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PEDAGOGICAL INTRODUCTION TO BRST*

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Abstract

We review some recent results in the BRST quantization of constrained systems. In particular, we explain how the Parisi-Sourlas extension of BRST supersymmetry emerges and how it implies formal equivalence with reduced phase space quantization. We construct the pertinent generators of Parisi-Sourlas superrotations for both first and second class constraint algebras, and as an explicit example we consider open bosonic strings.

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I. Introduction

Recently, progress has been made in understanding conceptual aspects of BRST invariance and its role in the quantization of constrained systems [1-6]. In particular, it is now clear that BRST quantization is conceptually quite different from the conventional reduced phase space approach [7,8]. In the latter, unphysical phase space degrees of freedom are eliminated using the constraints *i.e.* fixing the gauge, and the physical degrees of freedom are then quantized. On the contrary, in BRST quantization unphysical degrees of freedom are *not* eliminated. Instead, the quantum theory is formulated in an extended phase space which is a Parisi-Sourlas superspace [9,10] with ghost degrees of freedom viewed as negative dimensional coordinates. The BRST operator is identified with the generator of a particular Parisi-Sourlas superrotation in this phase space, and the equivalence to the reduced phase space quantum theory is a consequence of the Parisi-Sourlas mechanism [2-6].

In this article we shall review conceptual aspects of this new approach to the quantization of constrained systems, following the discussion in [3-6]. In particular, we shall explain how the Parisi-Sourlas supersymmetry extends the BRST supersymmetry, and the role that it plays in the covariant phase space quantization of constrained systems. We explain why Parisi-Sourlas supersymmetry ensures unitarity and positivity, and in particular we discuss the formal equivalence between extended and reduced phase space quantum theories. We construct the Parisi-Sourlas $Osp(1,1|2)$ extension of BRST for both first and second class constraint algebras, and as an explicit example we present this construction for free open bosonic strings.

II. Reduced Phase Space Approach

The basic idea in the reduced phase space quantization of constrained systems is very simple: We are interested in path integrals that can be conceptually viewed as infinite dimensional versions of ordinary $D-2$ dimensional Euclidean space integrals

$$\int d^{D-2}x F(x^2) \quad (2.1)$$

where, with no loss of generality we take F to be a $SO(D-2)$ rotation invariant function. In theories with constraints, the integration domain in the path integral version of (2.1) is specified only implicitly through constraints, as a subspace of some higher dimensional

space. In the simplest case we may view the constrained version of (2.1) to be an integral over the $x_1 = x_2 = 0$ subspace of a D dimensional Euclidean space,

$$\int d^D x \delta(x_1) \delta(x_2) F(x^2) \quad (2.2)$$

with F now a $SO(D)$ rotation invariant function. More generally, if we assume that the D-2 subspace is determined by two functions $\phi_1(x) = \phi_2(x) = 0$, instead of (2.2) we have integrals such as¹

$$\int d^D x \delta(\phi_1) \delta(\phi_2) \det \left| \frac{\partial \phi_a}{\partial x_b} \right| F(x^2) \quad (2.3)$$

which reproduces (2.2) at least locally, when we solve the constraints $\phi_1 = \phi_2 = 0$ for the variables x_1, x_2 . The equivalence between (2.1), (2.2), (2.3) then summarizes the conceptual foundations of the reduced phase space approach [7,8].

In order to write the path integral versions of (2.1)-(2.3) we consider a Hamiltonian system defined in a $2N$ dimensional² phase space Γ . Darboux theorem [11] ensures that locally we can view Γ as a $2N$ dimensional Euclidean space with canonical coordinates $q^i, p_i; i = 1, \dots, N$ and nontrivial Poisson brackets

$$\{p_i, q^j\} = -\delta_i^j \quad (2.4)$$

This phase space is then the analog of the D dimensional space in (2.2), (2.3).

The analog of the D-2 dimensional space in (2.1) is the physical phase space Γ^* which is a $2(N-M)$ dimensional subspace of Γ . As a phase space Γ^* is locally isomorphic to a $2(N-M)$ dimensional Euclidean space, and for this we introduce the (local) canonical coordinates q^*, p_* on Γ^* , with Poisson brackets as in (2.4). In order to characterize Γ^* as a subspace of Γ , we first introduce M conditions

$$\varphi_a(q, p) = 0 ; \quad a = 1, \dots, M \quad (2.5)$$

Momentarily we assume that the constraint functionals φ_a satisfy a first class Poisson bracket algebra

$$\{\varphi_a, \varphi_b\} = U_{ab}^c \varphi_c \quad \& \quad \{H_0, \varphi_a\} = V_a^b \varphi_b \quad (2.6)$$

with $H_0(p, q)$ the hamiltonian, and for simplicity we take U_{ab}^c, V_a^b to be constants. We also assume that these constraint functionals are irreducible, which means that locally

¹Here, and in the following we assume for simplicity that all Jacobian determinants are positive definite.

²In this paper we use formulation adequate for a system with a finite number of degrees of freedom. The transition to field theory is made in the usual way.

we can use (2.5) to solve for the M unphysical components q^a of the position variables,

$$\varphi_a(p, q) = 0 \Rightarrow q^a = q^a(q^*, p_i) \quad (2.7)$$

In this way we reduce Γ into a $2N-M$ dimensional manifold. In order to complete the reduction to the $2(N-M)$ dimensional Γ^* we eliminate the M unphysical components p_a of the momentum variables p_i . For this we introduce another set of M functionals ψ^a (gauge fixing conditions) which satisfy

$$\det|\{\psi^a, \varphi_b\}| \neq 0 \quad (2.8)$$

at least in the vicinity of the constraint surface (2.5). If we select the ψ^a such that their Poisson brackets vanish,

$$\{\psi^a, \psi^b\} = 0 \quad (2.9)$$

by a canonical transformation we can identify ψ^a with the unphysical components p_a of the momentum variables p_i ,

$$\psi^a \sim p_a \quad (2.10)$$

and the physical phase space Γ^* is specified by the conditions

$$\varphi_a = \psi^a = 0 \quad (2.11)$$

In the local canonical coordinates q^*, p_* in Γ^* , the quantum theory is defined by the path integral

$$Z \sim \int_{\Gamma^*} [dq^* dp_*] \exp\{i \int (\dot{q}^* p_* - H^*) dt\} \quad (2.12)$$

where H^* is the restriction of the original hamiltonian H_0 to the surface Γ^* . We view (2.12) as the path integral version of (2.1). The corresponding version of (2.2) is

$$Z \sim \int [dq^* dp_*] [dq^a dp_a] \delta(q^a) \delta(p_a) \exp\{i \int (\dot{q}^* p_* + \dot{q}^a p_a - H_0) dt\} \quad (2.13)$$

while the corresponding (2.3) is

$$Z \sim \int [dq^i dp_i] \delta(\varphi_a) \delta(\psi^a) \det|\{\varphi_a, \psi^b\}| \exp\{i \int (\dot{q}^i p_i - H_0) dt\} \quad (2.14)$$

In this sense, the reduced phase space quantization is indeed conceptually identical to (2.1)-(2.3).

Finally, if we introduce Lagrange multipliers λ^a , π_a and anticommuting Faddeev-Popov ghosts η^a , $\mathcal{P}_a$ we can write (2.14) in the equivalent form

$$Z \sim \int [dq^i dp_i] [d\lambda^a d\pi_a] [d\eta^a d\mathcal{P}_a] \exp\{i \int (\dot{q}^i p_i - H_0 - \lambda^a \varphi_a - \pi_a \psi^a - \eta^a \{\varphi_a, \psi^b\} \mathcal{P}_b) dt\} \quad (2.15)$$

From this expression covariant representations can be derived by changes of variables [8].

In the following it will be useful to eliminate the hamiltonian as follows: We rewrite (2.12) in the form

$$Z = \int [dq^* dp_*] [dq^0 dp_0] \delta(p_0 + H^*) \delta(q^0 - t(\tau)) \exp\{i \int (\dot{q}^* p_* + \dot{q}^0 p_0) d\tau\} \quad (2.16)$$

where τ now parametrizes the evolution instead of the original time t . Obviously, we can view (2.16) as a constrained integral of the form (2.2), (2.13) in an extended phase space with additional canonically conjugate variables q^0 and p_0 , and conditions

$$\varphi_0 \sim p_0 + H^* = 0 \quad \& \quad \psi^0 \sim q^0 - t = 0 \quad (2.17)$$

The previous discussion can be immediately generalized to second class constraint algebras: Instead of (2.11) the physical phase space Γ^* is now specified by $2M$ linearly independent conditions

$$\phi^a = 0 ; \quad a = 1, \dots, 2M \quad (2.18.a)$$

such that

$$\{\phi^a, \phi^b\} = C^{ab} \quad \& \quad \det|C^{ab}| \neq 0 \quad (2.18.b)$$

where C^{ab} is an antisymmetric matrix, nondegenerate on the surface (2.18.a). Clearly, ϕ^a can be viewed as a combination of φ_a and ψ^a , except that in the general case a separation into "constraints" φ_a and "gauge conditions" ψ^a might not be possible or practical, or then it violates *e.g.* manifest Lorentz invariance.

Using Darboux theorem we again introduce a locally defined change of variables $\phi^a \rightarrow q^a, p_a$ with q^a and p_a the unphysical components of the phase space variables. Locally, the physical phase space Γ^* is then determined by

$$q^a = p_a = 0 ; \quad a = 1, \dots, M \quad (2.19)$$

In these local coordinates the quantum theory is again defined by (2.12), (2.16) or equivalently by (2.13), but since ϕ^a contains both q^a and p_a , *i.e.* φ_a and ψ^a , instead of (2.14) we get

$$Z = \int [dq^i dp_i] \delta(\phi^a) \sqrt{\det|\{\phi^a, \phi^b\}|} \exp\{i \int p_i \dot{q}^i\} \quad (2.20)$$

where we have assumed the reformulation (2.16). The Faddeev-Popov representation (2.15) can also be introduced, except that now only a single self-conjugate ghost η_a appears.

The previous discussion summarizes the reduced phase space method [7,8]. In principle, it provides a solution to the problem of quantization for systems with irreducible first and second class constraint algebras. However, except for the simplest case with constant U_{ab}^c , V_a^b , C^{ab} this quantization usually fails to be practical, and alternative formulations become necessary. Further difficulties become apparent in quantization of relativistic constraint algebras. For example in applications to Yang-Mills theory the variables x_1, x_2 in (2.2) correspond to the timelike and longitudinal components of the gauge field and their canonical conjugates, and if we eliminate these variables we also sacrifice manifest Lorentz invariance. This can be viewed as an analog of breaking the manifest $SO(D)$ invariance of F and the measure in (2.2), (2.3). Similarly, in string theories x_1 and x_2 are the timelike and longitudinal components of the spacetime coordinates. Now elimination of these components also eliminates part of the Minkowski space, and while manifest covariance can be restored in the pertinent version of (2.15) by introducing proper changes of variables, at the level of hamiltonian operator formulation covariance fails to be manifest.

The difficulties with the reduced phase space approach become more serious when one tries to quantize Lorentz invariant reducible constraint algebras such as antisymmetric tensor fields that naturally appear in supergravity and superstring theories, or higher rank constrained systems such as (super)membranes. Similarly, the quantization of second class constraint algebras such as Green-Schwarz superstring or Brink-Schwarz superparticle has not been very succesful in the reduced phase space formalism. In order to quantize these and other complicated systems that are presently becoming popular, it is necessary to introduce manifestly covariant techniques. In the following we shall describe one such technique, the Parisi-Sourlas reformulation of the BRST quantization of constrained systems [3-6].

III. Parisi-Sourlas Supersymmetry

In this section we shall outline a conceptual reformulation of (2.1)-(2.3) that *preserves* the $SO(D)$ invariance of the measure and the function F in (2.2), (2.3), and reproduces (2.1) *without explicitly restricting* the integrals to D-2 dimensional subspaces. In appli-

cations to covariant constrained systems this implies that symmetries such as manifest Lorentz invariance can be preserved. Such unrestricted representations of (2.1) in higher dimensional spaces are possible in the Parisi-Sourlas formalism [9,10]: Instead of reducing the integral as in (2.2), (2.3) we shall directly reproduce (2.1) using analytic continuation in dimensions, *i.e.* something like

$$\int d^{D-2} F(x^2) \sim \lim_{d \rightarrow -2} \int d^D x d^d y F(x^2 + y^2) \quad (3.1)$$

We then realize the negative dimensional coordinates in (3.1) by anticommuting variables, and in this way we find a representation of the D-2 dimensional integral as an integral over a D+2 dimensional superspace. In particular, the D dimensional rotation invariance of the measure and F in (2.2), (2.3) does not have to be compromised, in fact it will be extended into a D+2 dimensional Parisi-Sourlas superrotation invariance. In phase space applications these Parisi-Sourlas superrotations then generalize the conventional BRST supersymmetry.

In order to explain how the negative dimensional coordinates in (3.1) can be realized by anticommuting variables, consider an extended D+2 dimensional superspace Σ which in addition to the D Euclidean (or Minkowskian) coordinates x^μ includes two anticommuting coordinates θ and $\bar{\theta}$ such that $\theta^2 = \bar{\theta}^2 = \theta\bar{\theta} + \bar{\theta}\theta = 0$. We denote coordinates on Σ by

$$y^\alpha = (y^\mu, y^\theta, y^{\bar{\theta}}) = (x^\mu, \theta, \bar{\theta}) \quad (3.2)$$

and generalize D dimensional rotations into D+2 dimensional superrotations by replacing x^2 by

$$y^2 = g_{\alpha\beta} y^\alpha y^\beta = x^2 + \bar{\theta}\theta \quad (3.3)$$

where the nonvanishing components of the metric tensor $g_{\alpha\beta}$ are

$$g_{\mu\nu} = \eta_{\mu\nu} \quad \& \quad g_{\bar{\theta}\theta} = -g_{\theta\bar{\theta}} = \frac{1}{2} \quad (3.4)$$

The invariances of (3.3) determine the group of orthosymplectic superrotations, or $\text{Osp}(D|2)$. Its bosonic parts are rotations $\text{SO}(D)$ that leave x^2 invariant and $\text{Sp}(2)$ that leave $\bar{\theta}\theta$ invariant. But in addition there are also graded transformations that mix x^μ with θ and $\bar{\theta}$, for example

$$x^\mu \rightarrow x^\mu + \frac{1}{2} \epsilon^\mu \theta \quad (3.5.a)$$

$$\bar{\theta} \rightarrow \bar{\theta} - \epsilon^\mu x_\mu \quad (3.5.b)$$

$$\theta \rightarrow \theta \quad (3.5.c)$$

with ϵ^μ (infinitesimal) anticommuting D-vector. By considering all invariances of (3.3) the graded Lie algebra of $\text{Osp}(D|2)$ can be abstracted [10],

$$[R^{\alpha\beta}, R^{\gamma\delta}] = g^{\beta\gamma} R^{\alpha\delta} - (-)^{\gamma\delta} g^{\beta\delta} R^{\alpha\gamma} - (-)^{\alpha\beta} g^{\alpha\gamma} R^{\beta\delta} + (-)^{\alpha\beta} (-)^{\gamma\delta} g^{\alpha\delta} R^{\beta\gamma} \quad (3.6)$$

where the grading $(-)^{\alpha\beta}$ equals minus one if both α and β are anticommuting indices, plus one otherwise.

Since (3.6) is a grading of ordinary rotations, its representations can be constructed by grading representations of ordinary rotations. For example, if we grade the differential representation of ordinary rotations we find the differential representation of (3.6),

$$R^{\alpha\beta} = (-)^{\alpha\beta} y^\alpha \partial^\beta - y^\beta \partial^\alpha \quad (3.7)$$

where

$$\partial_\alpha = g_{\alpha\beta} \partial^\beta = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}}) = \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar{\theta}} \right) \quad (3.8)$$

Notice in particular that generators such as

$$Q_{\mu\nu} = R^{\mu\theta} + i R^{\nu\bar{\theta}} ; \quad \mu \neq \nu \quad (3.9)$$

are nilpotent. In a Minkowski space with coordinates x^μ , we find similarly that light-cone generators such as $R^{-\theta}$ are nilpotent: In the following section we shall argue that in phase space applications the BRST operator can be identified with the nilpotent generator $R^{-\theta}$ of an $\text{Osp}(1,1|2)$ algebra.

We define integration over anticommuting variables by

$$\int d\bar{\theta} d\theta \begin{Bmatrix} 1 \\ \theta \\ \bar{\theta} \\ \theta\bar{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \quad (3.10)$$

and consider superspace integrals of functions F that depend on their variables in $\text{Osp}(D|2)$ invariant combinations

$$\int d^D x d\bar{\theta} d\theta F(x^2 + \bar{\theta}\theta) = \frac{1}{\pi} \int d^{D-2} x F(x^2) - \frac{1}{\pi} F(\infty) = \frac{1}{\pi} \int d^{D-2} x F(x^2) \quad (3.11)$$

where we have restricted to functions F with $F(\infty) = 0$. From this we conclude that in $\text{Osp}(D|2)$ invariant integrations the θ and $\bar{\theta}$ variables *cancel* two bosonic variables.

Consequently θ and $\bar{\theta}$ can be viewed as coordinates in a minus two dimensional space in the sense of analytic continuation as in dimensional regularization. Indeed,

$$\lim_{D \rightarrow -2} \int d^2 x d^D y F(x^2 + y^2) = \lim_{D \rightarrow -2} \int_0^\infty d\alpha \int d^2 x d^D y e^{-\alpha(x^2 + y^2)} \hat{F}(\alpha) = \pi \int d^2 x d\bar{\theta} d\theta F(x^2 + \bar{\theta}\theta) \quad (3.12)$$

The *l.h.s.* of (3.11) is then the desired representation of (2.1). In particular, since x_1 and x_2 are now cancelled against θ and $\bar{\theta}$ instead of being eliminated explicitly as in (2.2), (2.3), in the representation (3.11) of (2.1) the D dimensional rotation invariance in (2.2), (2.3) (corresponding to *e.g.* manifest Lorentz invariance in phase space versions) remains unbroken, and is in fact extended into a D+2 dimensional superrotation invariance.

In applications to constrained systems, the integrands in the path integral versions of (3.11) in general do not have the simple $\text{Osp}(D|2)$ form of (3.11). However, it turns out that in the *unphysical* sector, these integrands can always be related to integrands of the form (3.11) by a simple change of variables. In order to explain this change of variables, we consider a generalization of (3.5) with ϵ^μ now a nontrivial function of the coordinates y^α ,

$$x^\nu \rightarrow x^\nu + \frac{1}{N} [x^\nu, R^{\mu\theta}] \Psi = x^\nu + \frac{1}{N} \Psi \eta^{\mu\nu} \theta \quad (3.13.a)$$

$$\bar{\theta} \rightarrow \bar{\theta} + \frac{1}{N} [\bar{\theta}, R^{\mu\theta}] \Psi = \bar{\theta} + \frac{2}{N} \Psi x^\mu \quad (3.13.b)$$

$$\theta \rightarrow \theta + \frac{1}{N} [\theta, R^{\mu\theta}] = \theta \quad (3.13.c)$$

where N is an integer introduced for later convenience, and $\Psi(y)$ is an arbitrary function of the superspace coordinates y^α generalizing ϵ^μ to a coordinate dependent quantity. Even though the squared length (3.3) remains invariant under (3.13) the measure in (3.11) fails to be invariant. Instead we have a superJacobian

$$\int d^D x d\bar{\theta} d\theta F(y^2) \xrightarrow{R^{\mu\theta}} \int d^D x d\bar{\theta} d\theta \left\{ 1 - \frac{1}{N} [\Psi, R^{\mu\theta}] + \mathcal{O}\left(\frac{1}{N^2}\right) \right\} F(y^2) \quad (3.14)$$

We repeat (3.14) N times in a basis (*e.g.* light-cone basis) with $R^{\mu\theta}$ is nilpotent. In the limit $N \rightarrow \infty$ the Jacobian exponentiates,

$$\left(1 - \frac{1}{N} [\Psi, R^{\mu\theta}] \right)^N \xrightarrow{N \rightarrow \infty} e^{-[\Psi, R^{\mu\theta}]} \quad (3.15)$$

and we get

$$\int d^D x d\bar{\theta} d\theta F(y^2) = \int d^D x d\bar{\theta} d\theta e^{-[\Psi, R^{\mu\theta}]} F(y^2) \quad (3.16.a)$$

In particular, since the *l.h.s.* of (3.16.a) is independent of Ψ , the *r.h.s.* must also be independent of Ψ , which constitutes a proof³ of the Fradkin-Vilkovisky theorem [1] which states the gauge *i.e.* Ψ independence of the path integral in the phase space version of (3.16.a).

Notice that with $R^{\nu\bar{\theta}}$ another nilpotent operator such that $[R^{\nu\bar{\theta}}, [\Psi, R^{\mu\theta}]] = 0$ and Φ another arbitrary function of y^α , (3.16.a) generalizes into

$$\int d^D x d\bar{\theta} d\theta F(y^2) = \int d^D x d\bar{\theta} d\theta \exp\{[\Psi, R^{\mu\theta}] + [\Phi, R^{\nu\bar{\theta}}]\} F(y^2) \quad (3.16.b)$$

and so on.

In the following section we shall find that in the phase space versions the integrands always have the simple Osp structure of (3.16) in the unphysical sector of the theory, with the pertinent $\text{Osp}(1,1|2)$ superrotation invariance generalizing the BRST supersymmetry. Physical states are then identified with $\text{Osp}(1,1|2)$ singlet states, which ensures positivity and unitarity: Indeed, consider an arbitrary function $G(x, \theta, \bar{\theta})$ on Σ and expand it as

$$G(x, \theta, \bar{\theta}) = u(x) + \bar{v}(x)\theta + v(x)\bar{\theta} + w(x)\bar{\theta}\theta \quad (3.17)$$

and introduce a conjugation $\bar{G}$ by

$$\bar{G}(x, \theta, \bar{\theta}) = u^*(x) + \bar{v}^*(x)\theta + v^*(x)\bar{\theta} + w^*(x)\bar{\theta}\theta \quad (3.18)$$

with "*" complex conjugation, and define the inner product of two functions G_1 and G_2 by

$$\langle G_1 | G_2 \rangle = \int d^D x d\bar{\theta} d\theta \bar{G}_1 G_2 \quad (3.19)$$

Functions normalizable with respect to this inner product then determine a Hilbert space. Since this inner product is not positive definite the Hilbert space admits both negative and zero norm states. However, if we restrict to the subspace of $\text{Osp}(2|2)$ invariant functions we find

$$\langle F | F \rangle = \int d^D x d\bar{\theta} d\theta \bar{F} F = -\frac{1}{\pi} \int d^D x (F^* \frac{\partial F}{\partial x^2} + F \frac{\partial F^*}{\partial x^2}) = \int d^{D-2} x F^* F \quad (3.20)$$

As a consequence the subspace of $\text{Osp}(2|2)$ invariant functions determines a Hilbert space with a positive definite metric. In particular, this subspace is isomorphic to the Hilbert

³Here we assume that the function $\Psi(y)$ in (3.13) is such that the change of variables is (globally) invertible. Otherwise we have ambiguities related to the Gribov problem and (3.16.a) fails to be true; see [12].

space of square integrable functions in D-2 dimensions. In path integral applications this D-2 dimensional Hilbert space is then identified with the physical Hilbert space, implying that conceptually the previous arguments are sufficient to ensure both unitarity and positivity of the Osp quantized constrained quantum theory, and in particular its equivalence to the reduced phase space quantum theory.

IV. Osp(1,1|2) Quantization of First Class Constraints

We shall now construct the phase space version of the *r.h.s.* integral in (3.16). For this we assume that the Hamiltonian has been included in the constraints as in (2.16). We also assume that we have implemented a locally defined change of variables $\varphi_a \rightarrow G_a$, $\psi^a \rightarrow F^a$ such that

$$\{G_a, G_b\} = \{F^a, F^b\} = 0 \quad \& \quad \{G_a, F^b\} = -\delta_a^b \quad (4.1)$$

The existence of the variables G_a, F^a is ensured by Darboux theorem. The constraint surface is then locally characterized by

$$G_a = F^a = 0 \quad (4.2)$$

In Lorentz invariant theories the Lagrange multipliers λ^a in (2.15) generically combine with the original phase space variables into Lorentz covariant quantities. For example in Yang-Mills theories with φ_a Gauss law operators, λ^a correspond to the time component of the gauge field A_μ . Thus, in order to ensure manifest covariance we view the Lagrange multipliers λ_a as canonical variables on an equal footing with the original phase space variables q^i, p_i , and identify the Lagrange multipliers π_a in (2.15) as the momenta conjugate to λ^a ,

$$\{\pi_a, \lambda^b\} = -\delta_a^b \quad (4.3)$$

This is consistent with the symmetry between the constraint functionals $\varphi_a \rightarrow G_a$ and the gauge fixing functionals $\psi^a \rightarrow F^a$, manifest in (4.1), (4.2)

The phase space $q^i, p_i, \lambda^a, \pi_a$ is $2N+2M$ dimensional. In order to reduce it to the $2(N-M)$ dimensional physical phase space we introduce $4M$ anticommuting variables as negative dimensional coordinates. We denote these variables $\eta^a, \mathcal{P}_a$ and $\bar{\mathcal{P}}^a, \bar{\eta}_a$ with nonvanishing graded Poisson brackets

$$\{\eta^a, \mathcal{P}_b\} = \{\bar{\mathcal{P}}^a, \bar{\eta}_b\} = -\delta_b^a \quad (4.4)$$

These variables then complete the Parisi-Sourlas phase space. For each a its unphysical subspace is an eight-dimensional graded phase space, with an equal number of bosonic and fermionic variables. Half of these variables are coordinates and half are momenta, hence the unphysical coordinate and momentum subspaces each are four dimensional. This implies that for each a the relevant orthosymplectic supergroup must be either $\text{Osp}(2|2)$ or $\text{Osp}(1,1|2)$. We shall now establish that it is $\text{Osp}(1,1|2)$.

For each a we introduce a four dimensional Parisi-Sourlas superspace with light cone position variables

$$q^{\alpha a} = (\hat{x}^a, x^a, \theta^a, \bar{\theta}^a) \stackrel{\text{def}}{=} (F^a, -\pi^a, \eta^a, \bar{\mathcal{P}}^a) \quad (4.5)$$

and momentum variables

$$p_{\alpha a} = (p_a, \hat{p}_a, \bar{\alpha}_a, \alpha_a) \stackrel{\text{def}}{=} (G_a, \lambda_a, \mathcal{P}_a, \bar{\eta}_a) \quad (4.6)$$

The Poisson brackets of these variables are

$$\{p_{\alpha a}, q^{\beta b}\} = -\delta_a^b \delta_\alpha^\beta \quad (4.7)$$

and from (3.7) we get the following realization of $\text{Osp}(1,1|2)$, for each a

$$R_a^{\alpha\beta} = q_a^\alpha p_a^\beta - (-)^{\alpha\beta} q_a^\beta p_a^\alpha \quad (4.8)$$

Explicitly,

$$R_a^{\theta\theta} = 2\eta^a \bar{\eta}_a \quad \& \quad R_a^{\bar{\theta}\bar{\theta}} = 2\mathcal{P}_a \bar{\mathcal{P}}^a \quad \& \quad R_a^{\theta\bar{\theta}} = R_a^{\bar{\theta}\theta} = \bar{\mathcal{P}}^a \bar{\eta}_a + \mathcal{P}_a \eta^a \quad (4.9.a)$$

$$R_a^{-\theta} = -R_a^{\theta-} = \eta^a G_a + \bar{\eta}_a \pi^a \quad \& \quad R_a^{-\bar{\theta}} = -R_a^{\bar{\theta}-} = \bar{\mathcal{P}}^a G_a - \mathcal{P}_a \pi^a \quad (4.9.b)$$

$$R_a^{+\theta} = -R_a^{\theta+} = \eta^a \lambda_a - \bar{\eta}_a F^a \quad \& \quad R_a^{+\bar{\theta}} = -R_a^{\bar{\theta}+} = \bar{\mathcal{P}}^a \lambda_a + \mathcal{P}_a F^a \quad (4.9.c)$$

$$R_a^{+-} = -R_a^{-+} = -F^a G_a - \pi^a \lambda_a \quad (4.9.d)$$

If in (4.9.b) we sum over all a , the generator $R^{-\theta}$ of the direct sum representation,

$$R^{-\theta} = \sum_a R_a^{-\theta} = \eta^a G_a + \bar{\eta}_a \pi^a \quad (4.10)$$

is identical to the conventional BRST operator for the abelian constraint functionals G_a . Similarly, the direct sum $R^{-\bar{\theta}}$ reproduces the corresponding antiBRST operator, $R^{\theta\bar{\theta}}$ the ghost number operator, and finally $R^{\theta\theta}$ and $R^{\bar{\theta}\bar{\theta}}$ the conjugation operators $\bar{K}$, $\bar{K}$ introduced in [13]. This suggests that (4.8) is the representation of $\text{Osp}(1,1|2)$ that

ensures the equivalence between the reduced phase space theory and the extended phase space theory by the Parisi-Sourlas mechanism. We shall now argue that this is indeed the case.

In order to construct the phase space version of the *l.h.s.* integral in (3.11), we observe that there are three $\text{Osp}(1,1|2)$ invariant combinations of the phase space variables (4.5), (4.6): The squared length of the position variables,

$$q^{\alpha a} q_{\alpha a} = -2(\pi^a F^a + \eta^a \bar{\mathcal{P}}^a) \quad (4.11)$$

the squared length of the momentum variables,

$$p^{\alpha a} p_{\alpha a} = 2(\lambda_a G_a + \bar{\eta}_a \mathcal{P}_a) \quad (4.12)$$

and finally the inner product of position and momentum variables

$$q^{\alpha a} p_{\alpha a} = F^a G_a - \pi^a \lambda_a + \eta^a \mathcal{P}_a + \bar{\mathcal{P}}^a \bar{\eta}_a \quad (4.13)$$

Since each of these can be written as a BRST commutator,

$$q^{\alpha a} q_{\alpha a} = 2\{\hat{x}^a \bar{\theta}^a, R^{-\theta}\} \quad \& \quad p^{\alpha a} p_{\alpha a} = -2\{\hat{p}_a \bar{\alpha}_a, R^{-\theta}\} \quad \& \quad q^{\alpha a} p_{\alpha a} = \{p_a \bar{\theta}^a - \hat{x}^a \bar{\alpha}_a, R^{-\theta}\} \quad (4.14)$$

if we define

$$\Psi = \kappa_1 \hat{x}^a \bar{\theta}_a + \kappa_2 \hat{p}^a \bar{\alpha}_a + \kappa_3 (p^a \bar{\theta}_a - \hat{x}^a \bar{\alpha}_a) \quad (4.15)$$

with $\kappa_{1,2,3}$ some constants we conclude from (3.11), (3.16) that the following path integral

$$Z = \int [dp_{\alpha a} dq^{\alpha a}] \exp\{i \int \dot{q}^{\alpha a} p_{\alpha a} - \{\Psi, R^{-\theta}\}\} \quad (4.16)$$

must be independent of the constants $\kappa_{1,2,3}$.

We first set $\kappa_3 = 0$ and introduce the change of variables

$$\pi_a \rightarrow \frac{1}{\kappa_2} \pi_a \quad \& \quad \bar{\mathcal{P}}^a \rightarrow \frac{1}{\kappa_2} \bar{\mathcal{P}}^a \quad (4.17)$$

with trivial superJacobian. In the limit $\kappa_2 \rightarrow \infty$ we then get

$$Z = \int [dF^a dG_a] \delta(F^a) \delta(G_a) \det|\{F^a, G_b\}| \quad (4.18)$$

which is the expected result in the unphysical sector of the reduced phase space approach, *c.f.* (2.2), (2.13). Furthermore, setting $\kappa_1 = \kappa_2 = 0$ (4.16) yields

$$Z = \frac{\prod_a (\det\{i\partial_t - \kappa_3\})^2}{\prod_a (\det\{i\partial_t - \kappa_3\})^2} \sim 1 \quad (4.19)$$

as expected from (3.11). Thus we have established that (4.8) is indeed the desired phase space realization of $\text{Osp}(1,1|2)$ and (4.16) the corresponding Parisi-Sourlas integral (3.16), (3.11), at least in the local coordinates (4.5), (4.6)

We now argue that for a general first class constraint algebra (2.6) the pertinent representation (3.16), (3.11) of (2.12)-(2.16) is [1]

$$Z = \int [d(\text{Liouville})] \exp\{i \int \dot{q}^i p_i - \dot{\pi}^a \lambda_a + \dot{\eta}^a \mathcal{P}_a + \dot{\bar{\mathcal{P}}}^a \bar{\eta}_a - \{\Psi, R^{-\theta}\}\} \quad (4.20)$$

with Ψ the parameter for the change of variables in (3.13), and $R^{-\theta}$ the BRST operator in the original variables (2.6),

$$\Omega_{BRST} = \eta^a \varphi_a + \frac{1}{2} U_{ab}^c \eta^a \eta^b \mathcal{P}_c + \bar{\eta}_a \pi^a \quad (4.21)$$

Indeed, this representation of $R^{-\theta}$ can be derived by inverting the change of variables $\varphi_a, \psi^a \rightarrow G_a, F^a$: For this we first assume that φ_a differs from G_a only infinitesimally,

$$\varphi_a = G_a + \epsilon_a^b G_b \quad (4.22)$$

and introduce a (pseudo)unitary canonical transformation generated by

$$\Phi = \eta^a \epsilon_a^b \mathcal{P}_b + \bar{\mathcal{P}}^a \epsilon_a^b \bar{\eta}_b - \pi^a \epsilon_a^b \lambda_b + \eta^a \bar{\mathcal{P}}^b \{G_a, \epsilon_b^c\} \lambda_c \quad (4.23)$$

To first order in ϵ_a^b the transformation of the BRST operator $R^{-\theta}$ yields

$$R^{-\theta} \xrightarrow{\Phi} e^{-\Phi} R^{-\theta} e^{\Phi} = R^{-\theta} + \{R^{-\theta}, \Phi\} + \dots \quad (4.24)$$

$$= \eta^a \varphi_a + \frac{1}{2} U_{ab}^c \eta^a \eta^b \mathcal{P}_c + \pi^a \bar{\eta}_a = \Omega_{BRST} \quad (4.25.a)$$

with Ω_{BRST} the BRST operator (4.21). Similarly,

$$R^{-\bar{\theta}} \xrightarrow{\Phi} \bar{\mathcal{P}}^a \varphi_a - \mathcal{P}_a \pi^a + \frac{1}{2} U_{ab}^c \bar{\mathcal{P}}^a \bar{\mathcal{P}}^b \bar{\eta}_c + U_{ab}^c \bar{\mathcal{P}}^a \eta^b \mathcal{P}_c - U_{ab}^c \bar{\mathcal{P}}^a \pi^b \lambda_c \equiv \bar{\Omega}_{BRST} \quad (4.25.b)$$

$$R^{\theta\bar{\theta}} \xrightarrow{\Phi} \bar{\mathcal{P}}^a \bar{\eta}_a + \mathcal{P}_a \eta^a \equiv N_{ghost} \quad (4.25.c)$$

$$R^{\theta\theta} \xrightarrow{\Phi} 2\eta^a \bar{\eta}_a - U_{ab}^c \eta^a \eta^b \lambda_c \equiv K \quad (4.25.d)$$

$$R^{\bar{\theta}\bar{\theta}} \xrightarrow{\Phi} 2\mathcal{P}_a \bar{\mathcal{P}}^a + U_{ab}^c \bar{\mathcal{P}}^a \bar{\mathcal{P}}^b \lambda_c \equiv \bar{K} \quad (4.25.e)$$

These coincide with the anti-BRST, ghost number and conjugation operators introduced in [13]. Since the representations (4.8), (4.9) and (4.25) differ from each other only by a (locally defined) canonical transformation and since the Liouville measure is invariant under canonical transformations, we conclude that (4.20) indeed is the desired representation (3.16) of the reduced phase space path integral (2.12) for a first class constrained system.

In general the representation of the remaining $\text{Osp}(1,1|2)$ generators is not very instructive, unless additional conditions for the commutator of φ_a and ψ^a in (2.8) are introduced. Furthermore, we conclude that even though the representation (4.25) appears independent of ψ^a , it is in fact gauge *i.e.* ψ^a dependent: This follows since for a generic gauge fermion Ψ the BRST commutator in (4.20) fails to be an invariant of the representation (4.25). Since two different BRST commutators are related by a change of variables of the form (3.13), we conclude that the BRST commutator, even though in general not an invariant of (4.25), is an $\text{Osp}(1,1|2)$ invariant provided we use the representation obtained from (4.8), (4.9) by implementing the pertinent change of variables (3.13) also in the $\text{Osp}(1,1|2)$ generators and in the definition of Poisson brackets. In this sense the present construction is coordinate independent; for further discussion of the coordinate independence in BRST quantization see [5].

V. $\text{Osp}(1,1|2)$ Quantization of Second Class Constraints

In this section we shall generalize the previous construction to second class constraint algebras (2.18) in such a way that Dirac brackets are avoided [6]. This is desirable: In field theory applications Dirac brackets generically lead to nonlocal or noncovariant formulations.

We again apply Darboux theorem to introduce a local change of variables $\phi^a \rightarrow G_a, F^a$ with G_a and F^a M local phase space coordinates with

$$\{G_a, G_b\} = \{F^a, F^b\} = 0 \quad \& \quad \{G_a, F^b\} = -\delta_a^b \quad (5.1)$$

and the constraint surface is locally characterized by

$$G_a = F^a = 0 \quad (5.2)$$

as in (4.2). Clearly, in terms of these local variables there is no difference between the first class and second class systems. The only difference between these two constrained systems is that the original second class functionals ϕ^a in general do not admit any natural separation into constraints φ_a and gauge fixings ψ^a as in (2.11). However, since there is no difference at the level of the local variables (4.1), (4.2) and (5.1), (5.2) we conclude that (locally) the Parisi-Sourlas phase spaces for first and second class systems must be identical. In particular, in terms of the local variables (5.1), (5.2) the $\text{Osp}(1,1|2)$ algebra of a second class system is given by (4.8), (4.9). Thus the quantum theory is determined by (4.16), and our previous arguments are then sufficient to establish the equivalence between the Parisi-Sourlas extended BRST quantization and the restricted phase space quantization, at least in the local Heisenberg coordinates (5.1), (5.2).

Difficulties arise when we try to implement the inverse change of variables $G_a, F^a \rightarrow \phi^a$: Since the ϕ^a do not naturally separate into φ_a and ψ^a , (4.20) is not directly applicable as the BRST operator $R^{-\theta}$ only involves "half" of the constraint functionals $\phi^a \sim G_a, F^a$. For a covariant BRST quantization of a second class algebra we need a BRST operator that involves both functionals $G_a, F^a \sim \phi^a$ symmetrically.

The desired BRST operator and the rest of the $\text{Osp}(1,1|2)$ generators can be constructed by first implementing the following canonical transformation in (4.9),

$$G_a \rightarrow \frac{1}{\sqrt{2}}(G_a + \lambda_a) \quad \& \quad F^a \rightarrow \frac{1}{\sqrt{2}}(F^a - \pi^a) \quad (5.3.a)$$

$$\lambda_a \rightarrow \frac{1}{\sqrt{2}}(\lambda_a - G_a) \quad \& \quad \pi^a \rightarrow \frac{1}{\sqrt{2}}(\pi^a + F^a) \quad (5.3.b)$$

which is generated by the functional

$$\Phi = \frac{1}{2}(G_a F^a + \lambda_a \pi^a - \lambda_a F^a + G_a \pi^a) \quad (5.4)$$

Since canonical transformations preserve Poisson brackets, the transformed generators also determine a Poisson bracket realization of the $\text{Osp}(1,1|2)$ algebra, and in particular the transformed BRST operator $R^{-\theta}$ is *symmetric* in the generators G_a, F^a of the second class algebra,

$$R^{-\theta} \rightarrow \frac{1}{\sqrt{2}}\eta^a(G_a + \lambda_a) + \frac{1}{\sqrt{2}}\bar{\eta}_a(F^a + \pi^a) \quad (5.5)$$

This differs from $R^{-\theta}$ in (4.9.b) only in the sense that in addition to the generators G_a , F^a now also the Lagrange multipliers λ_a , π^a appear in such a way that the second class Heisenberg algebra (5.1) is effectively converted into (abelian) first class algebra, with generators $G_a + \lambda_a$ and $F^a + \pi^a$.

In order to reconstruct the original BRST operator in terms of the variables ϕ^a we need to invert the change of variables $\phi^a \rightarrow G_a, F^a$. As in (4.24), we shall implement this change of variables as a canonical transformation on the extended phase space. For this we introduce the following symmetric notation,

$$G_a, F^a \rightarrow T^a \quad \& \quad \lambda_a, \pi^a \rightarrow \xi^a \quad (5.6.a)$$

$$\eta^a, \bar{\eta}_a \rightarrow c_a \quad \& \quad \mathcal{P}_a, \bar{\mathcal{P}}^a \rightarrow d^a \quad (5.6.b)$$

where on the *r.h.s.* now $a = 1, \dots, 2M$ as in (2.18). In terms of these variables we have

$$\{T^a, T^b\} = -\{\xi^a, \xi^b\} = \omega^{ab} \quad (5.7)$$

the second class algebra (5.1) with $\omega^{ab} (= \delta^{ab})$ the Heisenberg central extension. The BRST operator (5.5) is now (ignoring an irrelevant factor of $\sqrt{2}$)

$$R^{-\theta} = c_a(T^a + \xi^a) \quad (5.8)$$

As in (4.22) we first assume that the original functionals ϕ^a differ from T^a only infinitesimally,

$$\phi^a = T^a + \epsilon_b^a T^b + \mathcal{O}(\epsilon^2) \quad (5.9)$$

and try to relate (5.8) to a BRST operator involving ϕ^a by the (pseudo)unitary transformation (4.24). With

$$\Phi = c_a \epsilon_b^a d^b \quad (5.10)$$

and to first order in ϵ_a^b this yields

$$\begin{aligned} R^{-\theta} &\xrightarrow{\Phi} c_a(T^a + \epsilon_b^a T^b + \xi^a + \epsilon_b^a \xi^b) + \frac{1}{2} c_a c_b d^c \left[\{T^a, \epsilon_c^b\} + \{\epsilon_c^a, T^b\} \right] + \dots \\ &= c_a(\phi^a + \rho^a) + \frac{1}{2} c_a c_b U_c^{ab} d^c + \dots \end{aligned} \quad (5.11)$$

where, to first order in ϵ_a^b we have defined

$$\phi^a = T^a + \epsilon_b^a T^b \quad \& \quad \rho^a = \xi^a + \epsilon_b^a \xi^b \quad (5.12.a)$$

$$\{\phi^a, \phi^b\} = C^{ab} \quad \& \quad \{\rho^a, \rho^b\} = -D^{ab} \quad (5.12.b)$$

$$\{\phi^a, \rho^b\} + \{\rho^a, \phi^b\} = C^{ab} - D^{ab} = U_c^{ab} \phi^c \quad (5.12.c)$$

$$C^{ab} = \left[\{T^a, \epsilon_c^b\} + \{\epsilon_c^a, T^b\} \right] \phi^c + \omega^{ab} + \epsilon_c^a \omega^{cb} - \epsilon_c^b \omega^{ca} + \dots \quad (5.12.d)$$

Notice in particular that on the constraint surface $\phi^a = 0$ we have $C^{ab} = D^{ab}$. On the constraint surface the *r.h.s.* in (5.12.b) then differs only by a sign, while the *r.h.s.* of (5.12.c) vanishes.

From the previous Ansatz we conclude that for a general constraint algebra (2.18) the BRST operator admits an expansion in powers of the ghost variables which is of the functional form

$$\Omega_{BRST} = R^{-\theta} = c_a(\phi^a + \rho^a) + \frac{1}{2!} c_a c_b U_c^{ab} d^c + \frac{1}{3!2!} c_a c_b c_c U_{de}^{abc} d^d d^e + \dots \quad (5.13)$$

with ρ^a functionals of the dynamical Lagrange multipliers and original phase space variables. These functionals, as well as the higher order structure functions U_{de}^{abc} etc. can then be determined order-by-order in powers of ghost variables by requiring nilpotency of (5.13). In particular, for the ρ^a this gives

$$\{\phi^a, \rho^b\} + \{\rho^a, \phi^b\} + \{\rho^a, \rho^b\} = U_c^{ab} \rho^c - C^{ab} \quad (5.14)$$

a differential equation that determines ρ^a up to a natural arbitrariness.

The quantum theory is then constructed as before,

$$Z = \int [dq^i \dots dc_a] \exp \{ i \int \dot{q}^i p_i + \dot{\lambda}^a \pi_a + \dot{c}_a d^a - \{ \Psi, R^{-\theta} \} \} \quad (5.15)$$

with $R^{-\theta}$ now the BRST operator (5.13) and Ψ an arbitrary ghost number minus one functional of the phase space variables. Since this differs from (4.20) essentially only by (5.3), it is clear that all our previous conclusions are directly applicable: (5.15) is a path integral version of the *r.h.s.* integral in (3.16), and its equivalence with the reduced phase space functional integral (2.12), (2.20) is a consequence of the Parisi-Sourlas mechanism (3.11), (3.16).

VI. An Example: Open Bosonic String

As an example we now construct the $\text{Osp}(1,1|2)$ generators for an open bosonic string [14]. For this we assume that the functionals φ_a and ψ^a in (2.8) satisfy the following Lie

algebra central extension, relevant to string theories

$$\{\varphi_a, \varphi_b\} = U_{ab}^c \varphi_c \quad \& \quad \{\psi^a, \psi^b\} = 0 \quad (6.1.a)$$

$$\{\varphi_a, \psi^b\} = -T_{ac}^b \psi^c - \delta_a^b \quad (6.1.b)$$

where the central extension δ_a^b ensures (2.8). Here U_{ab}^c and T_{ab}^c are Lie algebra structure constants and the pertinent Jacobi identities are

$$U_{ab}^d U_{dc}^e + U_{bc}^d U_{da}^e + U_{ca}^d U_{db}^e = U_{ab}^d T_{dc}^e + T_{ac}^d T_{bd}^e - T_{bc}^d T_{ad}^e = 0 \quad (6.2.a)$$

and

$$U_{ab}^c = T_{ab}^c - T_{ba}^c \quad (6.2.b)$$

We construct the $\text{Osp}(1,1|2)$ generators from (4.9), using the (pseudo)unitary canonical transformation (4.24) with

$$\Phi_\epsilon = G_a \epsilon_b^a F^b - \lambda_a \epsilon_b^a \pi^b + \mathcal{P}_a \epsilon_b^a \eta^b + \bar{\eta}_a \epsilon_b^a \bar{\mathcal{P}}^b \quad (6.3)$$

Selecting

$$\epsilon_b^a = -\epsilon F^c T_{cb}^a \quad (6.4)$$

where ϵ is an infinitesimal constant, we find

$$\{\Phi_\epsilon, \{\Phi_\epsilon, R^{\alpha\beta}\}\} = 0 \quad (6.5)$$

with $R^{\alpha\beta}$ any generator in (4.9) Hence the transformation (4.24) can be integrated in ϵ , and in this way we find a realization of the $\text{Osp}(1,1|2)$ algebra based on the following realization of (6.1)

$$\varphi_a = G_a + T_{ab}^c F^b G_c \quad \& \quad \psi^a = F^a \quad (6.6)$$

Explicitly,

$$R^{\theta\theta} = 2\eta^a \bar{\eta}_a \quad \& \quad R^{\theta\bar{\theta}} = \mathcal{P}_a \eta^a + \bar{\mathcal{P}}^a \bar{\eta}_a \quad \& \quad R^{\bar{\theta}\bar{\theta}} = 2\bar{\mathcal{P}}^a \bar{\mathcal{P}}_a \quad (6.7.a)$$

$$R^{-\theta} = \eta^a \varphi_a + \frac{1}{2} U_{ab}^c \eta^a \eta^b \mathcal{P}_c + \eta^a T_{ab}^c (\bar{\mathcal{P}}^b \bar{\eta}_c - \pi^b \lambda_c) + \bar{\eta}_a \pi^a \quad (6.7.b)$$

$$R^{-\bar{\theta}} = \bar{\mathcal{P}}^a \varphi_a + \frac{1}{2} U_{ab}^c \bar{\mathcal{P}}^a \bar{\mathcal{P}}^b \bar{\eta}_c + \bar{\mathcal{P}}^a T_{ab}^c (\eta^b \mathcal{P}_c - \pi^b \lambda_c) - \mathcal{P}_a \pi^a \quad (6.7.c)$$

$$R^{+\theta} = -\bar{\eta}_a \psi^a + \eta^a \lambda_a \quad \& \quad R^{+\bar{\theta}} = \mathcal{P}_a \psi^a + \bar{\mathcal{P}}^a \lambda_a \quad (6.7.d)$$

$$R^{+-} = -\psi^a (\varphi_a - T_{ab}^c \pi^b \lambda_c + T_{ab}^c \eta^b \mathcal{P}_c + T_{ab}^c \bar{\mathcal{P}}^b \bar{\eta}_c) - \pi^a \lambda_a \quad (6.7.e)$$

Substituting the original φ_a, ψ^a in (6.7) we then have the the desired representation of $\text{Osp}(1,1|2)$. Notice that this representation is related to the representation (4.25) by the (pseudo)unitary canonical transformation (4.24) with

$$\Phi = -T_{ab}^c \eta^a \bar{p}^b \lambda_c \quad (6.8)$$

A realization of the algebra (6.1) can be constructed in (super)string theories [4], and here we review this construction for an open bosonic string. The canonical string variables are coordinates $\mathcal{X}_A(\sigma)$ of a D dimensional Minkowski space with $\sigma \in [0, S]$ where S is the (dimensionless) string length, and conjugate momenta $\pi_A(\sigma)$. The nonvanishing Poisson bracket is

$$\{\pi_A(\sigma), \mathcal{X}_B(\sigma')\} = -\eta_{AB} \delta(\sigma - \sigma') \quad (6.9)$$

and the string theory is defined as a local, Lorentz and translation invariant Poisson bracket realization of the 1+1 dimensional diffeomorphism algebra,

$$\{\mathcal{H}^\pm(\sigma), \mathcal{H}^\pm(\sigma')\} = \pm\{\mathcal{H}^\pm(\sigma) + \mathcal{H}^\pm(\sigma')\} \delta'''(\sigma - \sigma') \quad \& \quad \{\mathcal{H}^+(\sigma), \mathcal{H}^-(\sigma')\} = 0 \quad (6.10)$$

where

$$\mathcal{H}^\pm = \frac{1}{2} \eta^{AB} \mathcal{D}_A^\pm \mathcal{D}_B^\pm \quad \& \quad \mathcal{D}_A^\pm = \frac{1}{\sqrt{2}} \left(\sqrt{T} \pi_A \pm \frac{1}{\sqrt{T}} \partial_\sigma \mathcal{X}_A \right) \quad (6.11)$$

with T a parameter with dimensions of length square that can be identified with the string tension. Introducing the Fourier representations

$$\mathcal{X}_A = \frac{1}{2} \mathcal{X}_A^0 + \frac{1}{2} \sum_{n \neq 0} \mathcal{X}_A^n e^{in\pi \frac{\sigma}{S}} \quad \& \quad \pi_A = \frac{2}{S} \pi_A^0 + \frac{1}{S} \sum_{n \neq 0} \pi_A^n e^{in\pi \frac{\sigma}{S}} \quad (6.12)$$

with $\mathcal{X}_A^n = \mathcal{X}_A^{-n}$ and $\pi_A^n = \pi_A^{-n}$ the Poisson brackets are

$$\{\pi_A^0, \mathcal{X}_B^0\} = \eta_{AB} \quad \& \quad \{\pi_A^n, \mathcal{X}_B^m\} = \eta_{AB} \delta(n - m) \quad (6.13)$$

The Virasoro generators

$$L_n \stackrel{def}{=} L_n^+ = L_{-n}^- = \frac{S}{\pi} \int_{-S}^S \mathcal{H}^\pm(\sigma) e^{-in\pi \frac{\sigma}{S}} \quad (6.14)$$

satisfy the constraint algebra

$$\{L_n, L_m\} = i(n - m) L_{n+m} \quad (6.15)$$

and from (6.1) we get the identification

$$U_{nm}^k = i(n-m)\delta(n+m-k) \quad (6.16)$$

The Jacobi identities (6.2) then have the unique solution

$$T_{nm}^k = -im\delta(n+m-k) \quad (6.17)$$

and with $\psi^a \rightarrow F^{-n}$ the remaining brackets in (6.1) become

$$\{F^n, F^m\} = 0 \quad \& \quad \{L_n, F^m\} = -i(n+m)F^{n+m} + \delta(n+m) \quad (6.18)$$

Explicitly, the F^n are constructed as follows: For $n = m = 0$ we have

$$\{L_0, F^0\} = 1 \quad (6.19)$$

with the solution [3]

$$F^0 = 2 \frac{k \cdot \chi^0}{k \cdot \pi^0} \quad (6.20)$$

where k is an arbitrary light-like vector, $k \cdot k = 0$. For $n \neq 0$ we compute F^n from

$$\{L_n, F^0\} = -inF^n \quad \Rightarrow \quad F^n = \frac{i\sqrt{2}}{n} \frac{k \cdot \alpha_n}{k \cdot \pi^0} \quad (n \neq 0) \quad (6.21)$$

where we have introduced

$$\alpha_{A,n} \stackrel{def}{=} \alpha_{A,n}^+ = \alpha_{A,-n}^- = \frac{1}{2S} \int_{-S}^S \mathcal{D}_A^\pm(\sigma) e^{-in\pi \frac{\sigma}{S}} \quad (6.22)$$

so that

$$L_n = \frac{1}{2} \eta^{AB} \sum_{-\infty}^{\infty} \alpha_{A,m} \alpha_{B,n-m} \quad (6.23)$$

Substituting (6.14), (6.20), (6.21) into (6.7) we then have the desired representation of the $\text{Osp}(1,1|2)$ algebra for an open bosonic string.

Notice that the generators F^n are essentially those introduced in [15] as projection operators on the transverse states, to prove unitarity of strings in the physical subspace determined by

$$L_n |phys\rangle = 0 \quad \text{for } n \geq 0 \quad \& \quad F^n |phys\rangle = 0 \quad \text{for } n > 0 \quad (6.24)$$

We shall now verify that these states are indeed related to the $\text{Osp}(1,1|2)$ singlet states,

$$R^{\alpha\beta}|s\rangle = 0 \quad (6.25)$$

as expected from Parisi-Sourlas. For this we first observe that it is sufficient to consider the four generators

$$R^{-\theta} \quad \& \quad R^{+\bar{\theta}} \quad \& \quad R^{\theta\theta} \quad \& \quad R^{\bar{\theta}\bar{\theta}} \quad (6.26)$$

since the remaining $\text{Osp}(1,1|2)$ generators can be represented in terms of these four,

$$R^{\theta\bar{\theta}} = \frac{1}{4}[R^{\bar{\theta}\bar{\theta}}, R^{\theta\theta}] \quad \& \quad R^{+-} = [R^{+\bar{\theta}}, R^{-\theta}] + \frac{1}{4}[R^{\bar{\theta}\bar{\theta}}, R^{\theta\theta}] \quad (6.27.a)$$

$$R^{-\bar{\theta}} = \frac{1}{2}[R^{\bar{\theta}\bar{\theta}}, R^{-\theta}] \quad \& \quad R^{+\theta} = -\frac{1}{2}[R^{\theta\theta}, R^{+\bar{\theta}}] \quad (6.27.b)$$

A direct substitution in (6.7) gives for (6.26)

$$R^{\theta\theta} = 2\eta^n \bar{\eta}_n \quad \& \quad R^{\bar{\theta}\bar{\theta}} = 2\mathcal{P}_n \bar{\mathcal{P}}^n \quad (6.28.a)$$

$$R^{-\theta} = \eta^n (L_n + J_n + \bar{J}_n + \Lambda_n) + \bar{\eta}_n \pi^n \quad \& \quad R^{+\bar{\theta}} = \mathcal{P}_n F^n + \bar{\mathcal{P}}^n \lambda_n \quad (6.28.b)$$

where we have defined

$$J_n = T_{nm}^k \eta^m \mathcal{P}_k \quad \& \quad \bar{J}_n = T_{nm}^k \bar{\mathcal{P}}^m \bar{\eta}_k \quad \& \quad \Lambda_n = -T_{nm}^k \pi^m \lambda_k \quad (6.29)$$

and for quantum consistency of the $\text{Osp}(1,1|2)$ algebra we assume the critical values $D = 26$ and $\alpha_0 = 1$.

The $\text{Osp}(1,1|2)$ singlet states are determined by first observing that $R^{-\theta}|s\rangle = 0$ together with (6.24) implies

$$\eta^n |s\rangle = 0 \quad \text{for } n < 0 \quad (6.30.a)$$

while $R^{+\bar{\theta}}|s\rangle = 0$ and (6.24) implies

$$\mathcal{P}_n |s\rangle = 0 \quad \text{for } n \geq 0 \quad (6.30.b)$$

From $R^{\theta\theta}|s\rangle = 0$ and $R^{\bar{\theta}\bar{\theta}}|s\rangle = 0$ we conclude that

$$\bar{\eta}_n |s\rangle = 0 \quad \text{for } n \geq 0 \quad \& \quad \bar{\mathcal{P}}^n |s\rangle = 0 \quad \text{for } n < 0 \quad (6.30.c)$$

and finally, $R^{+\bar{\theta}}|s\rangle = 0$ together with $R^{-\theta}|s\rangle = 0$ implies that

$$\lambda_n|s\rangle = 0 \quad \text{for } n \leq 0 \quad \& \quad \pi^n|s\rangle = 0 \quad \text{for } n > 0 \quad (6.30.d)$$

As a consequence the states $|s\rangle$ in (6.25) can be identified with the direct product states

$$|s\rangle = |phys\rangle \otimes |0\rangle_{\eta\mathcal{P}} \otimes |0\rangle_{\bar{\eta}\bar{\mathcal{P}}} \otimes |0\rangle_{\lambda\pi} \quad (6.31)$$

with $|0\rangle$ Fock vacua in the oscillators $\eta^n, \mathcal{P}_n; \bar{\mathcal{P}}^n, \bar{\eta}_n$ and π^n, λ_n respectively. This ensures that the Parisi-Sourlas approach to quantum strings is indeed equivalent with the reduced phase space approach, as expected from general arguments.

Notice in particular, that now the ghost number of the physical states vanishes, $R^{\theta\bar{\theta}}|s\rangle = 0$ instead of being one half which is the value obtained when the $\bar{\eta}_n, \bar{\mathcal{P}}^n$ oscillators are truncated [14].

VII. Summary

In this review we have considered the conceptual aspects of the Parisi-Sourlas extended BRST quantization of constrained systems, following the approach of [3-6]. In particular, we have explained how BRST quantization differs from the reduced phase space quantization and how the quantum theories obtained in the two formulations can be related by the Parisi-Sourlas mechanism. We have also constructed the $Osp(1,1|2)$ generators of canonical Parisi-Sourlas supersymmetries for both first and second class constraint algebras, and as an example we have explicitly presented these generators in the case of an open bosonic string.

References

1. E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. **55B** (1975) 224; and CERN Report TH-2332 (1977); I.A. Batalin and G.A Vilkovisky, Phys. Lett. **69B** (1977) 309; For a review, see M. Henneaux, Phys. Repts. **126** (1985) 1
2. A. Neveu and P. West, Phys. Lett. **182B** (1986) 343; and Nucl. Phys. **B293** (1987) 266

3. H. Aratyn, R. Ingermanson and A.J. Niemi, Phys. Rev. Lett. **58** (1987) 965; and Phys. Lett. **189B** (1987) 427; and Nucl. Phys. **B307** (1988) 157; A.J. Niemi, Phys. Rev. **D36** (1987) 3731
4. H. Aratyn, R. Ingermanson and A.J. Niemi, Phys. Lett. **194B** (1987) 506; and Phys. Lett. **195B** (1987) 149
5. A.J. Niemi, OSU preprint DOE/ER/01545-410 (1988) (to appear in Annals of Physics)
6. A.J. Niemi, CERN Report TH-5053 (1988) (to appear in Physics Letters B)
7. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press 1964)
8. L.D. Faddeev, Theor. Math. Phys. **1** (1969); L.D. Faddeev and A.A. Slavnov, *Gauge Theories - Introduction to Quantum Field Theory* (Benjamin-Cummings 1980)
9. G. Parisi and N. Surlas, Phys. Rev. Lett. **43** (1979) 744; B. McClain, A.J. Niemi, C. Taylor and L.C.R. Wijewardhana, Phys. Rev. Lett. **49** (1982) 252; and Nucl. Phys. **B217** (1983) 430
10. A. Kupiainen and A.J. Niemi, Phys. Lett. **130B** (1983) 380
11. See *e.g.* N. Woodhouse, *Geometric Quantization* (Clarendon Press 1980)
12. J. Govaerts, CERN Report TH-5010 (1988)
13. S. Hwang, Nucl. Phys. **B231** (1984) 386
14. For a review, see *e.g.* M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory* (Cambridge University Press 1986)
15. P. Goddard and C.B. Thorn, Phys. Lett. **40B** (1972) 235; C.B. Thorn, Nucl. Phys. **B286** (1987) 61; E. Del Giudice, P. Di Vecchia and S. Fubini, Ann. Phys. **70** (1972) 378