating (1.5) in this gauge, reads

$$G(X_2^\mu, X_1^\mu) = \int_{-\infty}^{\infty} dC \, C^{-\frac{D}{2}} \exp - \frac{i}{2} \left[\frac{(X_2 - X_1)^2}{C} + m^2 C \right] \quad (1.9)$$

It is expressed in terms of an integral over Teichmüller space, i.e., over the whole real C -axis. The correct final answer is obtained by restricting the C -integration to a fundamental domain of $\mathbf{Z}_2$, e.g. to $0 \leq C < \infty$. Then the RHS of (1.9) has a *unique* interpretation as a distribution (that amounts to replacing m^2 by $m^2 - i\epsilon$) and it coincides with the Feynman (or causal) propagator. Choosing the other fundamental domain $-\infty < C \leq 0$, leads to the anticausal (positive energy backward, negative energy forward) propagator. Teitelboim [3] gave a simple physical argument why one should integrate only over C (or

λ_0) values of a given sign. The argument goes as follows: any non-singular world-line parametrization $X^\mu(\tau)$ must have the property that two points on the trajectory have the same τ only if they coincide, so $X^\mu(\tau)$ must be a monotonic function of τ . Moreover in view of the Legendre-transformation between $\dot{X}^\mu$ and P^μ

$$\frac{dX^\mu}{d\tau} = 2\lambda P^\mu \quad (1.10)$$

we note that $\lambda(\tau)$ can never be zero for the relation (1.10) to be invertible, that means, (if we want a continuous function), λ must always be either positive or negative: this concludes the intuitive argument of Teitelboim.

In the following we shall study the “moding out” of $\mathbf{Z}_2$ in the context of the $IOSp(D,2|2)$ extension of the Poincaré group [6]. Instead of using the Feynman-amplitude approach to one-particle quantum mechanics reviewed above, we work in the Schrödinger picture which is a convenient starting point for second quantization. In this formulation λ becomes an argument of the wave-function Ψ and it is treated on the same footing as X^μ : $\Psi = \Psi(\lambda)$ (we ignore the other arguments of Ψ , such as X^μ or the ghosts, for the moment). In the work of Neveu and West [6][7] the Hilbert space of the system consists of square-integrable functions $\Psi(\lambda)$ defined for $\lambda \in [0, \infty)$. This amounts to dividing out the modular group from the very beginning, in fact in Schrödinger theory the action of $\mathbf{Z}_2$ on the wave-function is simply $\Psi(\lambda) \rightarrow \Psi(-\lambda)$. However, to make the OSp -invariance of the theory manifest, they have to extend the range of λ again. This is done by defining $\Psi(\lambda < 0)$ via the equation $\Psi(-\lambda) = \Psi^*(\lambda)$, $\lambda \in \mathbb{R}$. In the usual approach this equation is a *definition* introduced for mathematical convenience. Here we argue that one should base the quantization on a Hilbert space of functions $\Psi(\lambda)$, where $\lambda \in (-\infty, +\infty)$, and one then should implement the modular invariance condition as a constraint on the *physical states* of the theory. This means that a general wave-function depends on both positive and negative values of λ , but a physical-state is described by a function which does not distinguish between the two modular sectors $\lambda < 0$ and $\lambda > 0$ in an observable way. Therefore bilinears of the form $\Psi^* \hat{O} \Psi$, representing observables, should be even functions of λ . This is guaranteed if we require physical states to satisfy

$$\Psi(-\lambda) = \sigma \Psi^*(\lambda), \quad \lambda \in \mathbb{R} \quad (1.11)$$

where σ is an arbitrary but fixed phase. The reason for the complex conjugation of the RHS of (1.11) will become clear in the next section. In our treatment this condition (1.11) has

the same logical status as the requirement that physical states must be annihilated by the BRST operator. The latter condition implements gauge invariance with respect to “small” diffeomorphisms continuously connected to the identity, whereas (1.11) implements invariance under the “large”, orientation-reversing, diffeomorphisms. From (1.11) we recover the Neveu and West relation for $\sigma = +1$.

The $OSp(D,2|2)$ -invariance has remarkable consequences. We shall see, for instance, that the PCT-theorem[26] can be derived as a consequence of the $OSp(D,2|2)$ -Ward identities. Due to the extension of the Lorentz group, the PCT transformation becomes a special element of the group $O(D,2) \subset OSp(D,2|2)$ which lies in the connected component of $OSp(D,2|2)$.

The rest of this article is organized as follows: In Section (2) we repeat the main steps leading to the extension of the Poincaré algebra in a way which is suitable for our purposes. In Section (3) we discuss the cohomology of the BRST operator and the physical-state condition of the theory. Then, in Section (4), we write down the functional integral of the second-quantized field theory and investigate its dimensional reduction to the physical subspace. In particular we discuss the conditions under which dimensional reduction can occur in spaces with *Minkowski* signature. In Section (5) we construct the propagator of the $IOSp(D,2|2)$ -invariant field theory and show that its dimensional reduction automatically leads to causal, i.e. Feynmann boundary conditions. Finally Section (6) is devoted to the PCT-theorem as indicated above and to the related question of microcausality in field theory. The presentation might look too detailed but we feel that it is necessary for internal consistency and completeness and for fixing several points that were treated only superficially in the literature on the subject. A brief report of the work presented here will appear in ref.[27] .

2. BRST-QUANTIZATION

As a preparation we now derive the $OSp(D,2|2)$ symmetry of the spinless relativistic point particle[6] in a form which is suitable for the discussion of the following sections. The presentation partly follows ref.[7][8] .

As explained in the introduction, the extended phase-space is parametrized by the position coordinates X^μ , their conjugate momenta P^μ , the Lagrange multiplier λ , its conjugate momentum π , the ghosts η^i , $i=1,2$, and the antighosts Q^i . The canonical

(anti-)commutation relations are

$$\begin{aligned} [X^\mu, P_\nu] &= -i\delta_\nu^\mu \\ [\lambda, \pi] &= i \\ [Q_i, \eta^j] &= -i\delta_i^j \end{aligned} \quad (2.1)$$

Here $[\ , \]$ denotes the $\mathbf{Z}_2$ -graded commutator : $[A, B] \equiv AB - (-)^{[A][B]}BA$, where $[A]$ is equal to zero (one) if A is bosonic (fermionic). We now define new ghost variables

$$\begin{aligned} \eta^i &= (c, -i\wp) \\ Q_i &= (\bar{\wp}, i\bar{c}), \end{aligned} \quad (2.2)$$

with the ghost charges $\#(c) = \#(\wp) = 1$ and $\#(\bar{c}) = \#(\bar{\wp}) = -1$. ($\#$ denotes the ghost charge). The hermiticity assignments are such that c and $\bar{c}$ are hermitian and $\wp$ and $\bar{\wp}$ are antihermitian. The two commuting constraints $\phi = 0$ and $\pi = 0$ lead to the BRST operator

$$\Omega = c(P^2 - m^2) - i\wp\pi \quad (2.3)$$

We now make the following canonical transformation [7] :

$$c \rightarrow \lambda c, \ \wp \rightarrow \lambda\wp, \ \bar{c} \rightarrow \lambda^{-1}\bar{c}, \ \bar{\wp} \rightarrow \lambda^{-1}\bar{\wp}, \ \pi \rightarrow \pi + \{\bar{c}\wp - c\bar{\wp}\} \quad (2.4)$$

The BRST operator becomes

$$\Omega = \lambda[c(P^2 - m^2 - i\wp\bar{\wp}) - i\wp\pi] \quad (2.5)$$

To make contact with $OSP(D, 2|2)$ it is convenient to make a further canonical transformation to the new ghost variables

$$\begin{aligned} \tilde{x}_1 &= -i\sqrt{2}[c + \frac{1}{2}\bar{c}] \\ \tilde{p}^1 &= \frac{i}{\sqrt{2}}[\frac{1}{2}\bar{\wp} + \wp] \\ \tilde{x}_2 &= -i\sqrt{2}[c - \frac{1}{2}\bar{c}] \\ \tilde{p}^2 &= \frac{i}{\sqrt{2}}[\frac{1}{2}\bar{\wp} - \wp] \end{aligned} \quad (2.6)$$

with the hermiticity property $\tilde{x}_a^\dagger = -\tilde{x}_a$ and $\tilde{p}^{a\dagger} = \tilde{p}^a$ where $a=1,2$. Their anticommutators

are

$$\begin{aligned} [\tilde{p}^a, \tilde{x}_b] &= -i\delta_b^a \\ [\tilde{x}_a, \tilde{x}_b] &= 0 \\ [\tilde{p}^a, \tilde{p}^b] &= 0 \end{aligned} \quad (2.7)$$

In terms of the new variables the BRST charge reads

$$\Omega = \Omega_1 + \Omega_2 \quad (2.8)$$

where

$$\Omega_a = \frac{\lambda}{\sqrt{2}} \left[\frac{i}{2} \tilde{x}_a (P^2 - m^2 + i\varepsilon_{ab} \tilde{p}^a \tilde{p}^b) - i\tilde{p}^a \pi \right] \quad (2.9)$$

with $[\Omega_a, \Omega_b] = 0$. Similarly the anti-BRST operator $\bar{\Omega}$ is found to be

$$\bar{\Omega} = \Omega_1 - \Omega_2 \quad (2.10)$$

It can be shown that (Ω_1, Ω_2) transform according to the spinor representation of a $\text{Sp}(2)$ -group generated by certain bilinears in $\tilde{x}_a$ and $\tilde{p}^a$ (for details see ref. [8]). Since the classical Hamiltonian vanishes for the system under consideration, the BRST-invariant Hamiltonian is completely determined by the gauge fixing function χ :

$$H = [\chi, \Omega] \quad (2.11)$$

We shall make the choice

$$\chi = \frac{i}{2} \wp = \frac{1}{\sqrt{2}} (\tilde{p}^1 + \tilde{p}^2) \quad (2.12)$$

which, in the Fradkin-Vilkovisky path-integral, leads to the gauge condition $\dot{\lambda} = 0$ [24]. The resulting Hamiltonian reads

$$H = \frac{1}{2} \lambda [P_\mu^2 - m^2 + i\varepsilon_{ab} \tilde{p}^a \tilde{p}^b] \quad (2.13)$$

We now introduce the Schrödinger picture representation of the operators $X^\mu, P_\mu, \lambda, \pi, \tilde{x}_a, \tilde{p}^a$. The position variables $X^\mu, \lambda, \tilde{x}_a$ are realized as multiplicative operators, whereas the conjugate momenta are represented by the derivative operators

$$P_\mu = i\partial_\mu, \quad \pi = -i\frac{\partial}{\partial\lambda}, \quad \tilde{p}^a = -i\frac{\partial}{\partial\tilde{x}_a} \quad (2.14)$$

They act on wave-functions $\Psi(\tau)$ where τ is the evolution parameter (proper-time). As we discussed in the introduction, we have allowed for both positive and negative values of λ .

The τ evolution of Ψ is governed by the Schrödinger equation

$$\begin{aligned} i \frac{\partial \Psi}{\partial \tau} &= H \Psi \\ &= \frac{1}{2} \lambda [P^2 - m^2 + i \epsilon_{ab} \bar{p}^a \bar{p}^b] \Psi \end{aligned} \quad (2.15)$$

It can be derived by requiring stationarity of the following action

$$S = i \int d^D X \int_{-\infty}^{\infty} dp_+ \int_{-\infty}^{\infty} d\tau d\tilde{x}_1 d\tilde{x}_2 \Psi^* [2ip_+ \partial_\tau - \square + 2i\partial_{\tilde{x}_1} \partial_{\tilde{x}_2}] \Psi \quad (2.16)$$

where $\square \equiv -(\partial_\mu \partial^\mu + m^2)$. Here we have separated off a factor of $\lambda^{1/2}$ from Ψ and replaced λ by $p_+ \equiv \lambda^{-1}$ as the argument of the wave function. In the next section we shall see that requiring square-integrability of Ψ as a function of p_+ leads to the correct physical spectrum. Since Ψ has to vanish for $p_+ \rightarrow \infty$, it becomes zero for $\lambda = 0$. This also excludes problems of the type discussed in connection with equation (1.10). For later use we also note the superfield decomposition of Ψ

$$\Psi = \psi(X^\mu, p_+, \tau) + \tilde{x}_a \rho^a(X^\mu, p_+, \tau) + \tilde{x}_1 \tilde{x}_2 \alpha(X^\mu, p_+, \tau) \quad (2.17)$$

Equation (2.16) is the action of the BRST-field theory considered by Neveu and West [6]. It is invariant under the (anti-)BRST-transformation $\delta \Psi = (\epsilon \Omega + \bar{\epsilon} \bar{\Omega}) \Psi$ where ϵ and $\bar{\epsilon}$ are constant anticommuting parameters. Next we introduce the Fourier transform of Ψ with respect to p_+ . The conjugate variable will be called x_- . It also will prove convenient to set $x_+ \equiv \tau$:

$$\Phi(X^\mu, x_+, x_-, \tilde{x}_a) = \int_{-\infty}^{\infty} \frac{dp_+}{2\pi} e^{ip_+ x_-} \Psi(X^\mu, x_+, p_+, \tilde{x}_a) \quad (2.18)$$

(Note that contrary to the original treatment [6] p_+ is integrated over the whole real axis without constraining the p_+ dependence of Ψ at this stage.) Using the cartesian coordinates

$$\begin{aligned} X_D &= \frac{1}{\sqrt{2}}(x_+ - x_-) \\ X_{D+1} &= \frac{1}{\sqrt{2}}(x_+ + x_-) \end{aligned} \quad (2.19)$$

the action (2.16) can be written as

$$S = -(2\pi) \int d^{D+4} X \Phi^* \square_{bsp} \Phi \quad (2.20)$$

where $d^{D+4}X \equiv d^D X dX^D dX^{D+1} i d\tilde{x}_1 d\tilde{x}_2$ and

$$\begin{aligned}\square_{osp} &= -\square + 2\partial_{x_+}\partial_{x_-} + 2i\partial_{\tilde{x}_1}\partial_{\tilde{x}_2} \\ &= \eta^{AB}\partial_A\partial_B + m^2\end{aligned}\quad (2.21)$$

Indices A,B,... refer to coordinates $X^A = (X^\mu, X^D, X^{D+1}, \tilde{x}^a)$ of a $D+4$ dimensional space obtained from Minkowski space by adding two bosonic coordinates X^D and X^{D+1} and two fermionic coordinates $\tilde{x}^a$. We also use the notation $X^\alpha = (X^\mu, X^D, X^{D+1})$ for the bosonic coordinates. The metric tensor η_{AB} (or its inverse, η^{AB} , respectively) appearing in (2.21) is given by

$$\eta_{\alpha\beta} = \text{diag}[+1, -1, -1, \dots, -1, +1] \quad (2.22)$$

and

$$\eta_{ab} = i\varepsilon_{ab} \quad (2.23)$$

Equation (2.22) is the metric of a $D+2$ dimensional Minkowski space with two time-like coordinates, X^0 and X^{D+1} , while the other new coordinate, X^D , is space-like. The metric η_{AB} is left invariant by the supergroup $OSp(D,2|2)$ or, including translations, by $IOSp(D,2|2)$. The operator $\square_{osp}$ is nothing but the corresponding $OSp(D,2|2)$ -invariant Laplacian, hence the $IOSp(D,2|2)$ -invariance of the action (2.20) is manifest. The superfield Φ transforms according to the scalar representation of $OSp(D,2|2)$,

$$\Phi^\Lambda(X^A) = \Phi(\Lambda^A_B X^B) \quad (2.24)$$

with Λ^A_B being the representation matrices of the vector-representation. The generators, in the scalar representation, are

$$J_{AB} = X_A P_B - (-)^{[A][B]} X_B P_A \quad (2.25)$$

The inhomogeneous part of the group is generated by $P_A = (P_\mu, P_D, P_{D+1}, \tilde{p}_a)$ since it is easily verified that $[X_A, P_B] = -i\eta_{BA}$. Note in particular that $[x_-, p_+] = [x_+, p_-] = -i$. The commutation relations are given by

$$\left[J_{AB}, J_{CD} \right] = (-)^{[A][B]} i\eta_{CA} J_{BD} - i\eta_{CB} J_{AD} + i\eta_{DB} (-)^{[B][C]} J_{AC} + i\eta_{AD} (-)^{[A][B+C]} J_{BC} \quad (2.26)$$

$$\left[J_{AB}, P_C \right] = -i\eta_{CB} P_A + (-)^{[A][B]} i\eta_{CA} P_B \quad (2.27)$$

It has been observed [6][7][8][18] that the BRST and anti-BRST operators $\Omega = \Omega_1 + \Omega_2$

and $\bar{\Omega} = \Omega_1 - \Omega_2$ are contained in the $\text{OSp}(D, 2|2)$ algebra. In fact one finds

$$\Omega_a = \frac{-i}{\sqrt{2}} J_{a-} = \frac{-i}{\sqrt{2}} (x_- \tilde{p}_a - \tilde{x}_a i \frac{\partial}{\partial \tau}) \quad (2.28)$$

In the second step of (2.28) we used the fact that p_- is conjugate to $x_+ \equiv \tau$. Inserting the Schrödinger equation (2.15) for $i \frac{\partial}{\partial \tau}$, equation (2.28) becomes identical to equation (2.9)*. The nilpotency of Ω is a simple consequence of the $\text{OSp}(D, 2|2)$ algebra. To be precise, it follows from the $\text{OSp}(1, 1|2)$ -subalgebra acting on $(X^D, X^{D+1}, \tilde{x}^a)$ -space. This can be seen from (2.26) by inserting (2.28). It is also easy to see, using (2.26), that Ω and $\bar{\Omega}$ commute, in fact

$$[\Omega, \bar{\Omega}] = \frac{1}{2} [J_{1-}, J_{2-}] - \frac{1}{2} [J_{2-}, J_{1-}] = 0 \quad (2.29)$$

This last step follows from the fact that $[J_{1-}, J_{2-}]$ is even in the exchange of the two operators J_{1-}, J_{2-} . The other elements of this $\text{OSp}(1, 1|2)$ are the J_{+a} : they combine to give a second set of BRST-operators $\tilde{\Omega}$ and $\bar{\tilde{\Omega}}$ canonically equivalent to the first ones made of the J_{-a} [16] [23]. Next we have the (J_{ab}) whose various linear combinations give the ghost-charge and the two ghost-conjugation operators. Among the generators (2.26) there are others that we can immediately recognize [8], namely the $J_{a\mu}$ which generate Parisi-Sourlas [17] “superrotations” in superspace. Finally there are the $J_{\mu\pm}$ that together with the Lorentz generators $J_{\mu\nu}$ make the subgroup $O(4, 2) \subset \text{OSp}(D, 2|2)$. As far as BRST-cohomology and dimensional reduction are concerned the unification of $\text{OSp}(1, 1|2)$ with the Lorentz group $O(D-1, 1)$ is irrelevant. In this sense the $\text{OSp}(D, 2|2)$ symmetry comes as a surprise, since to project on the physical degrees of freedom, it is sufficient to work with the BRST and the ghost-charge-operator. Then, following Niemi [23], these operators can be extended to an $\text{Osp}(1, 1|2)$ algebra, and this holds [23] for all constrained systems. The reader might then wonder which is the reason for having this large algebra $\text{OSp}(D, 2|2)$: is it just a consequence of using the gauge $\dot{\lambda} = 0$? In particular, which is the physical role of the Parisi-Sourlas $J_{\mu a}$ generators of superrotation? Why do we have them? The answer might be that the Parisi-Sourlas generators $J_{\mu a}$ naturally appear if we require covariance of the BRST-quantization (that means of the generators $J_{ab}, J_{\pm a}$) with respect to the

* For this identification to hold one also has to define π in a slightly different way, i.e. as $\pi = -i \frac{\partial}{\partial \omega}$ where $p_{\pm} = \pm e^{\omega}$, the sign depending on the modular sector.

$O(4,2)$ extension $(J_{\mu\nu}, J_{\mu\pm}, J_{\pm\mp})$ of the Lorentz group $(J_{\mu\nu})$. We already know that the BRST-quantization is Lorentz -invariant, in fact

$$\left[J_{a\pm}, J_{\mu\nu} \right] = 0 \quad (2.30)$$

If we require that it is also $O(4,2)$ -covariant, then we naturally need the Parisi-Sourlas generators $J_{\mu a}$ of superrotations, since

$$\left[J_{a\pm}, J_{\mu\pm} \right] \propto J_{\mu a}, \quad \left[J_{\pm\mp}, J_{\mu\pm} \right] \propto J_{\mu\pm}, \quad \left[J_{ab}, J_{\mu\pm} \right] = 0 \quad (2.31)$$

From these relations we can conclude that the Parisi-Sourlas generators $J_{\mu a}$ are nothing else then the BRST-generators $J_{a\pm}$ “seen” in a $O(4,2)$ rotated “frame”. Of course we have not answered the original question of “Why the $Osp(D,2|2)$ -invariance?”, but we have reduced it to a more physical one “Why do we need BRST-covariance with respect to $O(4,2)$?”. So the problem is to understand the physics behind the $O(4,2)$. In Section (6) we will start digging out a piece of the physics hidden behind $O(4,2)$: we will in fact see how important the presence of this $O(4,2)$ is for our derivation the PCT-theorem of quantum field theory .

3. PHYSICAL STATES

In this section we investigate which subset of the wave-functions Ψ describes physical states. As usual these states are required to have vanishing ghost number, $\#\Psi = 0$, and to be annihilated by the BRST-operator: $\Omega\Psi = 0$. These conditions implement invariance of the model under “small” diffeomorphisms, *i.e.* those continuously connected to the identity. To enforce modular invariance in the observable sector we should first of all say how we represent the modular transformation on the states $\Psi(p_+)$. We know that on “space-time” $X^\mu, p_+, \tau, \tilde{x}^a$ the modular transformation acts as $p_+ \rightarrow -p_+$ and so, in the same way as it is usually done for the Lorentz group, we try to find a transformation law for Ψ such that the action is invariant. As it is easily checked the representation of this symmetry for the action (2.16) is given by

$$\Psi'(p'_+) = \sigma \Psi^*(p_+) \quad (3.1)$$

where $p'_+ = -p_+$ and σ is any complex number of modulus one. To have invariance of all bilinears $\Psi^* \hat{O} \Psi$ (with $\hat{O}$ a modular invariant operator) , we have to require the following

physical-state condition:

$$\Psi(X^\mu, -p_+, \tau, \tilde{x}_a) = \sigma \Psi^*(X^\mu, p_+, \tau, \tilde{x}_a) \quad (3.2)$$

The invariance of the action (2.16) under the transformation (3.1) has to be compared with its BRST-invariance. These two symmetries are the relics of the invariance of the classical action (1.1) under “large” and “small” diffeomorphisms, respectively. Here equation (3.2) has a deep physical meaning: it implements modular invariance in the physical sector, so it has the same status as $\Omega\Psi = 0$.

Besides being a square-integrable function of (X^μ, p_+, τ) , we also require Ψ to vanish for $p_+=0$. This is necessary to guarantee the hermiticity of operators like $-i\frac{\partial}{\partial p_+}$ on a single fundamental domain of $\mathbf{Z}_2$, i.e. on the half-line $p_+ \geq 0$ say. The analysis of the cohomology of Ω indeed confirms this definition of the Hilbert space. The condition (3.2) can be taken into account in various ways. One possibility, if we choose $\sigma=1$, is to observe that the Fourier transform Φ of the wave-functions satisfying^{*} $\Psi(-p_+)=\Psi^*(p_+)$ would be real. In Section (4) we shall see that this characterization of modular-invariant states leads to a *manifest* $\text{OSp}(D,2|2)$ -invariant second-quantized field theory. A second possibility to deal with (3.2) is to describe physical states by wave-functions $\Psi(p_+)$ which are defined only on the half-line $p_+ > 0$. The full wave function $\Psi(p_+)$, $p_+ \in (-\infty, +\infty)$, can be reconstructed from this “truncated” one using (3.2).

We now briefly discuss the cohomology of the BRST-operator. We work in the formalism with $\Psi(p_+)$, $p_+ \in [0, \infty)$. In this language, factoring out the standard equivalence [16]

$$\Psi \sim \Psi + \Omega\xi \quad (3.3)$$

and dividing by the modular group $\mathbf{Z}_2$ have a very similar status, (ξ is any function of ghost charge -1). In both cases we are interested in the information contained in Ψ which cannot be changed by the respective transformation. In the case of $\mathbf{Z}_2$ this information is provided by a function $\Psi(p_+)$ which is defined for positive values of p_+ only. Then, in a second step, one has to study to which extent this function can be changed by the transformations $\Psi \rightarrow \Psi + \Omega\xi$.

* We shall frequently omit the $(X^\mu, \tau, \tilde{x}_a)$ -arguments from the wave-function.

Acting on functions of $\tilde{x}_a$ the ghost-charge-operator has the representation

$$\# = \tilde{x}_1 \partial_{\tilde{x}_2} + \tilde{x}_2 \partial_{\tilde{x}_1} \quad (3.4)$$

Therefore the most general state with $\# \Psi = 0$ is of the form

$$\Psi = \psi(X^\mu, p_+, \tau) + \tilde{x}_1 \tilde{x}_2 \alpha(X^\mu, p_+, \tau) \quad (3.5)$$

Writing $\Omega \Psi = 0$ in component-form, one finds:

$$\frac{i}{2} \square \psi - \alpha - (p_+^2 \partial_{p_+} + \frac{1}{2} p_+) \alpha = 0 \quad (3.6)$$

To arrive at (3.6) we have properly symmetrized operator products like $\lambda \pi$. Next we identify the cohomology classes resulting from the identification (3.3). Since ξ has ghost number -1 it must be of the form

$$\xi = (\tilde{x}_1 - \tilde{x}_2) F(X^\mu, p_+, \tau) \quad (3.7)$$

In terms of the component fields, equation (3.3) reads

$$\psi \sim \psi - \sqrt{2} \left(p_+ \frac{\partial F}{\partial p_+} + \frac{1}{2} F \right) \quad (3.8)$$

$$\alpha \sim \alpha - \frac{i}{\sqrt{2} p_+} \square F \quad (3.9)$$

We now have to choose a specific representative of each cohomology class. Eq. (3.9) shows that α always can be chosen to vanish identically. The reason is that we may choose F to be a solution of the inhomogeneous wave-equation $\alpha - \frac{i}{\sqrt{2} p_+} \square F = 0$. After having set α to zero, the remaining freedom consists of the transformations (3.8) whose F (in order to maintain $\alpha=0$) now has to be a solution of the homogeneous equation $\square F = 0$. With the convenient redefinition $F \equiv (2p_+)^{-\frac{1}{2}} f$, equation (3.8) becomes:

$$(p_+)^{-\frac{1}{2}} \psi \sim (p_+)^{-\frac{1}{2}} \psi - \frac{\partial f}{\partial p_+} \quad (3.10)$$

Obviously the cohomology classes are characterized by the invariants of the transformation

(3.10) . The only candidate is

$$\psi_{phys}(X^\mu, \tau) \equiv \int_0^\infty dp_+ p_+^{-\frac{1}{2}} \psi(X^\mu, p_+, \tau) \quad (3.11)$$

Under the transformation (3.10) , this would change by an amount

$$f(p_+ = 0) - f(p_+ = \infty) \quad (3.12)$$

However, for admissible functions F , both $f(p_+ = 0)$ and $f(p_+ = \infty)$ vanish. This follows from the fact that ξ is an element of the same Hilbert space as Ψ . Consequently it has to vanish at the origin $p_+ = 0$. Moreover it has to be square integrable and this implies $f(p_+ = \infty) = 0$. Going now back to eq. (3.6) , for $\alpha = 0$ we observe that ψ_{phys} satisfies

$$\square \psi_{phys}(X^\mu, \tau) = 0 \quad (3.13)$$

Using this in the Schrödinger equation (2.15) , we find that ψ_{phys} actually does not depend on τ :

$$\frac{\partial \psi_{phys}}{\partial \tau} = 0$$

Thus the final answer for the physical field in terms of the wave-function Ψ is

$$\psi_{phys}(X^\mu) = \int_0^\infty \frac{dp_+}{\sqrt{p_+}} \Psi(X^\mu, p_+, \tau = 0, \tilde{x}_a = 0) \quad (3.14)$$

In (3.14) we decided formally to eliminate the τ dependence by putting $\tau = 0$. The form of the measure for the p_+ -integration has no physical meaning. It depends, for instance, on the precise definition of π (see, for example, the different conventions in [6][7][8]). Equation (3.14) is the desired relation by means of which we can determine for each Ψ to which cohomology class it belongs. As was to be expected, of course, these cohomology classes are in one to one correspondence with the solutions of the Klein-Gordon equation. Eq. (3.14) will be used in the discussion of the PCT-transformation in section (6). As was already noted in [6] , despite the $\text{OSp}(D,2|2)$ invariance of the action, the physical state condition is not $\text{OSp}(D,2|2)$ invariant. The physical sector of the theory, *i.e.* the Klein-Gordon equation $\square \psi_{phys} = 0$, has only a smaller $\text{O}(D-1,1)$ Lorentz-invariance.

4. SECOND QUANTIZATION AND DIMENSIONAL REDUCTION

We now want to formulate a second-quantized BRST-invariant field theory which has the Schrödinger equation (2.15) as its equation of motion. Therefore its (classical) action is given by equation (2.16). As we said previously the modular invariance condition (3.2) can be taken into account (with the choice $\sigma=1$) by requiring the Fourier transform (2.18) to be real. We therefore can represent the physical degrees of freedom by real fields Φ . Their dynamics is governed by an $\text{OSp}(D,2|2)$ -invariant action of the form (2.20). The generating functional for an $\text{OSp}(D,2|2)$ invariant field theory reads

$$Z[J] = \int [d\Phi] \exp\left(\frac{i}{2\pi} \int d^{D+4}X \{ \Phi \square_{\text{osp}} \Phi - V(\phi) + J\Phi \}\right) \quad (4.1)$$

with $d^{D+4}X \equiv d^D X dX^D dX^{D+1} i d\tilde{x}_1 d\tilde{x}_2$, and the normalization is chosen for later convenience. Note that in (4.1) we also have included an $\text{OSp}(D,2|2)$ invariant interaction term. Let us now discuss the Parisi-Sourlas dimensional reduction of the generating functional $Z[J]$ of equation (4.1). We shall show that, by suitably restricting the source function J , $Z[J]$ will describe a D -dimensional scalar theory. The usual Parisi-Sourlas dimensional reduction in *Euclidean* space follows from the formula

$$\int dx_1 dx_2 d\tilde{x}_1 d\tilde{x}_2 g(x_1^2 + x_2^2 + 2i\tilde{x}_1 \tilde{x}_2) = (2\pi i) g(0) \quad (4.2)$$

valid for all functions g depending on the bosonic variables (x_1, x_2) and fermionic variables $(\tilde{x}_1, \tilde{x}_2)$ only through the $\text{OSp}(2|2)$ -invariant combination $[x_1^2 + x_2^2 + 2i\tilde{x}_1 \tilde{x}_2]^*$. However, in our case, eq. (4.2) is not applicable since we are dealing with a pseudo-orthosymplectic group $\text{OSp}(1,1|2)$, whose invariant length-square is

$$-X_D^2 + X_{D+1}^2 + 2i\tilde{x}_1 \tilde{x}_2 = 2(x_+ x_- + i\tilde{x}_1 \tilde{x}_2) \quad (4.3)$$

(For the time being we can ignore the X^μ -coordinates.)

It can be shown that eq. (4.2) does not hold for a Minkowskian signature. In the literature [6][12][23] this problem was overcome by assuming that it was allowed to analytically continue X_D to purely imaginary values and then applying (4.2) to the Euclidean theory. (Here and in the following “Euclidean” and “Minkowskian” always refers to $(X_D, X_{D+1}$ -space.) We shall see that a priori this analytical continuation is not allowed.

* To be precise one also has to require that $g(\infty) = 0$.

A formula analogous to (4.2) for the length- element (4.3) can be derived, but in general it does not coincide with (4.2). Only when we use the additional physical input that the propagator of the theory is the Feynman one, we can then prove dimensional reduction for the Minkowski case. At first sight this seems to be a rather trivial point. However, the fact that dimensional reduction can be proven only for Feynman boundary conditions confirms the known fact that a sensible (*i.e.*, unitary, causal) relativistic D-dimensional quantum field theory can be constructed only with the Feynman propagator, but not with, for instance, the advanced or retarded propagator[†]. In section (5) we shall give another argument in favour of the Feynman boundary conditions, which shows that they are a consequence of the modular invariance condition.

We now derive the $OSp(1,1|2)$ analogue of equation (4.2). Working in the “light-cone” coordinates x_+ and x_- , we want to evaluate

$$I \equiv \int_{-\infty}^{\infty} dx_+ \int_{-\infty}^{\infty} dx_- \int d\tilde{x}_1 \int d\tilde{x}_2 g(x_+ x_- + i\tilde{x}_1 \tilde{x}_2) \quad (4.4)$$

Introducing its Fourier transform $\hat{g}$ by

$$g(y) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{-isy} \hat{g}(s) \quad (4.5)$$

we obtain

$$I = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \int_{-\infty}^{\infty} dx_- \hat{g}(s) \int d\tilde{x}_1 d\tilde{x}_2 e^{s\tilde{x}_1 \tilde{x}_2} \int_{-\infty}^{\infty} dx_+ e^{-isx_+ x_-} \quad (4.6)$$

The Grassman integration gives a factor $(-s)$ and the x_+ integral is equal to $\frac{(2\pi)}{|s|} \delta(x_-)$. Hence we end up with^{*}

$$I = - \int_{-\infty}^{\infty} ds \frac{s}{|s|} \hat{g}(s) \quad (4.7)$$

[†] Here and in the following the term “Feynman” or “causal” propagator always includes the possibility of the anti-causal propagator obtained with the substitution $m^2 \rightarrow m^2 + i\epsilon$ rather than with $m^2 \rightarrow m^2 - i\epsilon$. The standard choice of the causal rather than the anti-causal boundary conditions is a matter of convention.

^{*} Ref.[28] contains similar considerations but there the crucial sign factor $\frac{s}{|s|}$ has been ignored.

Obviously in general this integral is not proportional to

$$g(0) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \hat{g}(s) \quad (4.8)$$

We observe that a dimensional reduction formula analogous to (4.2) indeed holds if $\hat{g}(s)$ happens to vanish either for all positive or for all negative values of s . In this case one obtains

$$\int dx_+ dx_- d\tilde{x}_1 d\tilde{x}_2 g(x_+ x_- + i\tilde{x}_1 \tilde{x}_2) = \pm 2\pi g(0) \quad (4.9)$$

The upper (lower) sign holds if $\hat{g}(s) = 0$ for $s > 0$ ($s < 0$). It is easy to see that this condition is in fact fulfilled if $g(y)$ is holomorphic in the upper (lower) complex y -plane. Therefore the analyticity of $g(y)$ in either the upper or the lower half y -plane is a sufficient condition for the validity of (4.9).

Equipped with the Minkowski-space reduction formula (4.9) we now prove that for Feynman boundary condition and for sources restricted to the physical subspace,

$$J(X^\mu, X^D, X^{D+1}, \tilde{x}^a) = j(X^\mu) \delta(X^D) \delta(X^{D+1}) \delta(\tilde{x}^a) \quad (4.10)$$

the functional $Z[J]$ of (4.1) becomes identical to that of a D -dimensional, purely bosonic, theory

$$Z[J] = Z_D[j] \quad (4.11)$$

where

$$Z_D[j] \equiv \int [d\phi] \exp \left(i \int d^D X [\phi(-\partial_\mu^2 - m^2)\phi - V(\phi) + j\phi] \right) \quad (4.12)$$

Let us first introduce the superfield decomposition

$$\Phi = \phi(X^\mu, X^D, X^{D+1}) + \tilde{x}_a \rho^a(X^\mu, X^D, X^{D+1}) + \tilde{x}_1 \tilde{x}_2 \alpha(X^\mu, X^D, X^{D+1}) \quad (4.13)$$

and let us split the Lagrangian entering Z as

$$L_1(\Phi) = \Phi(-\partial_\mu^2 - m^2)\Phi - V(\Phi) \quad (4.14)$$

$$L_2(\Phi) = -2\Phi[\partial_{x_+}\partial_{x_-} + i\partial_{\tilde{x}_1}\partial_{\tilde{x}_2}]\Phi \quad (4.15)$$

Following an idea of Cardy [29] we define the interpolating-functional

$$Z^\omega[J] = \int [d\Phi] \exp\left(\frac{i}{2\pi} \int d^{D+4}X \left[[\omega + 2\pi(1-\omega)\delta(X^D)\delta(X^{D+1})\delta(\tilde{x}_a)] \cdot L_1(\Phi) + L_2(\Phi) \right]\right) \cdot \exp\left(i \int d^{D+4}X J\Phi\right) \quad (4.16)$$

Cardy [28] observed that for $\omega = 1$ this functional reduces to equation (4.1), *i.e.* $Z^{\omega=1} = Z$, and to Z_D for $\omega = 0$. The field $\phi(X^\mu)$ in $Z_D[j]$ is simply the lowest component field $\phi(X^\mu)$ of Φ for $X^D = X^{D+1} = 0$. All the other degrees of freedom decouple [28] due to the special form (4.10) of the sources. Relation (4.11) is established if we can show that Z^ω does not depend on ω , *i.e.*, that the following expression vanishes:

$$\frac{\partial}{\partial \omega} \ln Z^\omega = \frac{i}{2\pi} \int d^{D+4}X \left[1 - 2\pi\delta(X^D)\delta(X^{D+1})\delta(\tilde{x}_a) \right] \langle L_1(\Phi) \rangle_{\omega J} \quad (4.17)$$

Here the expectation value of L_1 has to be computed in the presence of the external sources. Since they are assumed to be of the form (4.12) they do not break the $\text{OSp}(1,1|2)$ -symmetry in $(X^D, X^{D+1}, \tilde{x}_a)$ -space. Therefore $\langle L_1 \rangle$ can depend on these variables only through the invariant combination $[x_+x_- + i\tilde{x}_1\tilde{x}_2]$ or through the δ -functions: $\delta(x_+)\delta(x_-)\delta(\tilde{x}_a)$. In this second case the dimensional reduction would be trivial, so we concentrate on the first case:

$$\langle L_1(\Phi(X^A)) \rangle_{\omega J} \equiv \hat{L}_{\omega J}(X^\mu, x_+x_- + i\tilde{x}_1\tilde{x}_2) \quad (4.18)$$

Equation (4.17) now reads in momentum space

$$\frac{\partial}{\partial \omega} \ln Z^\omega[J] = \frac{i}{2\pi} \int d^{D+4}P \left[2\pi\delta(p_+)\delta(p_-)\delta(\tilde{p}_a) - 1 \right] L_{\omega J}(P^\mu, p_+p_- + i\tilde{p}_1\tilde{p}_2) \quad (4.19)$$

The next step is to show that $L_{\omega J}(P^\mu, y)$, as a function of $y \equiv [p_+p_- + i\tilde{p}_1\tilde{p}_2]$, is holomorphic in the upper (or lower) y -plane so to be able to apply the analysis done for equation (4.9), which then implies that the RHS of (4.19) is zero. Let us first suppose that $L_{\omega J}$ is just a 2-point function and we choose Feynman b.c., then $L_{\omega J}$ becomes

$$\frac{1}{P_\mu^2 - m^2 + 2y + i\epsilon} \quad (4.20)$$

and we see that this function is holomorphic in the upper half plane in y . In general $L_{\omega J}$ will get contributions from Green's functions $\Gamma_{L_1}^n$ that have an L_1 insertion of

momentum P and n external currents with momenta $P_{(1)}, P_{(2)}, \dots, P_{(n)}$. In a perturbative expansion $\Gamma_{L_1}^n$ will have the form

$$\Gamma_{L_1}^n(P) = \sum_v N g^v \frac{1}{P_{(E)}^2 - m^2} \delta(P - P_{(1)} - \dots - P_{(n)}) \cdot \prod_I \int d^{D+4} K_{(I)} \frac{1}{K_{(I)}^2 - m^2} \delta(K_{(I)}, P, P_{(E)}) \quad (4.21)$$

where g is the coupling constant, $P_{(E)}$ are the momenta of the external currents, $K_{(I)}$ the internal loop-momenta, and $\delta(K_{(I)}, P, P_{(E)})$ is a formal expression to indicate the various δ -functions of momentum conservation at vertices of the internal loops, and N stands for all the combinatoric factors. What we have to show is the dimensional reduction of the Γ^n . A typical graph entering $\Gamma_{L_1}^n$ is

$$\int d^{D+4} K \frac{1}{(K^2 - m^2)((K - P)^2 - m^2)} \quad (4.22)$$

Let us now show that, if we use for the propagators the Feynman-propagator ($m^2 \rightarrow m^2 - i\epsilon$), this expression is holomorphic in the upper- y -plane. Let us rewrite (4.22) using the Feynman parameter x and let us make the change of variables $K' = K - P(1 - x)$, then (4.22) can be written

$$\int_0^1 dx \int d^{D+4} K \frac{1}{K^2 - m^2 + i\epsilon + P_{(D)}^2 x(1 - x) + y[x(1 - x)]} \quad (4.23)$$

where y is the usual expression and $P_{[D]}$ is the D -dimensional projection of P . It is easy to see that this graph is analytic in the upper half-plane in y because its coefficient $x(1-x)$ is positive. This kind of analysis can be done to all orders using the Feynman parameters and the corresponding changes of integration variables.

This completes the discussion of the dimensional reduction. It indicates that for a Minkowski-type "internal-space" (X^D, X^{D+1}) the Feynman boundary conditions are a *sufficient* conditions to have dimensional reduction. Of course they might not be a necessary one, but we will see in the next sections that there are other compelling reasons for using the Feynman-propagator.

5. THE $OSp(D,2|2)$ -FEYNMAN PROPAGATOR

In this section we discuss the propagator of the supersymmetric field theory defined in the last section. We shall see that the Feynman propagator can be obtained either from the requirement of modular invariance or of the OSp invariance. In D -dimensions, the conventional argument in favour of the Feynman boundary conditions relies upon the requirement of causality (See, for example, the nice explanation on p.61 of ref.[30]) . This establishes an intriguing connection between modular invariance, causality, and $OSp(D,2|2)$.

We want to construct a Green's function for the Schrödinger equation (2.15). In ordinary non-relativistic Schrödinger theory one would choose it to be the position-space representation of

$$\frac{1}{i\partial_\tau - H - i\epsilon} \quad (5.1)$$

The $i\epsilon$ term introduced in this way has the effect of propagating the wave function Ψ forward in τ , whereas its complex-conjugate Ψ^* is propagated backward. Inserting our Hamiltonian (2.13) and rescaling the propagator by $2p_+^*$, we obtain for (5.1)

$$\frac{1}{P_\mu^2 - m^2 + i\epsilon_{ab}\bar{p}^a\bar{p}^b - 2ip_+\partial_\tau + 2i\epsilon p_+} \quad (5.2)$$

Allowing for both signs of p_+ , this Green's function is inconsistent with the modular invariance condition

$$\Psi(-p_+) = \sigma\Psi^*(p_+) \quad (5.3)$$

because (5.2) propagates $\Psi(\Psi^*)$ forward (backward) irrespective of the value of p_+ . On the other hand, equation (5.3) forces us to treat $\Psi(p_+ > 0)$ and $\Psi^*(p_+ < 0)$ as well as $\Psi(p_+ < 0)$ and $\Psi^*(p_+ > 0)$ on an equal footing. This can be achieved by changing the propagation direction for negative values of p_+ . For (5.2) this simply means that the $i\epsilon p_+$ -term has to be replaced by an $i\epsilon$ -term. From a D -dimensional point of view the result is precisely the Feynman propagator: it can be obtained with the replacement $m^2 \rightarrow m^2 - i\epsilon$. From the higher-dimensional point of view, it has the effect of propagating $\Psi(p_+ > 0)$ and $\Psi^*(p_+ < 0)$ backward and $\Psi(p_+ < 0)$ together with $\Psi^*(p_+ > 0)$ forward

* This amounts to the same redefinition which led to the action (2.16).

in τ . The reversed direction for $p_+ < 0$ or $\lambda < 0$, was already suggested by the classical equation of motion

$$\frac{dX^o}{d\tau} = +2\lambda(\tau)P^o \quad (5.4)$$

A sign-change of λ without a concomitant sign-change of $\frac{d}{d\tau}$ would destroy the causal propagation pattern of positive P^o forward in time and negative P^o backward. We thus define the propagator in the following $\text{OSp}(D,2|2)$ -invariant way

$$\begin{aligned} G_F(X_2^A - X_1^A) &= \left\langle X_2^A \left| \frac{1}{\eta^{AB} P_A P_B - m^2 + i\epsilon} \right| X_1^A \right\rangle \\ &= \int \frac{d^{D+4}P}{(2\pi)^{D+2}} \frac{e^{-i\eta_{AB}(X_2^A - X_1^A)P^B}}{P_\mu^2 - m^2 + 2[p_+ p_- + i\tilde{p}^1 \tilde{p}^2] + i\epsilon} \end{aligned} \quad (5.5)$$

Introducing a Schwinger proper-time representation of the denominator of (5.5), G_F assumes the form

$$G_F(X^A) = \frac{i}{4\pi} \int_0^\infty ds (2\pi i s)^{-\frac{D}{2}} \exp -\frac{i}{2} \left[\frac{\eta_{AB} X^A X^B}{s} + (m^2 - i\epsilon)s \right] \quad (5.6)$$

The standard D -dimensional Feynman propagator is obtained by setting

$$X^D = X^{D+1} = \tilde{x}^a = 0 \quad (5.7)$$

This is what we expect to happen on the basis of the dimensional reduction analysis performed in the previous section. Also note that only with Feynman-type b.c., the Schwinger representation is possible in the way done here. The reader might have observed that the measure for the integration over modular space [24] is proportional to $s^{\frac{D}{2}}$, despite the fact that the theory is defined in $D + 4$ dimension. This is the consequence of a “one-particle Parisi-Sourlas mechanism” : bosonic and fermionic momenta contribute with opposite signs to the exponent of s . It is instructive to rewrite G_F in still another way. Performing the p_- integral in (5.5) one obtains

$$G_F(X^A) = \int \frac{d^D P}{(2\pi)^D} \int d\tilde{p}^1 d\tilde{p}^2 \int_{-\infty}^\infty \frac{dp_+}{4\pi p_+} e^{-i[P_\mu X^\mu + x_- p_+ + i\tilde{x}_1 \tilde{p}_2 - i\tilde{x}_2 \tilde{p}_1]} \tilde{G} \quad (5.8)$$

Where

$$\tilde{G} = [\theta(x_+) \theta(p_+) - \theta(-x_+) \theta(-p_+)] e^{\left(\frac{i x_+}{2 p_+} [P_\mu^2 - m^2 + 2i\tilde{p}^1 \tilde{p}^2 + i\epsilon]\right)} \quad (5.9)$$

Recalling that $x_+ \equiv \tau$, we indeed find that modes with positive p_+ (negative p_+) are

propagated forward (backward) in τ . A further evaluation of (5.8) leads to the proper-time representation (5.6) again.

A very important feature of the $D + 4$ -dimensional theory is that the bosonic subspace has two time-like directions, X^0 and X^{D+1} . Its isometry group is $IO(D, 2)$. It has a special $SO(2)$ subgroup of rotations in the “time-plane” (X^0, X^{D+1}) . This clearly shows that advanced or retarded boundary conditions, say, are incompatible with $OSp(D, 2|2)$ -symmetry since, by a continuous $SO(2)$ -rotation, one could transform the “forward” direction with respect to X^0 into the “backward” direction. In a certain sense, an $OSp(D, 2|2)$ -invariant propagator has to be symmetric between forward and backward. It is exactly the Feynman boundary condition which realizes this symmetry between future and past. Formally this can be seen as follows: consider a Green’s function of the form

$$\frac{1}{\eta^{AB} P_A P_B - m^2 + i\epsilon P_\alpha} \quad (5.10)$$

where P_α is any fixed bosonic momentum component. For $\alpha=0$ we would recover an extension of the D -dimensional advanced propagator. This Green’s function is inconsistent with $OSp(D, 2|2)$ -symmetry or, to be precise, with the $O(D, 2)$ subgroup of it. Let us assume $\alpha=0$, for instance, then a continuous $SO(2) \subset O(D, 2)$ rotation of π in the (X^0, X^{D+1}) plane changes the sign of P_0 and P_{D+1} and hence the sign of the $i\epsilon$ term. Obviously the same type of argument can also be applied when α is a spacelike index or to $\alpha=D + 1$. Also for a more general ansatz than (5.10) it is easy to convince oneself that only $O(4, 2)$ -invariant boundary conditions lead to the Feynman $i\epsilon$ -prescription. We observe that D -dimensional Minkowski space with only one extra time coordinate allows for other boundary conditions as well if one requires invariance only under the connected component of $O(D - 1, 1)$. In this case there are no continuous “time-rotations”, and hence the sign of a $i\epsilon P_0$ -term cannot be changed. This particular, and very crucial, subgroup $O(D, 2)$ will reappear again later on in the discussion of the PCT theorem.

Let us now go back to the propagator (5.5). Up to now we considered G_F as the propagator of a $D + 4$ dimensional field theory. It is amusing to note that it can also be interpreted in terms of a $[(D + 4) + 1]$ -dimensional one-particle quantum mechanics. Let us exponentiate the denominator of (5.5) as:

$$G_F(X_2^A - X_1^A) = -\frac{i}{2} \int_0^\infty ds e^{-\frac{i}{2}(m^2 - i\epsilon)s} \left\langle X_2^A \left| e^{-\frac{i}{2}\eta^{AB} P_A P_B s} \right| X_1^A \right\rangle \quad (5.11)$$

We have here the transition amplitude

$$\langle X_2^A, s | X_1^A, 0 \rangle \quad (5.12)$$

of a one-particle theory living in a world where X^A are the (graded) position coordinates and the auxiliary variable s plays the role of the time. The lagrangian of this theory is the graded generalization of a non relativistic free particle lagrangian

$$L = -\frac{1}{2}\eta_{AB}\dot{X}^A\dot{X}^B \quad (5.13)$$

(The dot denotes derivative with respect to s .) Using (5.13) we have a simple path-integral representation for the matrix-element appearing in equation (5.11) :

$$\langle X_2^A | e^{\frac{i}{2}\eta^{AB}P_AP_Bs} | X_1^A \rangle = \int [dX^A(s)] e^{-i\int_0^s ds \eta^{AB}\dot{X}^A\dot{X}^B} \quad (5.14)$$

This integral is over paths starting at $X^A(0)=X_1^A$ and ending at $X^A(s)=X_2^A$ and the propagator G_F is obtained by integrating the amplitude (5.12) over all proper times s .

6. PCT THEOREM AND MICROCAUSALITY

In this section we will use the tools developed in the previous parts of the paper to rederive the PCT-theorem of field theory. The modular invariance condition and the fact that the extended bosonic space has two time-directions will play a crucial role.

We begin our discussion with the one-particle Schrödinger wave-function and then extend the argument to the interacting field theory. Let us consider a Schrödinger wave-function $\Phi(X^A)$ that is the (in general complex) Fourier transform of $\Psi(p_+)$. We know that $\Phi(X^A)$ is equivalent to its $\text{OSp}(D,2|2)$ -transform:

$$\Phi^A(X^A) = \Phi(\Lambda^A_B X^B) \quad (6.1)$$

where Λ^A_B is a graded-orthogonal matrix representing an element of $\text{OSp}(D,2|2)$. We now consider a special group element that we indicate as Λ_{PCT} whose action on the bosonic coordinates

$$X^\alpha \equiv (X^\mu, X^D, X^{D+1}) \quad (6.2)$$

is given by

$$(\Lambda_{PCT})^\alpha_\beta X^\beta = -X^\alpha \quad (6.3)$$

and which acts on the ghost variables like the identity. In the following we assume $D = 4$; the case $D > 4$ is a trivial generalization. The transformation Λ_{PCT} changes the sign of all

six bosonic coordinates. It can be visualized as being composed of three rotations of angle π in the (X^0, X^{D+1}) , (X^1, X^2) and (X^3, X^{D+1}) -planes, respectively. As we already mentioned, the (X^0, X^{D+1}) -plane consists of two time-like directions. Due to the presence of the new X^{D+1} coordinate the time-reversal transformation $X^0 \rightarrow -X^0$ can be obtained by a continuous rotation, rather than by a discrete reflection as in $(3+1)$ -dimensional Minkowski space. Similarly, because of the extra space-like X^D direction, the reflection $X^3 \rightarrow -X^3$ is a simple rotation in the extended bosonic space. This together with a trivial π -rotation in (X^1, X^2) -space has the effect of a parity transformation in the physical X^μ -space. We stress that in the $\text{OSp}(D, 2|2)$ -sense, Λ_{PCT} is continuously connected to the identity: it consists of three special $SO(2)$ rotations. The additional time-like (space-like) coordinate $X^{D+1}(X^D)$ makes it possible to unify time-reflection T (parity P) with the connected component of the $O(3, 1)$ -Lorentz group as parts of the connected component of $O(4, 2) \subset \text{OSp}(4, 2|2)$. This fact is a special feature of the *relativistic* particle. For the non-relativistic particle the extension of the Galilei group leads to a smaller $\text{OSp}(D, 1|2)$ symmetry [11] which does not allow for such discrete transformations to be seen as continuous ones. In the “physical” X^μ -world Λ_{PCT} acts as a PT transformation:

$$X^\mu \rightarrow -X^\mu \quad (6.4)$$

In the “internal” space it induces the reflections

$$(X^D, X^{D+1}) \rightarrow -(X^D, X^{D+1}) \quad (6.5)$$

We now show that this latter transformation is related to the charge conjugation in the space of physical wave-functions. The Λ_{PCT} -transformed wave-function Φ^{PCT} is given by

$$\Phi^{PCT}(X^\mu, X^D, X^{D+1}, \tilde{x}_a) = \Phi(-X^\mu, -X^D, -X^{D+1}, \tilde{x}_a) \quad (6.6)$$

In terms of the Fourier-transformed state Ψ used in the physical-state analysis of section (3), eq. (6.6) reads (recall the relation (2.19) and $x_+ \equiv \tau$):

$$\Psi^{PCT}(X^\mu, p_+, \tau, \tilde{x}_a) = \Psi(-X^\mu, -p_+, -\tau, \tilde{x}_a) \quad (6.7)$$

We now require Ψ to be an element of the physical subspace of the Hilbert space. Recalling the cohomology discussion of section (3), it is clear that Ψ^{PCT} is in the physical subspace

too. Being a physical wave-function implies, in particular, that Ψ obeys the modular invariance condition:

$$\Psi(X^\mu, -p_+, \tau, \bar{x}_a) = \sigma \Psi^*(X^\mu, p_+, \tau, \bar{x}_a) \quad (6.8)$$

To divide out $\mathbf{Z}_2$ we adopt the point of view that p_+ assumes only positive values and that therefore the negative p_+ -argument of the RHS of equation (6.7) has to be interpreted as giving σ times the complex conjugate wave-function. So combining (6.7) with (6.8) we get:

$$\Psi^{PCT}(X^\mu, p_+, \tau, \bar{x}_a) = \sigma \Psi^*(-X^\mu, p_+, -\tau, \bar{x}_a) \quad (6.9)$$

To completely project this equation on the physical subspace, we apply the result of the BRST-cohomology analysis of section (3), which has been summarized in equation (3.14). To obtain a physical wave function from Ψ we have to set to zero the ghost coordinates and the τ argument, and we have to integrate p_+ (with a certain measure) over the positive half-line. Applied to equation (6.9), this prescription yields

$$\psi_{phys}^{PCT}(X_\mu) = \sigma \psi_{phys}^*(-X^\mu) \quad (6.10)$$

This is the desired result. It shows that Λ_{PCT} acts on the physical part of the wave-function by multiplication with a phase σ , reflection of the argument and complex conjugation. This transformation has to be compared with the action of the PCT-operator in second-quantized field theory on a (charged) scalar field operator $\hat{\psi}$:

$$(PCT)\hat{\psi}(X^\mu)(PCT)^{-1} = \bar{\sigma} \hat{\psi}^\dagger(-X^\mu) \quad (6.11)$$

The phase $\bar{\sigma}$ depends on the phases entering the definition of the P, C and T transformation. We observe that, identifying ψ_{phys} with $\hat{\psi}$, the transformations (6.10) and (6.11) coincide for $\sigma = \bar{\sigma}$. The above argument cannot be repeated for the P or T transformation separately, since in order to obtain the sign-change of p_+ in (6.7) both X^D and X^{D+1} had to change their sign. A priori equation (6.10) is a statement about the wave function ψ of first quantization. However in the Heisenberg-operator formulation of field theory, ψ would be transmuted into $\hat{\psi}$. We have thus shown that the connected component of $\text{OSp}(D, 2|2)$ contains a special transformation, namely Λ_{PCT} , which, when projected down to the physical subspace, acts like the familiar PCT-transformation. Again we would like to stress the role of the modular-invariance condition (6.8) in making this identification (between an element of OSp and the PCT-transformation) possible.

We now extend the discussion to the interacting case, *i.e.* to the field theory defined by equation (4.1). We will show that the PCT-theorem[26] can be obtained in a very simple way from the $OSp(4,2|2)$ -Ward-identities of this theory. The only slightly unusual point in obtaining these Ward identities is that the sources are restricted to have the special form (4.10). This is related to the fact, already mentioned in section (3), that, despite the $OSp(D,2|2)$ -invariance of the action, the physical state condition is not $OSp(D,2|2)$ invariant. In the path-integral formalism this manifests itself in the δ -functions appearing in (4.10). These δ -functions distinguish certain directions, in $(D+4)$ -dimensional space, along which the Parisi-Sourlas reduction acts. Thus our starting point is:

$$Z[j] = \int [d\Phi^\Lambda] \exp \left(i S_{osp}[\Phi^\Lambda] + i \int d^4 X j(X^\mu) \Phi^\Lambda(X^\mu, 0, 0, \tilde{0}) \right) \quad (6.12)$$

where we have indicated with $\tilde{0}$ the fact that both ghost variables have been put to zero. In this equation we wrote the integration variable as Φ^Λ , *i.e.*, the field obtained from Φ by a certain $OSp(4,2|2)$ -element Λ . By S_{osp} we denoted the $OSp(4,2|2)$ -invariant action of equation (4.1). Hence we have

$$S_{osp}[\Phi^\Lambda] = S_{osp}[\Phi] \quad (6.13)$$

Furthermore, excluding the possibility of anomalous $OSp(4,2|2)$ -breaking, we assume that also the measure is invariant:

$$[d\Phi^\Lambda] = [d\Phi] \quad (6.14)$$

Consequently the following equation holds for any $\Lambda \in OSp(D, 2 | 2)$:

$$\begin{aligned} Z[j] &= \int [d\Phi] \exp \left(i S_{osp}[\Phi] + i \int d^4 X j(X^\mu) \Phi(X^\mu, 0, 0, \tilde{0}) \right) \\ &= \int [d\Phi] \exp \left(i S_{osp}[\Phi] + i \int d^4 X j(X^\mu) \Phi^\Lambda(X^\mu, 0, 0, \tilde{0}) \right) \end{aligned} \quad (6.15)$$

The Ward identities are now easily derived by taking derivatives with respect to $j(X^\mu)$. Here we are only interested in the special case

$$\Lambda = \Lambda_{PCT} \quad (6.16)$$

acting as

$$\Phi^\Lambda(X^\mu, 0, 0, 0) = \Phi(-X^\mu, 0, 0, 0) \quad (6.17)$$

One obtains:

$$\langle 0 | T \phi(X_1^\mu) \cdots \phi(X_n^\mu) | 0 \rangle = \langle 0 | T \phi(-X_1^\mu) \cdots \phi(-X_n^\mu) | 0 \rangle \quad (6.18)$$

Here $\phi(X^\mu) \equiv \Phi(X^\mu, 0, 0, \tilde{0})$ is the physical lowest component part of Φ . After integrating out the other fields, that are decoupled from $J(X^\mu)$, one ends up with the conventional T-ordered (*i.e.* X^0 -ordered) n-point functions. From (6.18) it follows that:

$$\langle 0 | \phi(X_1^\mu) \cdots \phi(X_n^\mu) | 0 \rangle = \langle 0 | \phi(-X_n^\mu) \cdots \phi(-X_1^\mu) | 0 \rangle \quad (6.19)$$

A priori this equation holds only if

$$X_1^0 > X_2^0 \cdots > X_n^0 \quad (6.20)$$

However, (6.19) is even true for general Wightman-functions with any temporal relations among the X_i^0 . In fact when commuting the ϕ 's in (6.19), the resulting commutator-terms drop against each other. Equation (6.19) is nothing else than the PCT-theorem [26] expressed in terms of Wightman-functions for an interacting scalar theory. In its most general form it states that in any quantum field theory, (defined by a normal-ordered hermitian Lagrangian invariant under the *proper* Lorentz group, and quantized with the correct connection between spin and statistics), the Hamiltonian commutes with the PCT-operator. In the present case of a hermitian scalar field, we find from (6.19) that this operator acts as:

$$(PCT)\phi(X^\mu)(PCT)^{-1} = \phi(-X^\mu) \quad (6.21)$$

The phases entering the definition of the separate P, C, and T transformations are chosen here in such a way that there appears no phase or sign factor on the RHS of (6.21). It is *exactly* this definition of the PCT-operator, *i.e.*, this choice of phases, that Lüders used in his proof of the PCT- theorem [26]. In his language it is the product of “strong reflection” and hermitian conjugation^{*}. Since we obtain the PCT-transformation by rotating an

* It is interesting to note that Lüders [26] explicitly states that “*these are exactly the transformations which one would have if strong reflection could be obtained by a rotation.*” What he had in mind was a $O(4)$ rotation in an Euclideanized 4-space. In our derivation we indeed obtained the PCT-transformation by a continuous rotation. However, instead of going to Euclidean space, we added higher dimensions that appear naturally in the BRST quantization. Another direction would be to use the complex extension of the Lorentz group as it is done in the work of Jost [26]. We also would like to add that a crucial role is played by the Lorentz-group invariance. In our proof it is easy to understand where its features (contrasted to the one of the Galilei group for example) are used: this latter group admits an extension [11] that is only $OSp(D, 1|2)$ and so it is impossible to get the PCT-transformation as a continuous subgroup of it, not to mention that the charge-conjugation operation has no meaning in the non-relativistic physics.

OSp scalar, we obviously could not find any phase factor. One might wonder whether it is possible to apply a similar argument to separate P, C, or T transformations. After all, by a single SO(2) rotation in the spatial-plane (X^1, X^D), say, one could reverse the X^1 -direction which amounts to a parity-transformation. The answer to this question is negative. To see this we have to go slightly beyond our simple scalar model[†]. Consider, for instance, a scalar-spinor theory with interaction Lagrangian

$$L_{int} = ig\phi\bar{\psi}\gamma_5\psi \quad (6.22)$$

This theory is parity conserving, if the operator P acts on ϕ as :

$$P\phi(X)P^{-1} = -\phi(-X) \quad (6.23)$$

i.e. ϕ is a pseudoscalar. As we already mentioned, we represent all the transformations of a 4-dimensional scalar by the rotations of an OSp(4,2|2)-scalar. Hence we never can obtain the crucial minus sign on the RHS of (6.23). This shows that in general OSp(4,2|2)-invariance does *not* imply separate P, C and T invariance. What, on the contrary, is special about the product PCT is that it always can be defined as in (6.21) with no phase-factor appearing in the transformation law.

In the axiomatic discussion of the PCT-theorem, an important role is played by the postulate of microcausality. It states that the commutator-function

$$\langle [\phi(X^\mu), \phi(Y^\mu)] \rangle = i\Delta(X^\mu - Y^\mu) \quad (6.24)$$

has to vanish if the separation of the two points X^μ, Y^μ is space-like. By Lorentz invariance it is sufficient to require $\Delta(0, X^k) = 0$ where $k=1,2,3$. We now show that this property can also be obtained as an OSp(4,2|2)-Ward identity. To represent the commutator (6.24) inside the path-integral, we have to use the old Feynman trick of splitting the time of the two fields, so that (6.24) has the representation

$$i\Delta(0, X^k) = \int [d\Phi] \left(\Phi(\epsilon, X^k)\Phi(0, 0) - \Phi(0, X^k)\Phi(\epsilon, 0) \right) \exp(iS_{osp}[\Phi]) \quad (6.25)$$

Here we have omitted the X^D, X^{D+1} and $\tilde{x}^a$ arguments from Φ . They have to be set equal to zero at the end anyhow. Furthermore the limit $\epsilon \rightarrow 0^+$ is understood. By this

[†] In this paper we only consider the scalar case, however higher spin could be treated along the direction of [7] and [31].

ϵ -splitting we have broken the $SO(2)$ -symmetry in the (X^0, X^{D+1}) -plane. This gives a well defined meaning to the T-ordered product resulting from (6.25). Now we change the integration variable from Φ to Φ^Λ and exploit the $OSp(4,2|2)$ -invariance of the action and the measure. We choose Λ to be a π -rotation in the (X^1, X^2) - plane together with another π rotation in the (X^3, X^D) -plane. Hence the signs of (X^1, X^2, X^3, X^D) are reflected, but the time components X^0 and X^{D+1} (and the ghosts) remain unchanged. The only effect of Λ on the path-integral (6.25) is that X^k is reflected to $-X^k$. Hence we find that

$$\Delta(0, X^k) = \Delta(0, -X^k) \quad (6.26)$$

On the other hand, from the definition (6.24) we know that Δ is an odd function of its argument, therefore we may conclude that $\Delta(0, X^k)$ vanish identically. This establishes the validity of the microcausality condition in the 4-dimensional field theory as a consequence of the $OSp(4,2|2)$ invariance of the underlying 8-dimensional BRST-field theory. Let us now add another observation. For the above conclusion it was essential that Δ is an antisymmetric function and this follows from the commutator on the LHS of Eq. (6.24) . If we had tried to quantize a scalar field using anticommutation relations, Δ would be an even function and consequently our proof would not go through. In this case Δ would not vanish for space-like arguments (or at least we would not been able to prove it does) and causality would (or could) be violated. Actually, explicitly evaluating the anticommutator in a free bosonic theory, one finds that it does not vanish for space-like separated arguments. This suggests that it might be possible to shed some light on the general spin-statistics theorem by exploiting the $OSp(4,2|2)$ -invariance and micro-causality but more work is needed, especially for spin-1/2 and higher spin.* Here we have only indicated that, given the spin-statistics theorem, microcausality is a consequence of the OSp -invariance (at least for the scalar field). Anyhow it is interesting to observe that the microcausality condition which in the usual approach to quantum field theory has the status of a postulate, here appears on the same conceptual level as the PCT-invariance, which conventionally is considered a "theorem".

* It is curious to note that the authors of [7] , in doing the BRST quantization of the spinning particle, obtained the anticommuting nature of the second-quantized Fermi-fields, and this might indicate that, fully exploiting the OSp -invariance, a general spin-statistic theorem could be derived.

7. CONCLUSIONS

In this paper we have proposed a BRST formulation of the quantum mechanics of a massive spinless particle, in which the Schrödinger wave-function is defined for all real values of the modular parameter. BRST-invariance (related to “small” world-line diffeomorphisms) and modular invariance (related to “large” diffeomorphisms) are implemented in the same way, namely as constraints on the physical states of the theory. This choice of making the modular invariance manifest, makes also the $\text{OSp}(D,2|2)$ of Neveu and West [6] manifest. This mysterious interplay between the two symmetries has a further evidence in the fact that both of them lead to the Feynman-propagator as the unique choice of propagator. “Somehow” both these symmetries incapsulate in an intricate way the natural requirement of causality, which is so fundamental in field theory. It is in fact this requirement that automatically leads to the Feynman-propagator in the conventional approach to quantum field theory. That this OSp -invariance (like the modular-invariance) encodes causality, is further indicated by the fact that, once we have the OSp -invariance, the microcausality *postulate* of quantum field theory can be derived as a *theorem*. But what is more striking is that the PCT-invariance itself (a corner-stone of quantum field theory) can be derived in a very simple way from the requirement of OSp -invariance.

All these features indicate that something profound and universal is behind this OSp . We do not yet fully understand all its facets and the physical reason for requiring the $\text{OSp}(D,2|2)$ -invariance. We have indicated a possible one in section 2: there we showed that the full OSp is generated if one requires that the BRST-quantization is not only Lorentz-invariant, but also $O(D,2)$ -covariant. This subgroup $O(D,2)$ seems to be the crucial element in all these constructions. Note that this was also the relevant subgroup of $\text{OSp}(D,2|2)$ necessary for the proof of the PCT-theorem. Strangely enough, $O(D,2)$ is the conformal-group!

Clearly more work is needed to uncover the physics behind this $O(D,2)$. A first step could be to derive the same results presented here (especially the PCT-theorem and microcausality), for the spinning particle[7] and later on for strings and superstrings.

ACKNOWLEDGEMENTS

We want to thank P.Di Vecchia for having drawn our attention to the paper of Neveu and West [6] long ago, Carlo Becchi for discussions and, last but not least, Sergio Pugnetti for collaboration in the early stage of this work.

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