CERN-TH.4819/87

A REVIEW OF DUALITY, STRING VERTICES, OVERLAP IDENTITIES
AND THE GROUP THEORETIC APPROACH TO STRING THEORY^{*)}

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A B S T R A C T

A review of the group theoretic approach to string theory is presented. It is preceded by a discussion of some of the relevant string literature on duality, factorization, oscillator vertices and overlap identities.

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INTRODUCTION

There are three approaches to string theory; the way it was discovered which we will call the historical approach¹, the sum over world history approach² and the string field theory approach^{3,4}. Recently, there has been a desire to find a new formulation of string theory which incorporates the strengths of the previous approaches and is both complete and transparent, complete in the sense that it contains the whole of string theory including its perturbative and non-perturbative effects. While the perturbative effects are contained in both the string field theory and sum over world history approaches, only in the latter do the results emerge in an elegant and natural way. Alternatively, only in the former of these approaches is there the hope of computing non-perturbative effects. Since the classical string field equations in the light cone gauge and now in gauge covariant form are known, one could solve these to find classical solutions. In functional form, there is only one equation of motion, but in terms of component fields one has infinitely many equations. This makes finding solutions a formidable but not hopeless task. Given a classical solution one can, in the usual way in a field theory, find the quantum corrections. One good problem to start with is to attempt to shift the vacuum expectation values of the scalar fields to a new minimum.

Non-perturbative quantum effects are more difficult to contemplate, our experience of these effects, such as quark confinement, even in ordinary point field theory being somewhat limited. Nonetheless the simplicity of the string perturbative S matrix may give one hope that such effects may be calculable in string theory in an appropriate formalism.

Another drawback of the two known approaches is the explicit occurrence of the Minkowski metric. A theory of gravity, as envisaged by Einstein predicts the space-time metric and connection and it is unnatural to begin with any preferred choice. The problem in string theory is that the graviton is one of an infinite number of particles which sit in a kind of multiplet, and it is also not desirable to give a preferred role to the graviton with respect to all these other fields. One can, of course, take the more radical viewpoint, that the space-time manifold itself has no meaning at Planck length scales, that it is not a good concept, and it should be eradicated from the theory, to be found only as a macroscopic effect. In this respect, we note that in string field theories bear a much closer resemblance to Yang-Mills theories than to Einstein's theory of gravity on which our expectations about space-time structure are largely based.

The less radical efforts to solve the above problem of the occurrence of a preferred metric have included in the sum over world history approach the introduction of background fields⁵. It is possible that a similar device may be used in string field theory, but in this case one also has the possibility of dropping the kinetic term which is the only metric dependent part of the action and recovering it by a field shift⁶.

In the more radical proposals, one is led to consider dropping $x^\mu(\sigma)$ as a variable and basing string theory on different variables. In the old dual model, this was already the case, although the Minkowski metric made an appearance, the scatterings were described in terms of an integral over the Koba-Nielson variables⁷ and variables (the moduli) which arise from putting the propagators in parametric form. Several attempts to find new formulations have taken this path in the sense that they are based respectively on the universal moduli space⁸, Grassmannians⁹ and the space $\text{diff } S'/S$ ¹⁰. Part of these notes will review the new approach of Refs. 11-13 which also takes this line of attack on the problem of space-time.

The finiteness of string theories has still not been demonstrated and by transparency we mean a formulation of string theory in which this and its other properties required for consistency are apparent. The finiteness¹⁴ of certain supersymmetric theories was, with hindsight essentially obvious once they were formulated in a way in which the majority of their supersymmetry was manifest in the quantum calculations. The two-dimensional symmetries in the sum over world histories are relatively well understood, but although the vast space-time symmetries of gauge-covariant string field theories are known, which group they form and what is their significance is relatively obscure. To understand this aspect of gauge-covariant string theory is likely to be rather illuminating. Presumably the large symmetry in string theory and the principle, in which it is embodied, will play a very important role, which in the best formulation can hopefully be easily exploited.

The particular content of these notes arose as a result of the fact that some of the topics covered at length in the lectures are already the subject of a pedagogical review by the author⁴ and so are not discussed here. An example of this is a description of the gauge-covariant formulation of the free open bosonic string found in Refs. 15 and 16 and its interacting extension of Refs. 17 and 18 which were independent and virtually simultaneous. Part of the lectures also discussed the necessary extension of the above interacting gauge-covariant string theories to incorporate systematically the string lengths; for this the reader is referred to Refs. 19 and 20. A different method of considering string lengths has been considered in Ref. 21. This work does not use the Parisi-Sourlas mechanism nor introduce a τ dependence, but wishes to have the string length parameter being gauged away. So far these authors have achieved this at the free level. A different formulation of gauge covariant string theories which has been the subject of many reviews can be found in Ref. 16. The light cone gauge string field theory³ was also discussed in detail.

What is contained in these notes is the remaining content of the lectures which was a discussion of the historical approach to string theory and the group theoretical approach proposed recently in Refs. 11, 12 and 13.

In contrast to the gauge-covariant and sum over world history approaches to string theory, which are rather logical once the initial framework has been given, the historical approach like so often with the

path of scientific development is more ad hoc. However, it is an approach which, once one accepts the Veneziano formula²² and its generalization²³ for the scattering of an arbitrary number of spin zero particles, has a compelling logic of its own. Namely, once one has established the factorization and duality properties of these amplitudes, one can derive the whole of tree-level string scattering and also many of its quantum properties.

The historical approach in which the majority of string theory was discovered, presents a rather different picture that stresses some of its more stringy aspects such as duality and the simplicity of the final S-matrix. While the author would certainly not suggest that the historical approach is superior, in the current search for a new formulation of string theory, it is useful to know all three approaches. In particular, the historical approach is useful some aspects of one's string intuition and there does not exist, unlike for the other two approaches, any recent reviews which relate it to more recent developments.

In Sections 4 and 5 we consider the group theoretic approach to string theory proposed recently. The educated reader may begin with section 4; however, some of the features such as duality which are foremost in the historical approach are also key ingredients in this new formulation of string theory.

2. PREHISTORY

String theory arose from an attempt to understand hadronic interactions in the late 1960's. At that time it was felt that the complexity of hadronic phenomenon which included many states of spin two and above could not be described by a quantum field theory. The latter was to be replaced by a set of properties of the S matrix, which included some well-known properties such as unitary, factorization and Lorentz invariance, and some less familiar ingredients, the most important of which is duality. We will explain the latter as we go along, but historically duality arose in a particular approximation which enjoyed some success in accounting for the observed string interaction data²⁴. There began a general search²⁵ to find such an S matrix, but the first S matrix which actually realized some of these desirable properties was the Veneziano formula²² for the scattering of four spin zero states. We take

the spin zeros to have momenta p_i , $i = 1, \dots, 4$ and define

$$s = -(p_1 + p_2)^2, \quad t = -(p_2 + p_3)^2, \quad u = -(p_1 + p_3)^2 \quad (2.1)$$

which are the rest frame (energies)² in the various channels also called s , t and u (see Fig. 2.1)

Fig. 2.1

If we denote the mass of the spin zero particles, which are taken to have the same mass, by μ^2 (i.e., $-p_i^2 = \mu^2$) one finds that

$$s + t + u = 4\mu^2 = - \sum_{i=1}^4 p_i^2 \quad (2.2)$$

Neglecting the trivial term corresponding to no scattering, the S matrix for the four spin zero particles was postulated to be a sum of three terms

$$T^4(p_i) = F(s, t) + F(t, u) + F(u, s) \quad (2.3)$$

where

$$F(s, t) = g^2 \frac{\Gamma(-\alpha(s)) \Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \quad (2.4)$$

and

$$\alpha(s) = \alpha' s + \alpha_0 \quad (2.5)$$

The constants α' and α_0 are called the slope and intercept respectively. The Γ function has its usual integral representation

$$\Gamma(u) = \int_0^{\infty} dt \, e^{-t} t^{u-1} \quad (2.6)$$

which is valid for $\text{Re } u > 0$. In this region, one can show by integration by parts that

$$\Gamma(u+1) = u \Gamma(u). \quad (2.7)$$

One observes from Eq. (2.6) that $\Gamma(1) = 1$, while $\Gamma(n+1) = n!$ for any positive integer n .

We would like to define Γ for all u . This is achieved by demanding that Γ is given by Eq. (2.6) for $\text{Re } u > 0$ and also satisfies Eq. (2.7), which was derived for $\text{Re } u > 0$, but is now taken to hold for all u . Then we have from Eq. (7) for $u \neq 0, -1, -2, \dots$ we have that

$$\Gamma(u) = \frac{1}{(u)} \frac{1}{(u+1)} \dots \frac{1}{(u+n)} \Gamma(u+n+1) \quad (2.8)$$

As $\Gamma(u+n+1)$ is defined by Eq. (2.6) for $\text{Re } u > -n-1$, one finds a definition of $\Gamma(u)$ in this region. Since n can be as large as one likes we have defined $\Gamma(u)$ in the whole complex plane. At the points $u = 0, -1, -2, \dots$, $\Gamma(u)$ has a simple pole. Hence $\Gamma(u)$ is analytic at all points except in the complex plane at the points $u = 0, -1, -2$ at which it has simple poles and the point $|u| = \infty$ where it has an essential singularity.

Actually $F(s, t)$ is recognizable in terms of the well-known beta function which is defined by

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}. \quad (2.9)$$

and has the integral representation

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad (2.10)$$

This result follows from the integral representation of $\Gamma(u)$ of Eq. (2.6) after appropriate change of variables. Starting from the product of two Γ functions and changing the two integration variables to ξ^2 and η^2 and then changing to r, θ we find

$$\begin{aligned}
 \Gamma(x)\Gamma(y) &= 4 \int_0^\infty \int_0^\infty d\zeta d\eta \zeta^{2x-1} \eta^{2y-1} e^{-\zeta^2 - \eta^2} \\
 &= 4 \int_0^{\pi/2} \int_0^\infty r^{2x+2y-1} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} e^{-r^2} dr d\theta \\
 &= 2 \Gamma(x+y) \int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \quad (2.11)
 \end{aligned}$$

After the further substitution $\cos\theta = u^{\frac{1}{2}}$. We recognize the second term as the integral representation of Eq. (2.10). Although $B(x,y)$ is valid for $\text{Re } x > 0$, $\text{Re } y > 0$, we can define its value on the whole complex plane from its expression in terms of Γ functions with the above definition or equivalently by working with the equivalent recursion relations for $B(x,y)$.

As a result $B(x,y)$, which is a symmetric function of x and y , has, for y fixed, poles at $x = 0, -1, -2, \dots$; an essential singularity at $|x|$ is infinity, but is otherwise analytic. Note it has no poles in both x and y as then also $\Gamma(x+y)$ has a simple pole. Expanding the second bracket in Eq. (2.10), we find, for $\text{Re } y > 0$, that

$$B(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(x+n)} \frac{\Gamma(y)}{\Gamma(-n+y)} \quad (2.12)$$

This expression includes all the poles in $B(x,y)$ implied by the definition of $\Gamma(x)$ for $\text{Re } x < 0$ given above. For later convenience we record that

$$F(s,t) = g^2 \int_0^1 dx (x)^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} \quad (2.13)$$

Using Eq. (2.12), we find that

$$F(s,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\alpha(t)+1)(\alpha(t)+2)\dots(\alpha(t)+n)}{(n - \alpha(s) - \alpha_0)} \quad (2.14)$$

Hence we find poles, i.e., resonances in the s channel whenever

$$\alpha(s) = \alpha'(s) + \alpha_0 = 0, +1, +2, \dots \quad (2.15)$$

This corresponds to particles being exchanged in these channels of masses given by

$$m^2 = \frac{1}{\alpha'} (n - \alpha_0) \quad (2.16)$$

for n a positive integer. The fact that the residue is a polynomial of degree n indicates that one of the particles of this mass being exchanged is of spin $J = n$. This particle is the highest spin particle at this mass. We recall that a particle of spin J has J Lorentz indices and so couples to two scalars as $\phi \partial_{\mu_1} \dots \partial_{\mu_J} \phi$. Thus from two such vertices one finds J factors of momentum squared. Hence we arrive at the well-known statement that for these highest spin particles $m_J^2 = 1/\alpha' (J - \alpha_0)$, that is there exist in the theory particles of ever-increasing spin whose $(\text{mass})^2$ is linearly rising with their spin. Since we have a residue which is a polynomial in t , we also have, in general, other particles. If we were to identify the mass μ of the external spin zeros with the lowest mass particle in the spectrum which is also a scalar we would find that

$$\mu^2 = - \frac{\alpha_0}{\alpha'} \quad (2.17)$$

Although very natural, there is at this point no compelling reason to do this. We also note that unless $\alpha' > 0$, we will have an infinity of tachyons. However, even if $\alpha' > 0$, but $\alpha_0 > 0$ we will have one or more tachyons. Further, since F is symmetric in s and t and using Eq. (2.12), we may write it as

$$F(s, t) = \sum_{n=0}^{\infty} \frac{P_n(t)}{n - \alpha(s)} = \sum_{n=0}^{\infty} \frac{P_n(s)}{n - \alpha(t)} \quad (2.18)$$

where $P_n(x)$ is a polynomial of degree n in x which is given by

$$P_n(x) = \frac{1}{n!} (\alpha(x)+1)(\alpha(x)+2) \dots (\alpha(x)+n) \quad (2.19)$$

This is the expression of the remarkable property of duality which motivated the Veneziano formula, namely $F(s,t)$ can be expressed in terms of poles in s for fixed t , but also expressed in terms of poles in t for fixed s . Clearly, to get duality, one requires an infinite number of poles as otherwise the first term in Eq. (2.18) would be a finite polynomial in t and so analytic in t .

Another way of expressing duality is to denote $F(s,t) \equiv A(1,2,3,4)$. Cycling the variables, i.e., $p_i \rightarrow p_{i+1}$ we find $s \rightarrow t$ and $t \rightarrow s$ and so

$$A(2,3,4,1) = F(t,s) = F(s,t) = A(1,2,3,4) \quad (2.20)$$

That is, the symmetry of $F(s,t)$ under exchange of s and t , which in essence is duality is the same as $A(1,2,3,4)$ being the same if we cycle its indices. We also note that under $1 \leftrightarrow 2, 3 \rightarrow 3, 4 \rightarrow 4$, then $s \rightarrow s$ and $t \leftrightarrow u$ and so $F(s,u) = A(2,1,3,4)$. As such the three terms in the non-trivial S matrix can be written as

$$T^4(p_i) = A(1,2,3,4) + A(2,1,3,4) + A(1,3,2,4) \quad (2.21)$$

We note that the poles in the various channels arise from the end points of the integration regions. In Eq. (2.13) the s -channel and t -channel poles arise from the $x = 0$ and 1 end-points respectively.

The three terms for the S matrix are given by

$$g^2 \int_0^1 dx x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} + g^2 \int_0^1 dx x^{-\alpha(t)-1} (1-x)^{-\alpha(u)-1} \\ + g^2 \int_0^1 dx x^{-\alpha(u)-1} (1-x)^{-\alpha(s)-1} \quad (2.22)$$

They can be combined into one term, namely

$$g^2 \int_{-\infty}^{\infty} |x|^{-\alpha(s)-1} |1-x|^{-\alpha(t)-1} \quad (2.23)$$

by making use of the substitutions

$$\frac{1}{1-x} = \bar{x} \quad \text{and} \quad \bar{\bar{x}} = \frac{x-1}{x} \quad (2.24)$$

in the second and third terms respectively and using $s+t+u = 4\mu^2$. Here we have assumed the relation

$$4\alpha'\mu^2 + 3\alpha_0 = 1 \quad (2.25)$$

This in conjunction with Eq. (17), implies that $\alpha_0 = 1$ and $\mu^2 = -1/\alpha$. In fact, we will see that for the consistency of string theory $\alpha_0 = +1$ and $\mu^2 = -1/\alpha$.

The property of duality is not present in field theory. There the S matrix is made up of Feynman graphs. For example, in ϕ^3 theory the tree-level graph of Fig. 2 is equal to $g^2 s^{-1}$ which is not expressible as poles in t ; the poles in t arising from a different Feynman graph.

Fig. 2.2

Another remarkable property of the Veneziano formula is its high-energy behaviour which is discussed in many places¹.

Factorization for the four-point function means that if one picks a given channel, say the s channel, then the residues of the poles in s can be written as a product of two factors; one factor corresponds to the amplitude for states one and two to interact with the particle being exchanged, and the other corresponding to the amplitude for states three and four interacting with the particle being exchanged. In equations, this means

$$F(s,t) = \begin{array}{c} 2 \\ \diagup \\ \textcircled{g} \\ \diagdown \\ 1 \end{array} \text{---} \begin{array}{c} 3 \\ \diagup \\ \textcircled{g} \\ \diagdown \\ 4 \end{array} = \sum_{n=0}^{\infty} \sum_d \frac{g(1,2;n,d) g(3,4;n,d)}{(-\alpha' s^2 + \alpha_0 + n)} \quad (2.26)$$

where the sum over d denotes the sum over all the states with the same mass which are exchanged. The above discussion means that the first factor depends only on the properties (e.g., momenta) of particles 1 and 2 and the particle being exchanged, and does not depend on the properties of particles 3 and 4. An equivalent restriction also holds for the second factor. Using partial wave analysis, one can identify what are the spins of the states are being exchanged and also read off the on-shell scattering amplitudes between the two spin-zero states and all the particles being exchanged. The only problem occurs when more than two particles of the same spin and mass are exchanged in which case one cannot identify the amplitude for each spin separately. Of course not particles couple to two spin zeros and so one does not see these Three particles from such an amplitude. In an S matrix theory of which any field theory is an example, it can be shown from unitarity that in an arbitrary amplitude, the residues of any poles will factorize²⁶. In the beginning of string theory, one was in effect simply handed an S matrix and one did not know in advance whether it was unitary or that it would factorize. That factorization did take place was one of the features that greatly encouraged the early workers in string theory.

We now give a short illustration of the above discussion. In the centre-of-mass frame of the collision, let us denote the total energy by E and the scattering angle by θ ; we then find that

$$s = E^2, \quad t = -\frac{1}{2}(E^2 - 4\mu^2)(1 - \cos\theta)$$

where μ^2 is the mass of the external spin 0. The exchange of a resonance of spin J and mass m in the s-channel gives the behaviour

$$\frac{g^2 P_J(\cos\theta)}{s - m^2}$$

where P_J is the Legendre polynomial.

It is obvious that there is only a spin-zero particle exchange at the lowest mass, that is, when $\alpha_1 m^2 = (-\alpha_0)$. The residue of the pole at $s = m^2 = (1 - \alpha_0)/\alpha'$ is given by

$$\begin{aligned} g^2(\alpha' t + \alpha_0 + 1) &= g^2 \left[\left(-\frac{\alpha'_1 s}{2} + 2\alpha'_1 \mu^2 \right) (1 - \cos\theta) + \alpha_0 + 1 \right] \\ &= g^2 \left[\left(2\alpha'_1 \mu^2 + \frac{3}{2}\alpha_0 + \frac{1}{2} \right) - 2\alpha'_1 \mu^2 \cos\theta \right] \end{aligned}$$

Hence, in general we have one spin-0 and one spin-1; however, if $4\alpha'^{\frac{1}{2}}\mu^2 + 3\alpha_0 + 1 = 0$, which will turn out to be the case, we only have one spin-1.

Using momentum conservation and $s = 0$ we find that

$$\begin{aligned} (p_2 - p_1) \cdot (p_3 - p_4) &= 4 p_2 \cdot p_3 + 2 p_2^2 + 2 p_1 \cdot p_2 \\ &= (p_2 \cdot p_3) - \left(\frac{1 - \alpha_0}{\alpha'} \right) = \cos \theta \left[4\mu^2 - \left(\frac{1 - \alpha_0}{\alpha'} \right) \right] \end{aligned}$$

and hence the spin-1 residue at $s = 0$ is given by

$$- \frac{2\alpha'\mu^2}{\left[4\mu^2 - \left(\frac{1 - \alpha_0}{\alpha'} \right) \right]} \frac{g^2}{\alpha'} (p_2 - p_1) \cdot (p_3 - p_4)$$

which does indeed factorize into two pieces. In terms of Lagrange field theory, we recognize the spin-1-two spin-0 vertex to be of the form $\overleftrightarrow{g} \phi \partial^\mu \phi A^\mu \sqrt{\alpha'}$. The reader may find the spectrum and vertices at the next level if he desires. In fact one finds that there is in general one spin-2, one spin-1 and one spin-0. If we adjust α_0 as above, then the spin-1 disappears. Having done this in 26 dimensions, one finds only a spin-2, but in all other dimensions a spin-2 and spin-0. However for dimensions greater than 26 one finds that the spin-0 gives a negative contribution to the residue and so unitarity is violated. Thus, we learn even at this stage that the string must live in less than or equal to 26 dimensions with 26 dimensions having a different spectrum.

The next step taken in string theory was to guess the scattering of N spin zero particles²³. The key was to generalize the parametric form of Eq. (2.13) for the four-point scattering. Let us label the N external legs in cyclic order by $i = 1, \dots, N$ beginning with any leg and associate a real variable z_i , called the Koba-Nielson variable⁷, with leg i such that $z_{i+1} > z_i$. We next define the quantity

$$\begin{aligned} A(1, 2, \dots, N) &= g^{N-2} \int \dots \int_{z_1 < z_2 < \dots < z_N} dz_1 \dots [dz_a] \dots \\ &\dots [dz_b] \dots [dz_c] \dots dz_N |z_b^0 - z_c^0| |z_c^0 - z_b^0| |z_c^0 - z_a^0| \dots \end{aligned}$$

$$\prod_{1 \leq i < j \leq N} |z_i - z_j|^{2 p_i \cdot p_j \alpha'} \quad (2.27)$$

from which we will construct the S matrix. The notation in this formula means we omit the integrations over z_a , z_b and z_c which are assigned the arbitrary values z_a^0 , z_b^0 and z_c^0 respectively, provided $z_a^0 < z_b^0 < z_c^0$. We will often drop the index 0. At first sight, the formula is rather strange in that it singles out three z 's, i.e., z_a , z_b , z_c for special treatment and also gives them preferred values. This is, in fact, not the case and its demonstration requires the use of projective transformations which are transformations of the form

$$z \rightarrow \frac{az+b}{cz+d} \quad (2.28)$$

where z is any point on the half plane (see Appendix). Since any such transformation can be written as a product of an inversion, a dilation and translations, to prove the projective invariance of A it suffices to demonstrate it for each of these particular transformations. It is instructive to rewrite A as a measure and an integrand

$$g^{N-2} \int \dots \int_{z_1 < z_2 < \dots < z_N} d\mu \quad I \quad (2.29)$$

where the split is as follows

$$d\mu = dz_1 \dots [dz_a] \dots [dz_b] \dots [dz_c] \dots dz_N$$

$$|z_b^0 - z_c^0| |z_c^0 - z_a^0| |z_c^0 - z_a^0| \prod_{i=1}^N |z_{i+1} - z_i|^{-1}$$

$$I = \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} |z_i - z_j|^{2\alpha' p_i \cdot p_j} \prod_{i=1}^N |z_{i+1} - z_i|$$

(2.30)

where $z_{N+1} = z_1$.

For some purposes, it is more useful to write $A(1,2,\dots,N)$ as

$$\int \dots \int \prod_{i=1}^N dz_i \delta(z_a - z_a^0) \delta(z_b - z_b^0) \delta(z_c - z_c^0) \\ \prod_{i=1}^N |z_{i+1} - z_i|^{-1} \prod_{i=1}^N \theta(z_{i+1} - z_i) |z_b - z_c| \\ \cdot |z_c - z_a| |z_c - z_a| \quad I \quad (2.31)$$

where z_a^0 , z_b^0 and z_c^0 are the chosen values of these three co-ordinates. Clearly $d\mu$ and I are invariant separately under translations and $d\mu$ is manifestly dilation-invariant. Under a dilation, we find that, upon using momentum conservation,

$$I \rightarrow I \left\{ \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} |\lambda|^{2\alpha' p_i \cdot p_j} \right\} |\lambda|^N \\ = I |\lambda|^N \exp \sum_{\substack{i,j \\ 1 \leq i < j \leq N}} 2(p_i \cdot p_j) \alpha' \ln |\lambda| \\ = |\lambda|^N \exp \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \alpha' p_i \cdot p_j \ln |\lambda| \quad I \\ = I |\lambda|^{N - \alpha' \sum_{i=1}^N p_i^2} = |\lambda|^{N(1 + \alpha' \mu^2)} I. \quad (2.32)$$

Consequently, I is invariant provided $p_i^2 = 1/\alpha'$. That is, the mass of the spin zero is a tachyon, and the other particles have non-tachyonic masses. Identifying the mass of the spin zero with the lowest mass particle in the theory then implies $\alpha_0 = +1$. In fact, one can slightly generalize Eq. (2.27) for A in such a way that the restriction is not required. One puts in a factor $\prod_{i=1}^N |z_{i+1} - z_i|^{(-1 + \alpha' \mu^2)}$

Under an inversion, the measure $d\mu$ is easily found to be invariant (i.e., $z_i \rightarrow -1/z_i$) while

$$\begin{aligned}
 I &\rightarrow I \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} \left\{ \left| \frac{1}{z_i} \right| \left| \frac{1}{z_j} \right| \right\}^{2\alpha' p_i \cdot p_j} \\
 \prod_{i=1}^N \frac{1}{|z_i|} \frac{1}{|z_{i+1}|} &= I \exp \left\{ - \left[\sum_{i=1}^N \ln |z_i| + \ln |z_{i+1}| \right] \right\} \quad (2.33) \\
 &- \left\{ \sum_{\substack{i,j \\ 1 \leq i < j \leq N}} 2\alpha' (p_i \cdot p_j) (\ln |z_i| + \ln |z_j|) \right\} \Big\} \\
 &= I \exp 2 \sum_{i=1}^N \ln z_i \left\{ -1 + \alpha' p_i^2 \right\}
 \end{aligned}$$

which vanishes when $p_i^2 = 1/\alpha'$ as before. Thus A is projectively invariant. We observe that this is an unusual type of symmetry since it acts on variables which are integrated over and so are not physical. We recall that given three distinct points, there always exists a projective transformation which can take them to any other three distinct points (see Appendix). Since both the measure and the integrand are invariant, we may make a projective transformation to make z_a, z_b, z_c take on any desired values and the value of A is unchanged. Further, as the result is projective invariant we may move the z_i 's to be on any contour which is related to by a projective transformation to the real axis. One such example is the unit circle. We recall that the cyclic order of the z_i 's is always preserved under a projective transformation (see Appendix).

A little more subtle is the demonstration that we may choose any three points not to be integrated over and the result is the same. It suffices to show that we can choose z_{a+1} rather than z_a to deduce the general case. The idea is that starting from Eq. (2.27) where z_a is fixed, we may always perform the projective transformation T that takes z_{a+1} to a constant denoted z_{a+1}^0 and z_b and z_c to be as before and so they take to z_b^0 and z_c^0 respectively. That is

$$T \text{ is } \begin{pmatrix} z_{a+1} & z_b^0 & z_c^0 \\ z_{a+1}^0 & z_b^0 & z_c^0 \end{pmatrix} \quad (2.34)$$

Let us write this transformation as

$$z' = T(z) = \frac{az+b}{cz+d}$$

where a, b, c and d depend on $z_{a+1}, z_{a+1}^0, z_b^0$ and z_c^0 .

The transformation T has two fixed points $z_b = z_b^0$ and $z_c = z_c^0$ and so may also be written in the form

$$\frac{T(z) - z_b}{T(z) - z_c} = K \frac{(z - z_b)}{(z - z_c)} \quad (2.35)$$

Taking $z = z_{a+1}$ we have

$$K = \frac{(z_{a+1}^0 - z_b)}{(z_{a+1}^0 - z_c)} \frac{(z_{a+1} - z_b)}{(z_{a+1} - z_c)} \quad (2.36)$$

Although $T(z_{a+1})$ is fixed to be z_{a+1}^0 , a constant, z_a is now no longer fixed to be z_a^0 and is now a function of z_{a+1} . Differentiating Eq. (2.35) we find that

$$\frac{dT(z)}{dz_{a+1}} = \frac{(T(z) - z_b)}{(z_{a+1} - z_b)} \frac{(T(z) - z_c)}{(z_{a+1} - z_c)} \quad (2.37)$$

Let us write the expression for $A(1, 2, \dots, N)$ of Eq. (2.27) in the form

$$A(1, 2, \dots, N) = \int \dots \int \prod_{\substack{i=1 \\ i \neq a, b, c}}^N dz_i |z_a^0 - z_b^0| |z_a^0 - z_c^0| \\ \cdot |z_b^0 - z_c^0| f(z_1, \dots, z_N) \quad (2.38)$$

where z_a^0 , z_b^0 and z_c^0 are the fixed values of z_a , z_b , z_c . Since we have shown A is projective invariant and under

$$z_i' = \tilde{T}(z_i) = \frac{a z_i + b}{c z_i + d}$$

$$\prod_{\substack{i=1 \\ i \neq a, b, c}}^N dz_i' |z_a' - z_b'| |z_c' - z_b'| |z_a' - z_c'|$$

$$= \left\{ \prod_{i=1}^N \frac{d\tilde{T}(z_i)}{dz_i} \right\} \prod_{\substack{i=1 \\ i \neq a, b, c}}^N dz_i |z_b - z_a| |z_c - z_b| |z_a - z_c| \quad (2.39)$$

we must conclude that

$$f(z'_1, \dots, z'_N) = \left\{ \prod_{i=1}^N \frac{d\hat{T}(z_i)}{dz_i} \right\}^{-1} f(z_1, \dots, z_N).$$

$$= \prod_{i=1}^N \left\{ \frac{(ad-bc)}{|(cz_i+d)|^2} \right\}^{-1} f(z_1, \dots, z_N). \quad (2.40)$$

One can of course also find this result from the explicit form of f . We now make the change of variables $z'_i = T(z_i, z_{a+1})$ $i \neq a+1$ and $z_{a+1} \rightarrow z_{a+1}$. Since the Jacobian for this transformation is

$$\prod_{\substack{i=1 \\ i \neq a, b, c, a+1}}^N \left(\frac{dT}{dz_i} \right)^{-1} \quad (2.41)$$

we have

$$A = \int \dots \int \prod_{\substack{i=1 \\ i \neq a, b, c, a+1}}^N dz'_i |z_a - z_b| |z_c - z_b| |z_c - z_a| dz_{a+1}$$

$$\prod_{\substack{i=1 \\ i \neq a, b, c, a+1}}^N \left(\frac{dT}{dz_i} \right)^{-1} f(z_1, \dots, z_N)$$

$$= \int \dots \int \prod_{\substack{i=1 \\ i \neq a, b, c, a+1}}^N dz'_i dz_{a+1} |z'_a - z'_b| |z'_c - z'_b| |z'_c - z'_a| \quad (2.42)$$

$$\cdot \prod_{\substack{i=1 \\ i \neq a+1}}^N \left(\frac{dT}{dz_i} \right)^{-1} f(z_1, \dots, z_a, z_{a+1}, \dots, z_N)$$

$$= \int \dots \int \prod_{\substack{i=1 \\ i \neq a, b, c, a+1}}^N dz'_i dz_{a+1} |z'_a - z_b^0| |z_c^0 - z_b^0| |z_c^0 - z'_a|$$

$$\cdot \frac{(ad-bc)}{(cz_{a+1}+d)^2} f(z'_1, \dots, z'_a, z_{a+1}^0, \dots, z_b^0, \dots, z_c^0, \dots, z'_N).$$

Finally we must make the change of variables $z_{a+1} \rightarrow \omega = T(z_a)$. From Eq. (2.37), we note that

$$\frac{dw}{dz_{a+1}} = \frac{(z'_a - z'_b)}{(z_{a+1} - z'_b)} \frac{(z'_a - z'_c)}{(z_{a+1} - z'_c)} \quad (2.43)$$

and from explicit computations we find that

$$\begin{aligned} (z'_{a+1} - z'_b)(z'_{a+1} - z'_c) &= \frac{(ad-bc)^2}{(cz_{a+1}+d)^2} (z_{a+1} - z'_b) \\ &\cdot \frac{(z_{a+1} - z'_c)}{(cz'_b+d) \cdot (cz'_c+d)} \end{aligned} \quad (2.44)$$

From Eq. (2.35) we note that

$$\frac{dT}{dz} = K \frac{(T(z) - z_c)}{(z - z_c)} = \frac{1}{K} \frac{(T(z) - z_b)}{(T(z) - z_b)} \quad (2.45)$$

and consequently substituting z_b and z_c as appropriate we have

$$\begin{aligned} \frac{(ad-bc)}{(cz_b+d)^2} &= \left. \frac{dT}{dz} \right|_{z=z_b} = K \\ \left. \frac{dT}{dz} \right|_{z=z_c} &= \frac{ad-bc}{(cz_c+d)^2} = \frac{1}{K}. \end{aligned} \quad (2.46)$$

Putting all this together, Eq. (23) becomes

$$\begin{aligned} A(1, 2, \dots, N) &= \int \dots \int \prod_{\substack{i=1 \\ i \neq a, b, c, a+1}}^N dz'_i d\omega |z'_a - z'_b| \\ &|z'_a - z'_c| |z'_b - z'_c| f(z'_1, \dots, \omega, z'_{a+1}, \dots, z'_b, \dots, z'_c, \dots, z'_N) \end{aligned} \quad (2.47)$$

However, ω is a dummy variable and we can call it z_a which gives the required result. The reader may satisfy himself that this change from z_a to z_{a+1} is consistent with the cyclic ordering of the points and that we could have taken z_{a+1} to be any point between z_a and z_b and the discussion is unchanged.

The problem encountered here is rather similar to that encountered in gauge theories where one has an action which is invariant under a local symmetry, and one wants to integrate in the functional integral, not over all gauge field configurations, but only over this space when moded out by gauge equivalent field configurations. In the case here we could have started with the expression

$$\int \prod_{i=1}^N dz_i \prod_{1 \leq i < j \leq N} |z_i - z_j|^{2\alpha' p_i \cdot p_j}$$

This is, however, projectively invariant and so contains a factor corresponding to the volume of the non-compact projective group, $SL(2, R)$, which is infinite. To eliminate this degeneracy we can proceed as for gauge theories; we fix the projective invariance by fixing three of the z 's to have fixed values and then introduce the corresponding Faddeev-Popov determinant. This determinant is precisely the factor $|z_a^0 - z_b^0| |z_b^0 - z_c^0| |z_c^0 - z_a^0|$ if z_a, z_b, z_c are the three variables which have been fixed.

As a result the formula for $A(1, \dots, N)$ is cyclically symmetric

$$A(1, 2, \dots, N) = A(N, 1, 2, \dots, N-1) \quad (2.48)$$

Examining Eq. (2.27) we see that if $z_i \rightarrow z_{i+1}, p_i \rightarrow p_{i+1}$, the result is the same but the three fixed z 's move up by one. However, as we have just seen, the integral is the same no matter which three z 's are fixed and which value is chosen for them.

The expression for the non-trivial part of the S matrix for the scattering of N spin zero particles is given by

$$T^{(N)}(p_i) = \sum_{\substack{\text{non-cyclic and} \\ \text{non-reversed perms.}}} A(1, 2, \dots, N) \quad (2.49)$$

In this count we have not included terms in which the order of all the legs is reversed. For example, if we have $V(1, 2, \dots, N)$ we do not include $V(N, N-1, \dots, 2, 1)$ as well, since this is just a question of which way round the graph to label the external legs. There are $(n-1)!/2$ such non-cyclic permutations.

To recover the four-point function we can take $z_1^0 = 0$, $z_2 = x$ ($0 < x < 1$), $z_3^0 = 1$ and $z_4^0 = \infty$ in Eq. (2.27). One finds

$$\begin{aligned}
 A(1,2,3,4) &= g^2 \int_0^1 dx |1-0| |\infty-0| |\infty-1| \\
 &\quad |x|^{2\alpha' p_1 \cdot p_2} |1|^{2\alpha' p_1 \cdot p_3} |\infty|^{2\alpha' p_1 \cdot p_4} |x-1|^{2\alpha' p_2 \cdot p_3} \\
 &\quad |x-\infty|^{2\alpha' p_2 \cdot p_4} |1-\infty|^{2\alpha' p_3 \cdot p_4} \\
 &= g^2 \int_0^1 dx |x|^{-\alpha(s)-1} |1-x|^{-\alpha(t)-1} \quad (2.50)
 \end{aligned}$$

which is the desired result. The factor of ∞ comes to the power zero and we have used $\alpha' p_i^2 = -1$.

Duality for an N point function is the following. We subdivide the external legs labelled $1, \dots, i, \dots, N$ into two non-overlapping sets, each containing consecutive legs, say, $\{j, \dots, N, 1, \dots, k\}$ and $\{k+1, \dots, j-1\}$. The channel variable selected is then

$$s_{jk} = - \left(\sum_{i=j}^k p_i^\mu \right)^2 = - \left(\sum_{i=k+1}^{j-1} p_i^\mu \right)^2 \quad (2.51)$$

Then duality states that the amplitude $A(1,2,\dots,N)$ should be expressible in terms of a sum over poles in s_{jk} for any choice of j and k . Clearly, the cyclic symmetry of $A(1,2,\dots,N)$ is a related point, since given that it is a sum of poles in a given channel, it must be a sum of poles in all other channels related by cyclic symmetry.

Factorization then means that for a given channel, say s_{jk} that

$$A(1,2,\dots,N) = \sum_{p,q,c} \sum_d \frac{g(p;c,d) g(q;c,d)}{-s_{jk}^2 + c} \quad (2.52)$$

The sum over d means a sum over all the intermediate states with mass c_i and the labels p and q denote the dependence on the momenta on legs j to k and on legs $k+1, \dots, j-1$ respectively. As the formula implies, the first factor $g(p;c,d)$ depends only on the properties of the particles j to k and the intermediate state, and not on the properties of

the particles $k+1$ to $j-1$ and vice-versa for the second factor $g(q;c,d)$. A worry to early string theorists was the occurrence of more and more states at a given mass as one looked at higher and higher point functions. However, this is only a result of the constraints on the possible intermediate states that can occur and in fact explicit examination showed that for a given mass there is only a finite number of particles and their maximal spin rises linearly with their mass. The factorization discussed above is illustrated in Fig. 2.3

Fig. 2.3

One can also apply factorization to the resulting two pieces and so arrive at an amplitude with two simultaneous poles (see Fig. 2.4)

Fig. 2.4

By such repeated factorization, one can, provided there are suitable convergence properties of the sum over poles, arrive at the notion of duality which we now give. We divide the legs into m non-overlapping sets each containing consecutive legs, i.e., $\{j, j+1, j+2\}$, $\{j+3, j+4\}$, etc. The channel variables we denote by

$$s_{j, j+2}, s_{j+3, j+4}, \dots \quad (2.53)$$

Then the amplitude A is expressible as a sum of simultaneous poles in $s_{jj+2}, s_{j+3,j+4}, \dots$. Duality is now taken to mean that this is true no matter what the choice of channels.

Factorization states that for a given choice of channels, A can be written in the form

$$\sum_{\text{poles } C_n} \left\{ \prod_{n=1}^m \frac{1}{(-s_n + C_n)} g(p_n, C_n) H_{C_1, \dots, C_m} \right\} \quad (2.54)$$

where the degeneracy at a given mass level is to be understood. This particular factorization is shown in Fig. 2.5 for $m = 3$.

Fig. 2.5

In fact, it was shown that not only did $A(1,2,\dots,N)$ of Eq. (2.27) satisfy duality, but it also factorized. As mentioned before, while the latter is guaranteed by unitarity in an S matrix theory or field theory, it will not, in general, occur in any would-be amplitude.

Thus, given the amplitude for the arbitrary spin zero scattering is dual and factorizable, we may deduce not only the mass spectrum, but also the scattering amplitudes between any of the particles in the mass spectrum. For example, from duality and factorization we may rewrite the scattering amplitude for an arbitrary number of spin zeros, as in Fig. 2.6

Fig. 2.6

By examining the appropriate residues, one may deduce the three- four-, etc., point scattering amplitude for arbitrary states.

One must stress, however, that having deduced the arbitrary scattering amplitudes, one must verify that one gets the same result for a given amplitude, no matter how one factorizes it. That the amplitudes deduced from Eq. (2.27) pass this very stringent requirement was called the bootstrap condition.

3. THE PHYSICAL STATES AND GENERAL SCATTERING AMPLITUDES

In the previous section, we described how from the seemingly innocuous scattering formula for an arbitrary number of spin zero states, one can deduce not only the existence of an infinite number of other particles, but also the scattering amplitude between them. This was made possible by the duality and factorization properties of the amplitudes.

While this procedure may be straightforward in principle, in practice the presence of an infinite number of particles made it a forbidding task. What was required was a method of encoding this vast number of particle states. This was the oscillator formalism²⁷. One introduces operators α_n^μ , $n = 0, \pm 1, \pm 2$ which satisfy

$$[\alpha_n^\mu, \alpha_m^\nu] = n \eta^{\mu\nu} \delta_{n+m, 0} \quad (3.1)$$

and

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu. \quad (3.2)$$

We can regard the α_n^μ , $n > 1$ as destruction operators and α_n^μ , $n \leq 1$ as creation operators on a vacuum:

$$\alpha_n^\mu | \rangle = 0 \quad n \geq 1 \quad (3.3)$$

The most general state is then given by

$$| \psi \rangle = (\varphi(x) + \alpha_1^{\mu\dagger} A_\mu(x) + \alpha_1^{\mu\dagger} \alpha_1^\nu h_{\mu\nu}(x) + \alpha_2^{\mu\dagger} B_\mu(x) + \dots) | \rangle \quad (3.4)$$

It is very useful to introduce a $\alpha_0^\mu = p^\mu$, the momentum. Of course, these oscillators just correspond to the infinite modes of the string $x^\mu(\sigma, \tau)$ and it was in this way that it was first realized that the Veneziano amplitude had something to do with string theory²⁸, the precise correspondence being given by first considering the Fourier transform of $x^\mu(\sigma, \tau)$

$$x^\mu(\sigma, \tau) = \sum_{n=-\infty}^{\infty} x_n^\mu(\tau) e^{in\sigma} \quad (3.5)$$

where we have extended the range of σ from $0 \leq \sigma \leq \pi$ to $-\pi \leq \sigma \leq \pi$ by reflection [i.e., $x^\mu(-\sigma) \equiv x^\mu(\sigma)$]. The oscillators defined above then correspond to

$$\partial^\mu \equiv -i\hbar \frac{\delta}{\delta x^\mu(\sigma)} = \frac{1}{2\pi\alpha'} \dot{x}^\mu(\sigma) \equiv -\frac{1}{\sqrt{2\alpha'}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{in\sigma} \quad (3.6)$$

Physical states - particle spectrum

Let us first turn to the particle spectrum which was found by the process described above. To do this we are required to define the operator²⁹

$$L_n \equiv \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu}; n \neq 0 \quad (3.7)$$

and

$$L_0 \equiv \sum_{n=0}^{\infty} \alpha_n^\mu \alpha_n^\mu \quad (3.8)$$

which is normal ordered. The on-shell states were found to be given by the equations³⁰

$$(L_0 - 1) |\psi\rangle = 0 \quad (3.9)$$

$$L_n |\psi\rangle = 0 \quad n \geq 1 \quad (3.10)$$

We call states that obey these equations physical states. The first equation is the mass shell condition which is compatible with the observations made in the previous section. In particular, they imply that the lowest mass particle is a tachyon with $(\text{mass})^2 = -1/\alpha'$ and the highest spin, found at a given mass level, rises linearly with the $(\text{mass})^2$. The

meaning of the last equation is less obvious. At the first level, one finds that Eqs. (3.9) and (3.10) imply the presence of a massless spin one, namely the constraints

$$\partial^\mu A_\mu = 0 \quad ; \quad \partial^2 A_\mu = 0 \quad (3.11)$$

The reader may examine the restrictions on the higher states, but one obvious danger is apparent, namely there are in $|\psi\rangle$ many potential negative norm states due to the presence of a_n^0 (note that $\langle a_n^0 a_n^{0+} \rangle = -n$). One of the truly remarkable facts³¹ of string theory is that all the physical states have non-negative norms as is required for unitarity if the dimension of space-time is less than or equal to 26 and the intercept is one. Were one to change $(L_0 - 1)|\psi\rangle = 0$ to $(L_0 - c)|\psi\rangle = 0$, $c \neq 1$, then this no-ghost theorem would not work. Consequently we find that for unitarity we require the lowest mass state to have a mass $-1/\alpha'$, that is the intercept takes the value $\alpha_0 \equiv +1$. We recall that the projective invariance of the Veneziano scattering formula required the lowest mass states to have mass $-1/\alpha'$ and so we must identify the spin zero external states with the tachyon.

The Virasoro operators L_n obey the algebra³²

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}n(n^2-1)\delta_{n+m,0} \quad (3.12)$$

where the last term³³ results from the normal ordering requirement of L_0 . It can be shown from the Jacobi identity that the most general form for such a term is $an^3 + bn$ and an explicit evaluation of $\langle [L_n, L_{-n}] \rangle$, for $n = 1, 2$ fixes it to be as above.

Neglecting the central term the above algebra is recognizable as the Lie algebra of the $D = 2$ conformal group or equivalently the group of diffeomorphisms on the circle, $\text{diff } S^1$. Actually, to be precise, each of these groups contains two copies of the Virasoro group. If θ parametrizes the circle and $z = e^{-i\theta}$, then the generator of reparametrizations can be taken to be

$$L_n = z^{n+1} \frac{d}{dz}$$

which generate the above algebra. The identification with the $D = 2$ conformal group is through $z = e^{-i(\tau-\sigma)}$ and $\bar{z} = e^{-i(\tau+\sigma)}$, after an

appropriate Euclidean rotation. The appearance of these groups is to be expected from the string viewpoints since the L_n 's are the constraints of the classical theory in a Hamiltonian approach, and so are that subgroup of two-dimensional reparametrization remaining after we have selected one co-ordinate as a preferred time.

The norms of the physical states are not, however, all positive definite. Consider for example the state

$$|s\rangle = L_{-1} |\Omega\rangle \quad (3.13)$$

with $L_0 |\Omega\rangle = 0$, $L_n |\Omega\rangle = 0$, $n > 1$. It is easily verified that $|s\rangle$ is a physical state, that is, satisfies Eqs. (3.9) and (3.10), and so

$$\langle \chi | s \rangle = \langle \chi | L_{-1} |\Omega\rangle = 0 \quad (3.14)$$

for any other physical state $|\chi\rangle$ including $|s\rangle$ itself. At the next level we may consider the state

$$|s'\rangle = (L_{-2} + b L_{-1}^2) |\Omega\rangle \quad (3.15)$$

where now $L_n |\Omega'\rangle = 0$, $n > 1$ ($L_0 + 1$) $|\Omega'\rangle = 0$. We demand $|s\rangle$ be physical and in particular

$$L_1 |s'\rangle = 0 = (3L_{-1} + 2bL_1) |\Omega'\rangle \quad (3.16)$$

which implies that $b = +3/2$, while $L_2 |s'\rangle = 0$ implies that $D = 26$. Consequently if $D = 26$ and $b = 3/2$ we find that $|s'\rangle$ is a physical state and

$$\langle s' | \chi \rangle = 0 \quad (3.17)$$

for any physical state $|\chi\rangle$.

The existence of zero norm states of the form

$$(L_{-n_1} \dots L_{-n_r}) |\Omega''\rangle \quad (3.18)$$

with $L_n |\Omega''\rangle = 0$, $n > 1$, $L_0 |\Omega''\rangle = (\sum_i n_i - 1) |\Omega''\rangle$ can be read off from the Kac determinant. The reader may verify by explicit computation that at the next level such a state only exists if $D = 28$.

The no-ghost theorem asserts for $D = 26$ that the physical states have positive norm and so include states of zero norm and states with positive definite norm. To decompose a given physical state into these two types of states, we choose a Lorentz frame in which the momentum of the states is labelled by p^μ and specify a vector $\bar{k}^\mu$ such that $(\bar{k}^\mu)^2 = 0$ and $\bar{k}_\mu p^\mu \neq 0$. Then $|\psi\rangle$ can be uniquely decomposed as $|\psi\rangle = |\chi_1\rangle + |\chi_2\rangle$ where both $|\chi_1\rangle$ and $|\chi_2\rangle$ are each separately physical states and in addition $K_n |\chi_1\rangle = 0$, $n > 0$ where $K_n = \alpha_n^\mu \bar{k}_\mu$ and $\langle \text{any physical state} | \chi_2 \rangle = 0$. It follows that $\langle \chi_2 | \chi_2 \rangle = 0$, but it can also be shown that $\langle \chi_1 | \chi_1 \rangle > 0$. Further³¹, any zero norm physical state $|\chi\rangle_2$ can be written in the form $a|s\rangle + b|s'\rangle$ where $|s\rangle$ and $|s'\rangle$ are given in Eqs. (3.13) and (3.15) above. In this sense the discussion of zero norm null states given above is complete.

At least one null state is to be expected since we have in the spectrum a massless spin one state which is well known to have such a state. In the Lorentz frame, where the spin one has momentum $k^\mu = (k, 0, 0, k)$, the four photon states can be written as $\epsilon_\mu^{\lambda+}(k)|k\rangle$ and in particular a basis is given by the states

$$a^{i+}(k)|k\rangle, i=1, 2; (a^{0+}(k) - a^{3+}(k))|k\rangle \quad (3.19)$$

and

$$(a^{0+}(k) + a^{3+}(k))|k\rangle \quad (3.20)$$

The last state is unphysical as it does not obey $k^\mu \epsilon_\mu = 0$. The first three states are physical, but only the first two have positive definite norm. The third state has polarization vector k^μ and so has zero norm and also has zero scalar product with any physical state. The effect of this is to leave us with two states in accord with the irreducible representation theory of the Poincaré group.

In QED one knows that, on-shell, the above zero norm null state does not couple to states with positive definite norm states in the sense that, given any scattering amplitude for fermions and photons, all the photons of which are physical and only one of which has zero norm, then the process will vanish. This is a consequence of the $U(1)$ gauge invariance and in particular its Ward identity. Were it to fail, then unitarity would be violated. In string theory, however, we must verify that such decoupling of null states does indeed take place. This will

happen if we can push L_{-n} 's from the one leg of the amplitude on which the spurious state is present and what comes out on the other legs is a sum of terms involving L_n , $n \geq 1$ and $(L_0 - 1)$, which will vanish on the physical state there. We will discuss this in detail when we have the general amplitudes. However, it is desirable to have objects which we now discuss through which the L_n 's pass in a well-defined way.

An operator $R(z)$ is said to have conformal dimension d if

$$g R(z) g^{-1} = \left[\frac{z}{z'} \frac{dz'}{dz} \right]^d R(z') \quad (3.21)$$

where g is a conformal transformation and so is of the form $g = \exp \sum_{n=-\infty}^{\infty} a_n L_n$. In fact, the transformation $z \rightarrow z^1 = z^1(z)$ is the same as the one found by naively applying g when L_n is represented by

$$L_n = z^{n+1} \frac{d}{dz} \quad (3.22)$$

For an infinitesimal transformation this reduces to the usual definition

$$[L_n, R(z)] = z^n \left(z \frac{d}{dz} + nd \right) R(z) \quad (3.23)$$

One easily verifies using the relation $[L_n, \alpha_m^\mu] = -m \alpha_{n+m}^\mu$ that

$$P^\mu(z) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n} \quad (3.24)$$

and

$$\varphi^\mu(z) = - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^\mu z^{-n} + \alpha_0^\mu \ln z + \frac{2}{2\alpha_0^\mu} \quad (3.25)$$

have conformal weight 1 and 0 respectively. The object $L(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n}$ has conformal weight 2, were one to neglect the central term in the L_n algebra. We note that

$$P^\mu(z) = z \frac{d}{dz} \varphi^\mu(z) \quad (3.26)$$

The three-string scattering vertex

Sciuto³⁴, equipped with the N spin zero scattering S matrix and the oscillator formalism, was able to deduce the scattering for three arbitrary string states by the method of factorization discussed in Section 2. He wrote the N spin zero scattering amplitude in terms of a three-vertex scattering three arbitrary states as in Fig. 2.6. For a readable and explicit discussion on how in practice to factorize, see Ref. 35.

The result found by Sciuto was made cyclically symmetric by Caneschi, Schwimmer and Veneziano³⁶ and the result was found to be

$$V^{CSV} = \langle 0|_1 \langle 0|_2 \langle 0|_3 \exp - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^3 \left\{ \alpha_m^{(r)} \alpha_n^{(r+1)} \frac{(-1)^m \Gamma(n)}{\Gamma(1+m) \Gamma(1+n-m)} \right\} \quad (3.27)$$

This object has three vacua corresponding to the three external legs labelled by $r = 1, 2, 3$. It can be written in an alternative form

$$V^{CSV} = \langle 0|_1 \langle 0|_2 \langle 0|_3 \exp - \left\{ \sum_{r=1}^3 \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} p^{(r)}_{\mu}(z) \varphi^{(r+1)}_{+} \left(\frac{1}{1-z} \right) \right\} \quad (3.28)$$

where

$$\varphi^{(r)}_{+}(z) = \sum_{n=1}^{\infty} \frac{\alpha_n^{(r)}}{n} z^{-n} \quad (3.29)$$

The scattering of any three states $|\chi\rangle_1$, $|\chi\rangle_2$ and $|\chi\rangle_3$ is given by

$$V^{CSV} |\chi\rangle_1 |\chi\rangle_2 |\chi\rangle_3 \quad (3.30)$$

and the final result is a straightforward exercise in oscillator algebra once the $|\chi\rangle_i$ are given. For example, for three tachyon states $|p_1\rangle$,

$|p_2\rangle, |p_3\rangle$ with $p_2^2 = +1$, $i = 1, 2, 3$, the oscillators all annihilate on the vacuum on the right and the result is one. The massless spin one states, on the other hand, are described by

$$|\phi\rangle = \varepsilon^\mu(k) a_1^{\mu\dagger} | \rangle \quad (3.31)$$

with

$$k^\mu \varepsilon_\mu(k) = 0 \quad ; \quad k^2 \varepsilon^\mu(k) = 0 \quad (3.32)$$

This results from the Virasoro conditions $L_n |\phi\rangle = (L_0 - 1) |\phi\rangle = 0$, $n > 1$. A little oscillator algebra yields the result

$$\begin{aligned} & (k_1 \cdot \varepsilon_3)(\varepsilon_1 \cdot \varepsilon_2) + (k_2 \cdot \varepsilon_1)(\varepsilon_2 \cdot \varepsilon_3) + (k_3 \cdot \varepsilon_2)(\varepsilon_1 \cdot \varepsilon_2) \\ & + (k_1 \cdot \varepsilon_3)(k_2 \cdot \varepsilon_1)(k_3 \cdot \varepsilon_2) \end{aligned} \quad (3.33)$$

which agrees with that found from Yang-Mills after one drops the last term in the zero slope limit. The reader may amuse himself by computing other scattering processes.

Since this vertex contains all possible three string scatterings, it must also contain the vertex for the tachyon, spin one, etc., emission from two arbitrary string states. In the tachyon case, we must consider $V^{CSV} | \rangle_3$ and we should expect to find the well-known result $e^{-k \cdot Q}$ plus factors that give one on on-shell states. In fact, to find only $e^{-k \cdot Q}$, we should consider

$$V(1, 2) = V^{CSV} | \rangle_3 \Omega^{(2)} \quad (3.34)$$

where Ω is the twist operator³⁶ and is given by Ω

$$\Omega = e^{-L_1(-1)} (L_0 - \frac{1}{2} p_1^2) \equiv e^{-L_1(-1)} R \quad (3.35)$$

We take $e^{-L_1^{(2)}}$ through V^{CSV} , but as Q_+ only has half of its oscillators, it is not a conformal weight zero operator. However, by explicit computations we find that

$$e^{-L_1} \varphi_+(z) e^{-L_1} = \varphi_+\left(\frac{z}{1-z}\right) + \varphi_+(-1) \quad (3.36)$$

We also require the result

$$\langle 0 | e^{z L_1} = \langle 0 | \exp \left\{ \alpha_0^\mu \varphi_+^{(\mu)} \left(\frac{1}{z} \right) \right\} \quad (3.37)$$

Putting all this together we find

$$\begin{aligned} V(1, z) &= \langle 0 | \langle 0 | e^{-L_1^{(2)}} (-1)^R \exp - \frac{1}{2\pi i} \left\{ \oint_{\bar{z}} \frac{dz}{z} P_{\mu}^{(1)}(z) \left(\varphi_+^{(2)\mu} \left(\frac{1}{z} \right) - \varphi_+^{(2)\mu}(1) \right) + \oint_{z=0} \frac{dz}{z} \alpha_0^{(3)\mu} \varphi_+^{(1)\mu} \left(\frac{1}{1-z} \right) \right\} \\ &= \langle 0 | \langle 0 | \exp \left\{ - \frac{1}{2\pi i} \oint_{\bar{z}} \frac{dz}{z} P_{\mu}^{(1)}(z) \varphi_+^{(2)\mu} \left(\frac{1}{z} \right) \right. \\ &\quad \left. - \alpha_0^{(3)\mu} \varphi_+^{(1)\mu}(1) + \left(\alpha_0^{(1)\mu} + \alpha_0^{(2)\mu} \right) \varphi_+^{(2)\mu}(1) \right\} \quad (3.38) \\ &= \langle 0 | \langle 0 | \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n^{(1)\mu} \alpha_n^{(2)\mu} \right. \\ &\quad \left. + \alpha_0^{(3)\mu} \left(\varphi_+^{(1)\mu}(1) + \varphi_+^{(2)\mu}(1) \right) \right\} \end{aligned}$$

After turning two around and identifying the one and two oscillators, we recover the well-known result. We define the operator $V(k)$ with line one turned around by

$$V(1, z) |x\rangle_1 |x\rangle_2 = \langle x | V(k) |x\rangle_2 \quad (3.39)$$

for any states $|x\rangle_1$ and $|x\rangle_2$.

It is easily seen that $V(k)$ is the usual expression namely

$$V(k) = e^{-k^\mu \varphi_+^{\mu\dagger}(1)} e^{-ik^\mu q^\mu} e^{+k^\mu \varphi_+^\mu(1)} \quad (3.40)$$

We can also consider the operator

$$V(k, z) = e^{-k^\mu \varphi_+^{\mu\dagger}(z)} e^{-ik \cdot q - k \cdot p \ln z} e^{+k^\mu \varphi_+^\mu(z)} \quad (3.41)$$

and one can verify that it has conformal weight one if $k^2 = 1$, which it does for the tachyon.

The photon emission vertex follows from

$$V^{CSV} \propto \epsilon^{\mu+}(k) \epsilon_{\mu}(k) |k\rangle_3 \Omega^{(2)} \quad (3.42)$$

with $k^{\mu} \epsilon_{\mu} = 0$; $k^2 \epsilon_{\mu} = 0$.
and similarly for the higher mass states.

We can deduce the arbitrary amplitude by factorization from the spin zero amplitude; however, one can also, using factorization and duality, write it in terms of the three-point amplitude and sum over intermediate states. That is we can construct any tree amplitude out of V^{CSV} and the propagator $D = (L_0 - 1)^{-1}$ which encodes the correct sum over the poles. For example, for the arbitrary four-point amplitude we have the graph in Fig. 3.1 as well as two other contributions. In this way the amplitudes

Fig. 3.1

look like a ϕ^3 Feynman field theory with vertex V^{CSV} and propagator $(L_0 - 1)^{-1}$. However, the combinatoric rule is not that which follows from a Feynman path integral. By construction each term is cyclically symmetric and instead we sum over the inequivalent cyclic permutations not including terms in which all the labels are reversed order. By the latter we mean that if we include $V(1,2,3,\dots,N)$ we do not also include $V(N,N-1,\dots,2,1)$. This gives $(N-1)!/2$ terms. This is the same combinatoric rule as we used in Section 2 for the terms which make up the N tachyon scattering formula.

A specific example is the recovery of the scattering of N tachyons which may now be written as

$$A(1, \dots, N) = \begin{array}{c} \text{---} p_N \text{---} \text{---} p_1 \end{array}$$

$$= \langle 0, -p_N | V(p_{N-1}) D \dots D V(p_3) D V(p_2) | 0, p_1 \rangle \quad (3.43)$$

where $V(k)$ is given in Eq. (3.40) and $|0, p\rangle$ is the state annihilated by α_n^μ , $n > 1$, but carries momentum p . To evaluate this integral we write the propagator in parametric form

$$D = \int_0^1 \frac{dx}{x} x^{(L_0-1)} \quad (3.44)$$

and then Eq. (3.43) becomes

$$\begin{aligned} \int \prod_{i=2}^{N-2} \frac{dx_i}{x_i} <0, -p_N | V(p_{N-1}) x_{N-2}^{(L_0-1)} V(p_{N-2}) \\ \dots x_3^{(L_0-1)} V(p_4) \dots x_2^{(L_0-1)} V(p_2) | 0, p_1 \rangle \end{aligned} \quad (3.45)$$

Since the final state is on-shell, i.e. $(L_0-1)|0, p_N\rangle = 0$ and using the fact that $V(k, z)$ has conformal weight one, we have

$$\begin{aligned} \int_0^1 \prod_{i=1}^{N-2} \frac{dx_i}{x_i} <0, -p_N | V(p_{N-1}, 1) V(p_{N-2}, x_{N-2}) \\ \dots V(p_2, x_2 \dots x_{N-2}) | 0, p_1 \rangle \end{aligned} \quad (3.46)$$

We now define

$$z_i = \prod_{j=i}^{N-2} x_j \quad i = 2, \dots, N-2 \text{ and } z_{N-1} = 1 \quad (3.47)$$

and as a result

$$z_{N-1} = 1 > z_{N-2} > z_{N-1} \dots > z_2 > 0 \quad (3.48)$$

and

$$\prod_{i=2}^{N-2} \frac{dz_i}{z_i} = \prod_{i=2}^{N-2} \frac{dx_i}{x_i} \quad (3.49)$$

Consequently, Eq. (3.46) now reads

$$\begin{aligned} \int \dots \int \prod_{i=2}^{N-2} dz_i \prod_{i=2}^{N-2} \Theta(z_{i+1} - z_i) <0, -p_N | V(p_{N-1}, z_{N-1}) \\ V(p_{N-2}, z_{N-2}) \dots V(p_2, z_2) | 0, p_1 \rangle \end{aligned} \quad (3.50)$$

We can evaluate this expression by standard techniques in oscillator algebra. We use the relation

$$e^A e^B = e^{[A,B]} e^B e^A. \quad (3.51)$$

valid if $[A,B]$ commutes with A and B and in our case

$$\begin{aligned} e^{k \cdot \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n z^{-n}} e^{-k' \cdot \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n^\dagger (z')^{-n}} \\ = e^{-k \cdot k' \ln(1 - z/z')} \end{aligned} \quad (3.52)$$

provided $z > z'$. As a result for the scattering of N tachyons we find the result

$$\begin{aligned} A(1, 2, \dots, N) = \int \prod_{i=2}^{N-2} dz_i \\ \cdot \prod_{1 \leq i < j \leq N} |z_i - z_j|^{-p_i \cdot p_j} \end{aligned} \quad (3.53)$$

This agrees with Eq. (2.27) if we choose the three preferred z 's to be z_1 , z_{N-1} and z_N and to have the values $z_1 = 0$, $z_{N-1} = 1$, $z_N = \infty$.

One can write the expression of Eq. (3.53) in the form

$$\begin{aligned} \lim_{z_N \rightarrow \infty} \lim_{z_1 \rightarrow 0} \int \prod_{i=2}^{N-2} dz_i \theta(z_{N-1} - z_i) < 0, 0 | V(p_N, z_N) \\ z_N V(p_{N-1}, z_{N-1}) \dots \dots V(p_2, z_2) \\ \cdot \frac{V(p_1, z_1) | 0, 0}{z_1} \end{aligned} \quad (3.54)$$

since

$$\langle 0, -k_N | = \lim_{z_N \rightarrow \infty} \langle 0, 0 | z_N V(p_N, z_N) \quad (3.55)$$

and

$$| 0, k_1 \rangle = \lim_{z_1 \rightarrow 0} \frac{1}{z_1} V(k_1, z_1) | 0, 0 \rangle \quad (3.56)$$

In this way, it is the vacuum expectation value of a number of V's and is recognizable as the same result obtained from the Polyakov approach.

One can now verify in this example of tachyon scattering, that if one replaces one of the states, say the end state, by the zero norm state of Eq. (3.13) or that of Eq. (3.15), then the result is zero. In particular, one must consider the expression

$$\langle s | V(p_{N-1}, z_{N-1}) V(p_{N-2}, z_{N-2}) \dots V(p_2, z_2) | 0, p_1 \rangle \quad (3.57)$$

Writing

$$\langle s | = \langle \Omega | (L_1 - L_0) = \langle \Omega | L_1 \quad (3.58)$$

and commuting the $L_1 - L_0$ factor to the left past the V's using the fact that they have conformal weight one, one verifies that it vanishes. The computation for the state of Eq. (3.15) is similar. In fact, in this case one can also show that any state of the form

$$| s \rangle = L_{-n} | \Omega \rangle \quad n \geq 1 \quad (3.59)$$

with

$$L_m | \Omega \rangle = 0 \quad m \geq 1 \text{ and } (L_0 + n - 1) | \Omega \rangle = 0 \quad (3.60)$$

decouples.

When one considers even tree graphs constructed out of V^{csv} and D, not all the legs of V^{csv} are external and so some legs are not sandwiched against physical states. In general, the decoupling property of a vertex does not ensure that the N point vertex constructed in this way will decouple zero norm physical states, since if one external leg is zero norm some of internal legs may not involve only the propagation of

physical states. This phenomenon occurs with the old fermion emission vertex of the superstring, but there was not such a problem in the bosonic case. By this method and by factorization, the tree-level scattering vertices for N arbitrary string states were computed^{45,46}.

Any operator for the emission of a physical state should have conformal weight one, for the appropriate mass-shell condition, since in this case the zero norm states will decouple. One can indeed verify that the vertex for photon emission of Eq. (3.42) has conformal weight one if $k^2 = 0$. In fact, in conformal field theory this provides the method for constructing emission vertices. However, V^{csv} contains the vertex operators for the emission of all possible physical states and one can ask whether zero norm states will in general decouple. This is related to the conformal properties of V^{csv} and other such vertices. To understand this behaviour was one of the motivations for the work discussed in the next sections.

One can also imagine constructing graphs corresponding to loops out of V^{csv} and $D = (L_0 - 1)^{-1}$ by sewing together sufficiently many legs. In this case we have V^{csv} vertices which have two legs internal to the graph and so are not sandwiched onto physical states. As a result, we will have spurious, i.e., non-physical states and zero norm states propagating round the loop. This will occur even if these states decouple in the sense previously described, since now two legs of the vertex can be non-physical while before only one of the legs was of this form. In the light of our knowledge about quantization in field theories, we should expect such a Lorentz-covariant, Feynman-graph-like procedure to have problems without the incorporation of ghosts. In the old days, it was shown how to insert physical state projectors at one loop³⁷ and produce the required result. At higher loops the situation was not so clear. Nevertheless, multi-loop amplitudes were computed⁵⁶ without any projections and the results are not so very different from the correct results found more recently.

It is instructive to compute the number of parameters that occur in such a multi-loop graph. We get for each propagator one variable z and therefore the number of such variables is

$$3M + E - 3 \quad (3.61)$$

where M is the number of loops and E the number of external lines. The result will be of the form

$$\int \prod_i dz_i \bar{f}(z_i) \mathcal{V} \quad (3.62)$$

where

$$\mathcal{V} = \langle 0 | \dots \langle 0 | g(\alpha_n^{(n)\mu}, z_i) \quad (3.63)$$

The above count of Eq. (3.61) agrees with that found in the Polyakov approach where it results from the number of inequivalent complex structures that exist on a Riemann surface of genus M . Expressions of the generic form of Eq. (3.63) will form the starting point for the new formulation of string theory discussed in the later sections.

It may be useful for the reader to summarize the above discussion in terms of the more modern language where one includes anticommuting coordinates as well as the x^μ 's. In the notation of Ref. 4 the physical states are defined by

$$Q|\chi\rangle = 0, \quad \beta_0|\chi\rangle = 0$$

Clearly if $|\chi\rangle$ is physical so is $|\chi'\rangle = |\chi\rangle + Q|\Lambda\rangle$ as $Q^2 = 0$. Further $|\chi\rangle$ and $|\chi'\rangle$ have the same scalar product with any physical state. We can use Q to set up an equivalence relation; we say $|\chi_1\rangle \sim |\chi_2\rangle$ if $|\chi_1\rangle = |\chi_2\rangle + Q|\Lambda\rangle$. States which are physical and are not Q of something we refer to as being in the non-trivial cohomology class of Q . It can be shown that these states (i.e., the equivalence classes) are none other than the physical states with positive definite norm discussed above. In other words if $|\chi\rangle$ is such a state, then $|\chi\rangle$ will satisfy

$$L_n |\chi\rangle = \alpha_n |\chi\rangle = 0 \quad n \geq 1, \quad (L_0 - 1) |\chi\rangle = 0$$

and

$$\beta_n |\chi\rangle = 0 = \bar{\beta}_n |\chi\rangle = \bar{\beta}_0 |\chi\rangle$$

Clearly, any physical state can be written as

$$|\chi\rangle + Q|\Lambda\rangle$$

where $|\chi\rangle$ is a positive norm and the state $Q|\Lambda\rangle$ has zero scalar product with all physical states including itself.

A vertex V is physical, in the sense of the decoupling of zero norm physical states, if

$$\int \prod_i dz_i \bar{f}(z_i) \cup \varphi^{(1)} | \lambda \rangle_1 | \chi \rangle_2 \dots | \chi \rangle_n = 0$$

where all the states are physical. This is true if and only if

$$\int \prod_i dz_i \bar{f}(z_i) \cup \sum_{i=1}^N \varphi^i = 0$$

The extension of V^{CSV} to include ghosts was given in Ref. 17 where it was also shown to satisfy the above equations. This vertex is the basis for a new dual model which does not have the drawbacks described above.

4. INTRODUCTION TO OVERLAPS

In the next section we will give a systematic method of deducing the perturbative string S matrix from three assumptions which specify how to encode into the theory duality, the existence of overlap conditions and unitarity. The purpose of this section is to familiarize the reader with overlap conditions which have previously had only limited use in string theory. Some of the discussion will be rather generic, but the reader can find the details in Ref. 17 and the review of Ref. 4.

In the light-cone gauge it has been known for a long time how to construct a field theory of strings³. This is a field theory built out of the functionals $x^i(\sigma)$, $i = 1, \dots, 24$ as well as the additional centre-of-mass co-ordinates $x^\pm = x^0 \pm x^{26}$. It was found that strings interacted in a very simple way by simply joining at their end points^{38,3}. The corresponding world-sheet of the string is given in Fig. 4.1

Fig. 4.1

We also observe that the point at which the interaction takes place is observer-dependent. As shown in Fig. 4.1, the observers of the two frames of reference consider the interaction to have taken place at different space-time points marked A and B. The above picture means that at the moment of the interaction, the incoming string can be identified, in the appropriate regions, with the two outgoing strings. By time reversal we can, of course, have two strings becoming one by joining at their end points. The most natural parametrization for such a process is to assign the strings a length. For the r^{th} string of length $\alpha_r \pi$, we have

$$0 \leq \sigma_r \leq \alpha_r \pi$$

The lengths in the interaction will be preserved, i.e.,

$$\sum_{r=1}^3 \alpha_r = 0$$

In fact, in the light-cone gauge, the string lengths can be chosen to be p^+ and are therefore automatically conserved.

Interacting gauge covariant string theories have now been constructed. There are two such known theories depending according to how the strings join. A formulation in which the strings join at their end points was independently and virtually simultaneously found in Refs. 17 and 18. A different formulation in which the strings join at their mid points was given in Ref. 16. Both these formulations have as co-ordinates not only $x^\mu(\sigma)$, but also anticommuting co-ordinates $c(\sigma)$ and $\bar{c}(\sigma)$.

The string interaction by touching at their end points, can be made mathematically precise as follows. We consider the case of two strings joining to form a third and parametrize this process as a strip (Fig. 4.2)

Fig. 4.2

Rather than use σ_r , we use $\eta_r = \sigma_r \pi / |\alpha_r|$. The interaction point is given by $\eta_3 = \beta_2 \pi$ since at the top of string two $\alpha_3 \eta_3 = \alpha_2 \pi$ where $\beta_2 = \alpha_2 / \alpha_1 + \alpha_2$ and $\beta_1 = \alpha_1 / \alpha_1 + \alpha_2$. That is, the η 's on the three strings are related by

$$\eta_3 = \begin{cases} -\beta_2 \eta_2 + \beta_2 \pi & \text{for } 0 \leq \eta_3 \leq \beta_2 \pi \\ -\beta_1 \eta_1 + \pi & \text{for } \beta_2 \pi \leq \eta_3 \leq \pi \end{cases} \quad (4.1)$$

The interaction in co-ordinate space is given by

$$U(x) = V|x\rangle = \delta(x_3^\mu(\eta_3) - \theta(-\eta_3 + \beta_2 \pi) x_2^\mu(\eta_2) - \theta(\eta_3 - \beta_2 \pi) x_1^\mu(\eta_1)) \quad (4.2)$$

For the second-quantized light-cone theory, $\mu = 1$ to 24, while for gauge-covariant string theory $\mu = 0$ to 25. In the latter case, this is only part of the vertex, the remainder being constructed from c and $\bar{c}$. At the moment they touch, the δ function implies that they overlap, that is,

$$x_3^{\mu}(\eta_3) = \begin{cases} x_1^{\mu(1)}(\eta_1) & \beta_2 \pi \leq \eta_3 \leq \pi \\ x_2^{\mu(2)}(\eta_2) & 0 \leq \eta_3 \leq \beta_2 \pi \end{cases} \quad (4.3)$$

the cubic interaction term in the light-cone gauge field theory being given by

$$A^{\text{Int}} = \frac{g}{3} \int dx^+ dx^- U(x) \mathcal{X}(x^{i(1)}(\eta_1)) \cdot \mathcal{X}(x^{i(2)}(\eta_2)) \mathcal{X}(x^{i(3)}(\eta_3)).$$

The gauge covariant interaction term is of the same form but V now involves anticommuting co-ordinates. Since the interaction is given by a δ -function, we can expect it to obey identities³⁹ such as

$$U(x^{3\mu}(\eta_3) - x^{\mu(1)}(\eta_1)) = 0 \quad (4.4)$$

for $\beta_2 \pi \leq \eta_3 \leq \pi$. This is none other than the identity $x \delta(x) = 0$. We also have the identity

$$U \left\{ \frac{1}{\alpha_3} \overleftarrow{\frac{\partial}{\partial x^{\mu(3)}(\eta_3)}} - \frac{1}{\alpha_2} \overleftarrow{\frac{\partial}{\partial x^{\mu(1)}(\eta_1)}} \right\} = 0 \quad (4.5)$$

for $\beta_2\pi < \eta_3 < \pi$. For both cases we also have similar relations for $x^{(3)\mu}$ and $x^{(2)\mu}$.

Consequently, it follows that

$$U \left\{ \frac{1}{\alpha_3} \mathcal{O}^{\mu(3)}(\eta_3) - \frac{1}{\alpha_1} \mathcal{O}^{\mu(1)}(\eta_1) \right\} \quad (4.6)$$

where

$$\begin{aligned} \mathcal{O}^{\mu(n)}(\eta_r) &= -i\hbar \frac{\partial}{\partial x^{\mu(n)}} - \frac{1}{2\pi\alpha'} x^{\mu(n)}(\eta_r) \\ &\equiv \frac{1}{\sqrt{2\alpha'}} \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \alpha_n^{\mu(n)} z_r^{-n} \end{aligned} \quad (4.7)$$

and

$$z_r = e^{-i\eta_r} \quad (4.8)$$

This relation, however, will be of more use to us than Eqs. (4.4) and (4.5), as P^μ is a conformal operator of weight one. Integrating Eq. (4.6) with respect to η_3 and finding the constant of integration, which is the centre-of-mass position, we also have

$$U \{ \mathcal{Q}^{\mu(3)}(z_3) - \mathcal{Q}^{\mu(1)}(z_1) \} = 0 \quad (4.9)$$

We have used the fact that

$$\mathcal{Q}^\mu(z) = z \frac{d}{dz} \mathcal{Q}^\mu(z) = i \frac{d}{d\eta} \mathcal{Q}^\mu(\eta) \quad (4.10)$$

There also exists a relation¹⁷ between the

$$L(z) \equiv \sum_{n=-\infty}^{\infty} L_n z^n \quad (4.11)$$

on different legs.

In the case of the gauge covariant theory, one has also similar types of identities for quantities related to the anticommuting coordinates. For a further discussion of these as well as the above identities, the reader is referred to Ref. 17 and the review of Ref. 4.

The reader may be wondering what is the relation between the δ -function interaction of Eq. (4.2) and the known answer for three-string scattering, i.e., V^{csv} discussed in the previous section. The δ function overlap may be put into the oscillator basis by the standard procedure for a change of basis. In fact, this computation originally performed in Refs. 38 and 40 for the light-cone case is best carried by first going to a momentum basis, then a coherent state basis and finally recognizing this expression in terms of oscillators.

The result is of the form⁴¹

$$V^{cov} = \langle 0|_1 \langle 0|_2 \langle 0|_3 \exp \sum_{r,s} \sum_{n,m=0}^{\infty} \left\{ \alpha_n^{\mu(r)} N_{nm}^{rs} \alpha_m^{\mu(s)} \right\} \quad (4.12)$$

where N_{nm}^{rs} , $n, m = 0, 1, 2, \dots$ and $r, s = 1, 2, 3$ are rather complicated functions of the string lengths. In the light-cone gauge μ is summed over only 1 to 24. In the gauge covariant theory, one also has anticommuting oscillators. In the formulation of Refs. 17 and 18, the vertex is given by Eq. (4.12) with now $\mu = 0, 1, \dots, 25$ and an additional ghost piece which is a exponential of an expression bilinear in ghost oscillators and also involves the N_{nm}^{rs} coefficients. Carrying out manipulations in functional form is very non-rigorous as the expressions in this formalism are not well defined; indeed it is not difficult to arrive at incorrect results. This is not true in the oscillator formalism where the identities of Eqs. (4.6) and (4.9) can be verified explicitly.

Even though the vertex is in oscillator form, a glance at the N_{nm}^{rs} coefficients gives little encouragement that on-shell this vertex yields the same result as V^{csv} . Since physical states satisfy the Virasoro conditions $L_n |\chi\rangle = 0$, $n > 1$, and $(L_0 - 1) |\chi\rangle = 0$, the two vertices will lead to the same on-shell scattering if they are related by

$$V^{csv} = V^{cov} \prod_{r=1}^3 g_r. \quad (4.13)$$

where

$$g_v = \exp \left\{ \sum_{n=0}^{\infty} a_n^{(v)} (Q_n^{(v)} - \delta_{n0}) \right\} \quad (4.14)$$

In fact, this is the case and it was explicitly shown in Ref. 17 that the conformal mapping, g_r , is none other than the mapping between the strip and the upper half plane. This mapping is a Schwartz-Christoffel mapping which was first used in this context by Mandelstam⁴¹. It is reviewed in the same notation as that used here in Ref. 4.

Given that V^{cov} and V^{vcs} are related by a conformal transformation, it is now straightforward to see that V^{csv} will also satisfy an overlap equation. Applying $g = \prod_{r=1}^3 g^r$ to Eq. (4.9) say we find that

$$V^{\text{cov}} g g^{-1} \left\{ \varphi^{\mu(3)}(z_3) - \varphi^{\mu(1)}(z_1) \right\} g = 0 \quad (4.15)$$

As $Q^\mu(z)$ has conformal weight zero, its argument only changes as given by Eq. (3.21) and we may recognize Eq. (4.15) in the form⁴²

$$V^{\text{csv}} \left\{ \varphi^{\mu(3)}(z) - \varphi^{\mu(1)}(\hat{T}z) \right\} = 0 \quad (4.16)$$

Computation shows that

$$\hat{T}(z) = \frac{1}{1-z} \quad (4.17)$$

as the map which takes $z \rightarrow z_3$ also takes

$$\hat{T}(z) = \frac{1}{1-z} \rightarrow z_2 \quad \text{and} \quad \hat{T}^2(z) = \frac{z-1}{z} \rightarrow z_1 \quad (4.18)$$

In fact, it is straightforward to verify that Eq. (4.18) holds; one simply pushes the creation operators in $Q^{\mu(3)}(z)$ past the exponential in V^{csv} given in Eq. (3.27) and onto the vacuum where they annihilate. What is brought down from the exponential involves α_m^2 and α_m^1 for $m \geq 1$. The latter we recognize as being part of $Q^{\mu(1)}(\hat{T}(z))$ while the former and remaining parts of $Q^{\mu(3)}(z)$ are cancelled against the similar operation of pushing the creation operators in $Q^{\mu}(\hat{T}(z))$ onto the vacuum. That V^{csv}

and V^{cov} are related by Eq. (4.13) was shown¹⁷ not by explicit computation, but by verifying that the two overlaps did indeed hold and went over into each other.

In the open bosonic string for the light cone string field theory^{3,41,38} and in the gauge covariant theory of Refs. 17 and 18, one has a four-point string interaction which is also an overlap δ function. One could apply the same arguments to find overlap identities for this vertex. Higher point vertices are computed using the three- and four-point vertices joined together with propagators, according to the usual Feynman rule combinatoric instruction. However, an object of a given conformal weight, say Q^μ or P^μ can be moved in a well-defined way through the propagator in parametric form. Consequently, starting with a conformal operator on a given leg we may move it through the vertex, then through the propagator, etc., until it emerges out on another leg. As such, one can see that there will exist overlap identities for N point vertices. The same conclusion can be drawn from thinking about the dual model construction of vertices from V^{CSV} and $D = (L_0 - 1)^{-1}$, since both of these building blocks obey overlaps.

The reader should therefore come to expect the existence of overlap identities for vertices in string theory. As we shall see, these overlap identities, which in fact uniquely determine the corresponding vertex, are much simpler than the explicit expression for the vertex.

Given the gauge covariant vertex, one can imagine the conformal mapping which cycles the three sections of the strip of Fig. 4.1 into each other. One may expect that the vertex will remain the same; in practice this means that there exist conformal maps $e^{(r)}$ such that

$$V^{\text{cov}} \prod_{r=1}^3 e_r = V^{\text{cov}} \quad (4.19)$$

Using Eq. (4.13) we then have

$$V^{\text{CSV}} \prod_{r=1}^3 S^{(r)} = V^{\text{CSV}} \quad (4.20)$$

where $S^r = g^{r-1} e^r g^r$. As one might expect, S^r is none other than $\hat{T}^r$. One could prove this explicitly, but since the overlap identity of Eq. (4.16) determines V^{csv} uniquely, one can use the overlap identity itself. Equation (4.16) can be written as

$$0 = V^{csv} \hat{T}^{-1} \hat{T} \left\{ \phi^{(3)}(z) - \phi^{(1)}(\hat{T}(z)) \right\} \hat{T}^{-1} \quad (4.21)$$

Using the fact that Q^μ has conformal weight zero we then have

$$0 = V^{csv} \hat{T}^{-1} \left\{ \phi^{(3)}(\hat{T} z) - \phi^{(1)}(\hat{T}^2(z)) \right\} \quad (4.22)$$

which implies, after $z \rightarrow \hat{T}^{-1} z$, that

$$\left\{ V^{csv} \hat{T}^{-1} \right\} \left\{ \phi^{(3)}(z) - \phi^{(1)}(\hat{T}(z)) \right\} = 0 \quad (4.23)$$

Hence $V^{csv} \hat{T}^{-1}$ has the same overlap as V^{csv} and so we have shown that

$$V^{csv} \hat{T}^{-1} = V^{csv} \quad (4.24)$$

This argument also demonstrates that the transformations T that occur in the overlaps and those which leave the vertex inert will always be the same. Although the three point-function is very special, as we shall see, most of the above discussion carries over to the case of any vertex, the analogous discussion of the three-string Neveu-Schwarz vertex being given in Ref. 43.

5. GROUP THEORETIC APPROACH TO STRING THEORY

5.1 Introduction

It does not seem likely that Einstein could have discovered general relativity or Yang-Mills discovered Yang-Mills theory by guessing part of the S matrix for particle scattering. That string theory began with just such a guess is a reflection of the simplicity of the string S matrix. It is clear from the historical approach discussed in the earlier sections that the remarkable duality properties of string theory are partly responsible for this simplicity. In the course of constructing tree-level bosonic and superstring scattering vertices, we also

realized^{42,43} that although the vertices were complicated, they satisfied the overlap properties discussed in generic form in the previous section.

Not only were these overlap properties very simple, being determined by only particular $SL(2, R)$ transformations, they uniquely determined the vertex. The approach¹¹⁻¹³ we are about to consider assumes the existence of duality and overlap properties as well as a certain aspect of unitarity, namely the decoupling of physical zero norm states. From this we are able to deduce the entire perturbative S matrix whose computation then becomes an exercise in conformal group theory.

The principal character in this approach is the vertex which depends on the moduli, the actual scattering amplitude being obtained by integrating over the moduli with a suitable measure. By moduli we mean the external moduli, i.e., Koba-Nielsen co-ordinates and the parameters associated with loops. As we shall see these arise in connection with the duality properties of the vertex. In a Feynman-graph type of approach, which the method discussed here is not, the latter parameters arise from the usual parametric form of the propagator used in string theory, while in the sum over Riemann surface approach², they arise as the Tiechmuller parameters.

The duality property well known in string theory states that one can permute the legs of an amplitude and the result is the same provided that one maintains their cyclic order at least for the case of the open string. In the approach advocated here, the vertex can have its dependence on the external moduli cycled by the application of appropriate conformal transformations on all its legs. There also exists, for each loop, a cyclic transformation which leaves the vertex inert and corresponds to going around the given loop. In fact, the moduli of the vertex are really the parameters of the cycling transformations which characterize the vertex. A general conformal transformation when applied to the vertex will change the values of the moduli on which the vertex depends. Indeed, the behaviour under the cycling transformations allows one to compute the specific conformal transformation which changes the moduli in a given way. Such a change can only be achieved by a conformal transformation that contains constant terms or poles. This is analogous to the procedure of boosts in the theory of induced representations. the rôle of the little group being played by the transformations which leave the moduli alone.

A convenient representation of the vertex is in oscillator basis. In this case, the conformal transformation is implemented by the L_n 's, and one that changes the moduli is a transformation which involves L_n , $n < 0$ as well as L_n , $n > 0$.

It is the knowledge of this moduli-changing transformation and the enforcement of unitarity in the guise of the decoupling of zero norm physical states that allow for a deduction of the measure with which the vertex is integrated to yield the actual scattering amplitude. For physical states, L_n , $n > 1$ vanish and $L_0 = 1$. Consequently, using an infinitesimal moduli-changing transformation one finds, after integration by parts, an equation relating the derivative of the measure with respect to the moduli and a known function of the moduli.

5.2 General strategy

The central object in our consideration is a vertex V which depends on parameters z_i , $i = 1, \dots, N$ corresponding to the N external strings and sets of parameters v_r , $r = 1, \dots, M$ corresponding to the M loops. The number of parameters depends on the type of strings being considered. For the open and closed bosonic strings, for example, the z_i are real and complex, respectively, and each v_r denotes three real and three complex parameters respectively for each loop.

The most useful representation of the vertex for current purposes is in the oscillator basis, namely it will be of the generic form

$$U(z_i, v_r) = \langle 0 | \dots \langle 0 | h(\alpha_m^{u(m)}, z_i; v_r). \quad (5.1)$$

The on-shell scattering amplitude is obtained by integrating the above vertex over the parameters, i.e.,

$$W = \int \prod_i dz_i \prod_r dv_r \bar{f}(z_i, v_r) V(z_i, v_r). \quad (5.2)$$

where the method also determines the function $\bar{f}$.

The on-shell scattering amplitude follows from the assumptions:

A1) There exist cycling transformations associated with cycling the external legs and going around each loop. The cycling of the external legs is achieved by

$$V(z_1, \dots, z_N, \sigma_v) \prod_{k=1}^N (T_{k+1}^{(k)})^{-1} = V(z_2, \dots, z_N, z_1, \sigma_v) \quad (5.3)$$

The T 's obey the equation

$$T_N \dots T_2 T_1 = 1 \quad (5.4)$$

and T_j is obtained from T_{j-1} by cycling its dependence on the z 's.

For each loop, the vertex satisfies a relation of the form

$$V(z_j, \sigma_v) P_n^{(i)} = V(z_j, \sigma_v) \quad n=1, \dots, M \quad (5.5)$$

The $P_n^{(i)}$ differ, for a given n , from line to line and this is labelled by the upper index. As we shall see, the v_n are in fact the parameters of the conformal transformation P_n^i .

A2) The vertex obeys overlap identities. The one corresponding to cycling the external legs has the generic form

$$V(z_i, \sigma_v) \left\{ \sum_{i=1}^N A_i(z) R^{(i)}(T_i T_{i-1} \dots T_1(z)) \right\} = 0 \quad (5.6)$$

subject to one condition on the A_i 's which is of the form

$$\sum_{i=1}^N \frac{A_i(z)}{f_i(z)} = 0 \quad (5.7)$$

The f_i are determined once the conformal weight of R is known.

For each loop we have an overlap identity of the form

$$V(z_i, \sigma_v) \left\{ \sum_m B_m^{(i)} ((P_n^i)^m(z)) \right\} = 0 \quad (5.8)$$

where the B_m are also subject to one condition

$$\sum_m \frac{B_m^{(i)}(z)}{g_m^{(i)}(z)} = 0 \quad (5.9)$$

A3) The theory is unitary, which in this case means that zero norm physical states decouple.

Following the same arguments as in I, one finds that

$$f_i(z) = S_{T_i \dots T_3}(T_2 T_1 z), \quad f_2 = 1, \quad f_1 = \frac{1}{S(T_1 z)} \quad (5.10)$$

and

$$g_m^{(i)}(z) = S_{(p_n^i)^m}(z) \quad (5.11)$$

where

$$S_g(z) = \left[\frac{z}{g(z)} \frac{d}{dz} g(z) \right]^d \quad (5.12)$$

Care must be taken to include only so many non-zero A's or B's that lead to convergent expressions in z for some region of z . In practice this means that only two of them are non-zero. For the conformal operator

$$Q^\mu(z) = - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^\mu z^{-n} + \alpha_0^\mu \ln z + \sum \frac{1}{2\alpha_0^\mu} \quad (5.13)$$

which has conformal dimension zero, the above overlaps are

$$\bigvee \left\{ Q^{(i)\mu}(z) - Q^{(j)\mu}(T_j \dots T_{i+1} z) \right\} = 0 \quad (5.14) \\ 1 \leq i < j \leq N$$

and

$$\bigvee \left\{ Q^{(k)\mu}(z) - Q^{(k)\mu}(p_n^k(z)) \right\} = 0 \quad (5.15)$$

The corresponding identities for $P^\mu(z) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n}$ are easily found by computing the above S's for $d = 1$. This result is in agreement with differentiating the above identities with respect to z d/dz . The identities for $Q^\mu(z)$ and $P^\mu(z)$, unlike for $L(z) = \sum_n L_n z^{-n}$, do not involve normal ordering and so no constants occur in the identities. However, $L(z)$ is not, due to the central charge, exactly an object of weight 2 and we must add in constants whose values will be found below. As for the

tree case, one can find integrated identities. These have a particularly elegant form and will be discussed later in this section.

Let us now consider the form of the cycling transformations in more detail. In fact, the loop-cycling transformations are simplest to discuss since they are just arbitrary members of $SL(2, R)$ for the open bosonic string. These may be parametrized by three real parameters for each loop which can be chosen to be their two fixed points α_n^i and β_n^i and their multiplier ω_n^i .

The external state cycling transformations, T 's, are also elements of $SL(2, R)$ for the open bosonic string and we will take them to be given by

$$T_j = \frac{z-1}{z} \frac{1}{(1-u_{j-1,j})} \quad (5.16)$$

where $u_{j-1,j}$ is the Chan variable between lines $j-1$ and j computed by considering not only the external points z_j but also all their images under the action of the loop-cycling transformations. Since the T 's depend only on the cross-ratios of the z 's, the resulting expression will be $SL(2, R)$ invariant.

It is instructive to rewrite Eq. (5.16) in the form

$$T_j = (V^j)^{-1} (V^{j-1}) \quad (5.17)$$

where

$$V^j = \begin{pmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{pmatrix} \quad (5.18)$$

One then realizes that

$$T_j \dots T_{i+1} = (V^j)^{-1} V^i \quad (5.19)$$

and in particular if we define

$$\bar{z}_j = T_j \dots T_2 (\bar{z}_1) \quad (5.20)$$

then

$$\bar{z}_j = (V^j)^{-1} V^i (z_i) \quad \forall i, j$$

We were guided to the above cycling transformations by examining the Lovelace⁴⁵-Olive⁴⁶ vertex for open bosonic string scattering.

In fact the P's and T's are not all independent. Applying P_n^i to V and using Eqs. (5.3) and (5.5) or equivalently using the Q^μ overlaps of Eqs. (5.14) and (5.15), we find that

$$P_n^{i+1} = T_{i+1} P_n^i (T_{i+1})^{-1}$$

Iterating this equation gives

$$P_n^j = (T_j \dots T_{i+1}) P_n^i (T_j \dots T_{i+1})^{-1} \quad (5.21)$$

Consequently, given P_n^i , i.e., the n^{th} loop-cycling transformation on leg i, we can find it on all legs once the T's are specified. The most convenient way to express this fact is to take $P_n^i = (V^i)^{-1} P_n^{-1} (V^i)$ where P_n is arbitrary and then we find

$$P_n^j = (V^j)^{-1} P_n^{-1} V^j \quad \forall j \quad (5.22)$$

The parameters required to label P_n can be taken to be the v_n . Written in this way we can see the interpretation of P_n . Let us write $P_n V^j = V^{j+N+1}$ in the sense that the points z_j are moved by the transformation P_n to their images z_{j+N+1} . As a result, the formula for P_n^j is of the same form as Eq. (5.19) except that it is between the original points and their immediate images. The resulting transformation P_n^j is the same even if we compute it by considering image points and points related to these by P_n , as the transformations T_j only involve cross ratios.

Let us now count the number of parameters that occur in V. There are N real z_i 's and three M real parameters in the P_n 's. However, the overall result is $SL(2, R)$ invariant and so we may make three choices. An $SL(2, R)$ transformation, however, induces a similarity transformation on P_n (i.e., $P_n \rightarrow S P_n S^{-1}$), which transforms the fixed points $\alpha_n \rightarrow S \alpha_n$, $\beta_n \rightarrow S \beta_n$, but leaves the multiplier ω_n inert. It is customary to choose $z_1 =$

1 and to choose the fixed points of one of the loops, say 0 and ∞ . Consequently, we find that V depends on $N + 3M - 3$ real parameters. The case $N = 0, M = 1$ is special in the sense that we have two fixed points which we may choose and one multiplier (one parameter) which cannot be chosen. The fact that the number changes by three if we change M by one is in accord with the trivial observation that changing M means adding three propagators in the dual model graph and so in its parametric form adding three extra parameters.

The above discussion generalizes to the closed bosonic string and superstring by replacing $SL(2, R)$ by $SL(2, C)$ for the closed bosonic string and by prefacing these groups with the word graded for the superstring. For the closed string, one finds $N + 3M - 3$ complex parameters in V . The case $N = 0, M = 1$ is again special and one can only choose both the fixed points but not the multiplier giving in all one complex parameter. This of course agrees with the propagator count and also the number of inequivalent complex structures on a Riemann surface.

The behaviour of the vertex under the $SL(2, R)$ transformations given above allows us also to find the effects for general conformal transformations which change the parameters z_r and v_r . This is reminiscent of the method of induced representations. Let us consider a general change in all the parameters which is implemented by a conformal transformation

$$V(\hat{z}_i, \hat{v}_r) = V(z_i, v_r) \prod_{i=1}^N \mathcal{H}^i \quad (5.23)$$

The vertex $V(z, v_r)$ must satisfy the appropriate cycling equations for $\hat{T} = \hat{T}(\hat{z}, \hat{v}_r)$ and $\hat{P} = P(\hat{z}, \hat{v}_r)$ in Eqs. (5.3) and (5.5). Consequently we find that

$$(\mathcal{T}_{i+1})^{-1} \mathcal{H}^{i+1} \hat{T}_{i+1} = \mathcal{H}^i \quad \forall i \quad (5.24)$$

and

$$(\mathcal{P}_n^i)^{-1} \mathcal{H}^i \hat{P}_n^i = \mathcal{H}^i \quad \forall i, n \quad (5.25)$$

If we take $\mathcal{H}^i$ to be an infinitesimal conformal transformation, i.e.,

$$\mathcal{H}^i(z) = z + \epsilon f(z). \quad (5.26)$$

then the above equations become

$$p_n^i(z) + \varepsilon f_i(z) \frac{dp_n^i}{dz} = \hat{p}_n^i(z) + \varepsilon f_i(p_n^i z) \quad (5.27)$$

and

$$T_{i+1}(z) + \varepsilon f_i(z) \frac{dT_{i+1}(z)}{dz} = \hat{T}_{i+1}(z) + \varepsilon f_{i+1}(T_{i+1} z) \quad (5.28)$$

at lowest order. It is clear from the above equation and in accord with previous arguments that given $f_k(z)$ on the k^{th} leg which must satisfy (5.27), then all the other f_i 's are determined by (5.28). The remaining equations (5.27) for $i \neq k$ are automatically satisfied. For the infinitesimal case, Eq. (5.23) becomes

$$\left(\sum_{i=1}^N \delta z_i \frac{\partial}{\partial z_i} + \sum_{r=1}^M \delta v_r \frac{\partial}{\partial v_r} \right) V(z_i, v_r) =$$

$$V(z_i, v_r) \sum_{k=1}^N \left\{ \oint \frac{dz}{z} \in f_k(z) \frac{L^{(k)}(z)}{z} + \varepsilon C^{(k)} \right\} \quad (5.29)$$

There also exist conformal transformations which do not change the moduli. These are given in Eqs. (5.27) and (5.28) by setting $T = \hat{T}$ and $P = \hat{P}$.

The above equations are used to compute the measure that occurs in Eq. (5.2). This comes about because the $f_i(z)$ which shifts z_i or v_r has a pole or constant on one leg and is an analytic function which vanishes at $z = 0$ on all the other legs. Translated in L_n 's this means on one leg that we have $a_{-2}L_{-2} + a_{-1}L_{-1} + \sum_{n=0}^{\infty} a_n L_n$, while on the other legs we have only L_n $n > 0$. Unitarity, however, tells us that one spurious state acting on the physical vertex W does not couple on-shell if all states are physical and hence the terms $a_{-2}L_{-2} + a_{-1}L_{-1}$ should vanish. This relates a derivative of the vertex, and by integration by parts the measure, with respect to the parameters (i.e., moduli) to a known function of the parameters.

To be concrete, let us consider two examples, the first being the three-string scattering vertex $V^{(3)}$. In this case $z_{j+3} = z_j$ for all j and we may choose $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ by an $SL(2, R)$ choice and one

finds the transformations T_r are the same and are given by

$$T_r \equiv T = \frac{z-1}{z} \quad (5.30)$$

It is easily seen that $T^3 = 1$ and the duality condition on the vertex reads

$$V^{(3)}[T^{(1)} T^{(2)} T^{(3)}]^{-1} = V^{(3)} \quad (5.31)$$

While the Q^μ overlap reads

$$V^{(3)} \left\{ \varphi^{\mu(3)}(z) - \varphi^{\mu(1)}\left(\frac{z-1}{z}\right) \right\} = 0 \quad (5.32)$$

The reader will observe that $V^{(3)}$ does not obey the same overlap or duality condition as V^{csv} . In fact, V^{csv} obeys the equivalent relations, given in the previous section, in which the cycling transformation T above is replaced by $T^{-1} = \hat{T} = T^2$. This change can be achieved by taking the above identities and swapping the legs 1 and 2 say, and consequently

$$V^{(3)}(1, 2, 3) = V^{csv}(2, 1, 3) \quad (5.33)$$

We now demonstrate that these two vertices give the same result on-shell as they are related to each other by a twist. Consequently, it is irrelevant which vertex one considers. The vertex $V^{(3)} \left[\prod_{r=1}^3 \mathcal{X}^{(r)} \right]^{-1}$, will obey the overlap

$$0 = V^{(3)} \left[\prod_{r=1}^3 \mathcal{X}^{(r)} \right]^{-1} \left\{ \varphi^{(3)\mu}(z) - \varphi^{\mu(1)}(\mathcal{X} T \mathcal{X}^{-1} z) \right\}$$

which coincides with that for V^{csv} provided

$$\mathcal{X} T \mathcal{X}^{-1} = \hat{T} = T^{-1} = T^2$$

By explicitly computing the conformal transformations, one finds that this equation holds if

$$\mathcal{X}(z) = \frac{-z}{1-z}$$

or that

$$\mathcal{O} = (-1)^{L_0-1} e^{L_1} = e^{-L_1} (-1)^{L_0-1}$$

when written in terms of L_n 's. This we recognize as essentially the twist Ω of Eq. (3.36). Since the overlaps uniquely determine the vertex, we may conclude that

$$V^{(3)} \left\{ \prod_{r=1}^3 e^{-L_1^{(r)}} (-1)^{L_0^{(r)}-1} \right\} = V^{CSV}$$

On on-shell states the twist Ω gives 1. We could have found the same result by directly examining Eq. (5.31) and the corresponding equation for V^{CSV} .

To verify that V^3 (or V^{CSV}) is a physical vertex, we must show that if all three states are physical and only one state is zero norm, the result vanishes. To demonstrate this, one must consider the appropriate identity found by taking $R(z)$ to be $L(z)$. This is discussed in Ref. 42 and a more detailed account is given in the review of Ref. 44).

As another example we consider the case of the one-loop tadpole graph V . We may use the $SL(2, R)$ invariance to choose both of the fixed points and the one Koba-Nielson variable associated with the only leg to be 0, ∞ and 1 respectively. Then the loop-cycling transformation is given by $P(z) = \omega z$ where, since the graph is planar, $0 < \omega < 1$. The one "charge" at 1 and its images are given in Fig. 5.1.

Fig. 5.1

Using Eq. (5.16) one then finds that

$$P^{(1)}(z) = \frac{(z-1)(1+\omega^2)}{z\omega} \quad (5.34)$$

where, since $u_{j-1,j}$ is a cross-ratio, it is the same no matter what two adjacent changes we choose. The vertex is then labelled by ω and satisfies

$$V(\omega) P^{(1)}(\omega) = V(\omega). \quad (5.35)$$

and

$$0 = V(\omega) \{ \varphi^{(1)}(\omega) - \varphi^{(1)}(P'(\omega)) \} \quad (5.36)$$

5.3 Applications

For the detailed computation of both tree and loop contributions to the open bosonic string S matrix, the reader should consult Refs. 11-13 and here we only repeat some illustrative examples.

The open bosonic trees vertex

Let us now derive the N string vertex from its overlap properties. The Q^μ overlap equations are of the form

$$V^{(N)} \{ \varphi^{(\mu)}(\xi_i) - \varphi^{(\mu)}(\xi_j) \} = 0 \quad \forall i, j \quad (5.37)$$

where

$$\xi_j = ((V^j)^{-1} V^i) (\xi_i) \quad (5.38)$$

By differentiating with respect to ξ_i $d/d\xi_i$ or by following the general theory of the overlaps given above we have the equation

$$0 = V^{(N)} \left\{ p^{(\mu)}(\xi_i) - \left[\frac{\xi_i}{\xi_j} \frac{d\xi_j}{d\xi_i} \right] p^{(\mu)}(\xi_j) \right\} \quad (5.39)$$

We now consider the quantity

$$V^{(N)} \oint_{\xi_i=0} \frac{d\xi_i}{\xi_i} p^{(\mu)}(\xi_i) \phi(\xi_i). \quad (5.40)$$

where $\phi(\xi_i)$ is a function which is analytic everywhere except possibly at the points $\xi_j = 0$, including $j = i$ where it can have poles. We may now

deform the contour of the above expression from $|\xi_i| \ll 1$ to go around the other singularities which can only be at $\xi_j = 0 \quad j \neq i$. As one moves away from $\xi_i = 0$, one must use the overlap of Eq. (5.39) to keep expressions which are still convergent. In this process, we may change the variable of integration from ξ_i to the point around which the contour now goes. This change is exactly compensated for by the additional factor in Eq. (5.39). It is clear that the result of all this is namely

$$0 = V^{(N)} \left\{ \sum_{j=1}^N \oint_{\xi_j=0} d\frac{\xi_j}{\xi_j} P^{(N)}(\xi_j) \varphi(\xi_j) \right\} \quad (5.41)$$

The convergence discussion is, however, rather subtle.

From Eq. (5.41) it is trivial to determine most of V^N . Let us take

$$\begin{aligned} \varphi &= \frac{1}{(\xi_i)^n} = \frac{1}{[(V_i)^{-1}(V^k) \cdot (\xi_k)]^n} = [\Gamma(V_i)^{-1}(V^k)](\xi_k) \quad (5.42) \\ &\equiv \sqrt{n} \sum_{m=1}^{\infty} \frac{[\Gamma(V_i)^{-1}(V^k)]_{nm} (\xi_k)^m}{\sqrt{m!}} + \sqrt{n} [\Gamma(V_i)^{-1}(V^k)(0)]^{-n} \quad (5.43) \end{aligned}$$

where the infinite matrix $[\Gamma(V_i)^{-1}V^k]_{nm}$ is defined by the last line, Eq. (5.41) now becomes

$$\begin{aligned} V^{(N)} \left\{ \alpha_{-n}^{(i)} + \sqrt{n} \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{m=0}^{\infty} \frac{\alpha_m^{(k)}}{\sqrt{m!}} [\Gamma(V_i)^{-1}(V^k)]_{nm} \right. \\ \left. + [\Gamma(V_i)^{-1}(V^k)(0)]^{-n} \alpha_0^{(k)} \right\} \Big\} = 0 \quad (5.44) \end{aligned}$$

and so

$$\begin{aligned} V^{(N)} &= \langle 0 | \dots \langle 0 | \exp - \left\{ \sum_{1 \leq i < k \leq N} \right. \\ &\quad \left. \left\{ \sum_{n,m=1}^{\infty} \frac{\alpha_n^{(i)}}{\sqrt{n!}} (\Gamma(V_i)^{-1}(V^k))_{nm} \frac{\alpha_m^{(k)}}{\sqrt{m!}} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha_n^{u(i)}}{\sqrt{n}} \frac{[(V^i)^{-1} V^k(0)]^{-n}}{\sqrt{n}} \alpha_0^{u(k)} \\
 & + \alpha_0^{u(i)} \frac{[(V^k)^{-1} V^i(0)]^{-n}}{\sqrt{n}} \frac{\alpha_n^{u(k)}}{\sqrt{n}} \} \\
 & + \frac{1}{2} \sum_{i,j=1}^N \alpha_0^{(i)u} \alpha_0^{(j)u} \mathcal{O}^{ij} \}
 \end{aligned} \tag{5.45}$$

The term bilinear in momentum can be determined from the Q^μ overlap of Eq. (5.37). In Ref. 11, this is done and it also shows explicitly that Eq. (5.37) holds. Equation (5.45) for V^N is in agreement with the old result for the N string vertex of Lovelace⁴⁵ and Olive⁴⁶. The actual scattering is found by an integration over the Koba-Nielsen variables with a particular weight function $\bar{f}$. This function is also deduced from the above axioms in Ref. 11.

The one-loop measure

The vertex and measure for a general one-loop planar graph was found in Ref. 13 and here we find the measure for the special case of the tadpole graph whose overlap and cycling transformations we found in Eqs. (5.35) and (5.36).

While it would be correct to start from these equations, it is more convenient, for comparison with past literature, to carry out a gauge transformation of the form $G = \alpha^{L_0 - L_1 - 1}$ on $V(w)$, and so modify the cycling transformation. Since this gauge transformation leaves physical states inert, we are not affecting the final result. Using the relation

$$G R(z) G^{-1} = R(z') \left[\frac{z}{z'} \frac{dz'}{dz} \right]^d \tag{5.46}$$

where

$$z' = \frac{\alpha z}{1 - (1-\alpha)z} . \tag{5.47}$$

for an operator R of conformal weight d . We find that after the gauge transformation, Eq. (5.36) becomes

$$V(\omega) [\varphi^u(z) - \varphi^u(1+\omega z)] = 0 \quad (5.48)$$

For this simple case, Eq. (5.27) for the infinitesimal transformation which changes the modulus from ω to $\omega+\epsilon$ is given by

$$z + \epsilon f(1+\omega z) = \epsilon \omega f(z) \quad (5.49)$$

while the transformations that leave the vertex alone obey

$$f(1+\omega z) = \omega f(z) \quad (5.50)$$

We now must solve these equations, beginning with Eq. (5.49). Defining $t = \ln(z-1/1-\omega)$, and

$$f(t) \equiv e^t j(t) - \frac{1}{(1-\omega)^2} \quad (5.51)$$

we find that Eq. (5.49) is equivalent to

$$\frac{1}{\omega} + j(t + \ln \omega) = j(t) \quad (5.52)$$

However, we must also demand that the solution be single valued in z , in which case we also have

$$j(t + 2\pi i) = j(t) \quad (5.53)$$

Of course, such functions are well known. One example of interest to us which has only one simple pole in its fundamental domain is constructed from the Weierstrass ζ -function⁴⁷

$$j(t) = \frac{1}{\omega} \overline{\zeta}(t-r) + \text{const} \quad (5.54)$$

where

$$\overline{\zeta}(t) = \zeta(t) - \frac{t}{2\pi} \eta \quad (5.55)$$

r is a constant, and $\eta = \zeta(i\pi)$. The "periods" of ζ are $2i\pi$ and $\ln \omega$.

Equation (5.50) is solved by taking

$$f(t) = e^t \bar{j}(t) \quad (5.56)$$

provided that

$$\bar{j}(t + \ln w) = \bar{j}(t), \quad \bar{j}(t + 2\pi i) = \bar{j}(t) \quad (5.57)$$

That is, $\bar{j}$ is an automorphic (in this case elliptic) function, and so can be constructed also out of combinations of ζ with appropriate arguments. Taking $\bar{j}(t) = 1$ in Eq. (5.56), we find the resulting identity:

$$V \left(L_0 - \frac{L_{-1}}{(1-w)} \right) = 0 \quad (5.58)$$

Using $j(t)$ in Eq. (5.54), we find the identity

$$\frac{\partial V}{\partial w} = V \oint_{\bar{z}} dz \left\{ \left(z - \frac{1}{1-w} \right) \frac{1}{w} \bar{j}(u) - \frac{1}{(1-w)^2} \right\} \frac{L(\bar{z}) - c}{(5.59)}$$

where

$$u = \ln(1 - (1-w)z) \quad (5.60)$$

corresponds to the choice of r required to place the pole of $\bar{\zeta}$ at $z = 0$ and c is the normal ordering constant. Using Eq. (5.55), we find that the identity becomes

$$\begin{aligned} \frac{\partial V}{\partial w} = V \left\{ \frac{L_2}{w(1-w)^2} - \frac{3L_1}{2w(1-w)} - \frac{L_{-1}}{(1-w)^2} \right. \\ \left. + L_0 \left(\frac{1}{2w} - \frac{\eta}{i\pi w} - \frac{1}{12w} \right) - c \right. \\ \left. + \text{terms involving } L_{n,n \geq 1} \right\} \end{aligned} \quad (5.61)$$

The constant c is determined by putting the vacuum with zero momentum on each side of the equation, and using the explicit form of V ; see Ref. 12. One finds

$$c = - \frac{D}{w} \sum_{n=1}^{\infty} \frac{w^n}{(1-w^n)^2} + \frac{D}{2w \ln w} \quad (5.62)$$

This corresponds to $V(w)$ having a weight factor in its conformal properties.

To find the integration measure, we place the one external line on-shell and demand that W of Eq. (5.1) vanish when the line is a zero norm physical state. Taking $L_1|\phi\rangle$ trivially gives zero as a result of Eq. (5.58), and so the first Virasoro gauge gives no condition on the integration measure. However, taking $L_2|\phi\rangle$ leads to the on-shell result

$$\frac{\partial V}{\partial w} = V \left\{ + \frac{D}{2w \ln w} + \frac{1}{1-w} + \frac{L_2}{w(1-w)^2} + \frac{1}{w} + (D-2) \frac{\partial \ln f(w)}{\partial w} \right\} \quad (5.63)$$

Here, we have used the result

$$\gamma = -i\pi \left[2w \frac{\partial}{\partial w} \ln f(w) + \frac{1}{12} \right] \quad (5.64)$$

where

$$f(w) = \prod_{n=1}^{\infty} (1-w^n) \quad (5.65)$$

Consequently, we find that the second Virasoro gauge condition determines the integration measure to be

$$\overline{f}(w) = w^2 [f(w)]^{-D+2} (\ln w)^{-\frac{D}{2}} (1-w)^{-1} \quad (5.66)$$

This is a heuristic derivation, since there is no state at zero momentum with $L_0 = -1$. It nevertheless gives the correct answer.

It is interesting to note that no mention has been made of ghosts, but still the result has the correct reduction in the number of factors of the partition function. When a multiperipheral tree is attached to this tadpole, it is straightforward to verify that the standard answer is recovered for the one-loop planar diagram, in all details; in particular, the $(1-w)^{-1}$ factor of Eq. (5.66) is reabsorbed in a change of variables⁴⁸. The actual value of D is not determined by the planar graph. Rather, $D = 26$ comes from demanding that physical closed string poles occur in the orientable non-planar diagram⁴⁹. This minimal dependence on $D = 26$ of the present method contrasts sharply with that found in string field theory and may bode well for applications to string theories outside 10 and 26-dimensions.

The reader will notice that we have neglected one important term in the derivation of the measure given in Eq. (5.66), namely, when we integrate by parts, there is the boundary term. Due to the fact that the tadpole has only one leg, these terms are difficult to discuss in its case. These boundary effects for an arbitrary one-loop graph have been evaluated in Ref. 12 and by other means to that given here in Ref. 50. They are non-zero except for the gauge group $SO(2^{13})$ when the contribution from the planar graph cancels against the non-orientable graph. This corresponds to an anomaly, not in the gauge current which is anomalous when we use L_{-1} , but in the gauge invariance which first has an effect in the states at level 2. The reader will recall that at this level there is a massive pure spin 2 state. In more modern language this is an anomaly in the BRST current, as it means that Q , the BRST charge, does not satisfy the equation

$$W \left\{ \sum_{j=1}^N Q^{(j)} \right\} = 0 \quad (5.67)$$

It had been previously noted⁵¹ that only for $SO(2^{13})$ did the divergences cancel in this string theory. It has been shown in Ref. 52 that at one-loop the divergences can be removed by going to de Sitter space. The same mechanism may also remove the above anomaly⁵⁰.

The multi-loop vertex

As a final example, we compute the vertex for N external legs and M loops. In this case, we have cycling transformations T_j for the external legs and loop-cycling transformations P_n^j , $n = 1, \dots, M$. The latter we take to form a Schottky group whose fundamental domain is the area external to the isometric circles of P_n^j and $(P_n^j)^{-1}$.

The Koba-Nielsen points z_j , $j = 1, \dots, N$ for the external particles lie within the fundamental domain as shown in Fig. 5.2

Fig. 5.2

Let us consider the quantity

$$V \oint_{\substack{\xi_j=0 \\ \bar{\xi}_j}} d\bar{\xi}_j \phi^j(\bar{\xi}_j) P^{\mu(j)}(\bar{\xi}_j) \quad (5.68)$$

where $\xi_j = 0$ corresponds to the points $z_j = 0$ and the function ϕ^j is to be discussed.

We may deform this contour around z_j so that it goes around the edge fundamental region, i.e., around the $2M$ isometric circles and also around the other points z_i , $i \neq j$. The validity of this contour distortion relies on the fact that $\phi^j(\xi_j)$ only has possible poles at $\xi_k = 0$. If it only has poles at ξ_j it can be written in the form

$$\phi^j(\bar{\xi}_j) = \bar{\xi}_j^{-n} + \sum_{m=0}^{\infty} E_{n,m}^j (\bar{\xi}_j)^m \quad (5.69)$$

As for the case of the trees discussed earlier, one must use the P^μ overlap identities to retain an expression which is still convergent as one moves across the complex plane. However, Eq. (2.14) shows that this change of $P_\mu^{(j)}(\xi_j)$ to $P_\mu^{(j+1)}(T_{j+1}\xi_j)$ involves the factor $S_{T_{j+1}}(\xi_j)$ which is precisely the factor required to change the integration variable from ξ_j to ξ_{j+1} . Consequently, we find that

$$\begin{aligned} V \oint_{\substack{\xi_j=0 \\ \bar{\xi}_j}} d\bar{\xi}_j \phi^j(\bar{\xi}_j) P^{\mu(j)}(\bar{\xi}_j) &= V \left\{ - \sum_{\substack{i=1 \\ i \neq j}}^N \oint_{\substack{\xi_i=0 \\ \bar{\xi}_i}} d\bar{\xi}_i \phi^i(\bar{\xi}_i) \right. \\ &\cdot P^{\mu(i)}(\bar{\xi}_i) + \sum_{n=1}^M \left\{ \oint_{C_n} d\bar{\xi}_j \phi^j(\bar{\xi}_j) P^{\mu(j)}(\bar{\xi}_j) \right. \\ &\left. + \oint_{C'_n} d\bar{\xi}_j \phi^j(\bar{\xi}_j) P^{\mu(j)}(P_n^j \bar{\xi}_j) \right\} \left. \right\}. \end{aligned} \quad (5.70)$$

We now examine the conditions under which the last term vanishes. We recall that P_n^j maps C_n to C'_n and so if we change a variable, these two terms will cancel provided

$$\phi^j(P_n^j \bar{\xi}_j) = \phi^j(\bar{\xi}_j). \quad (5.71)$$

We therefore adopt this condition.

However, when performing the above contour integrals, we must beware of any multi-valued behaviour of ϕ^j . Let us suppose that ϕ^j is not single-valued but when going around a circle C_n , and so C'_n , it changes by a constant, c . We can implement this by taking a cut between the circles C_n and C'_n . The integration contour shown in Fig. 5.3 is the one that actually results from the contour deformation which lies around C_n and C'_n . We may write this contribution as

$$\begin{aligned} & V \left\{ \int_A^B \frac{d\bar{z}_j}{\bar{z}_j} \phi^j(\bar{z}_j) P^{\mu(j)}(\bar{z}_j) + \int_C^D \frac{d\bar{z}_j}{\bar{z}_j} \phi^j(\bar{z}_j) P^{\mu(j)}(\bar{z}_j) \right\} \\ &= V c \int_A^B \frac{d\bar{z}_j}{\bar{z}_j} P^{\mu(j)}(\bar{z}_j) = c V \left\{ \varphi^{\mu(j)}(A) - \varphi^{\mu(j)}(B) \right\} \\ &= 0 \end{aligned} \quad (5.72)$$

Fig. 5.3

However, A and B may be chosen so that they are related by the action of P_n^j and then by Eq. (5.15) we obtain zero. We can summarize the result as:

$$V \left\{ \sum_{k=1}^N \oint_{\bar{z}_k=0} \frac{d\bar{z}_k}{\bar{z}_k} \phi^j(\bar{z}_k) P^{\mu(k)}(\bar{z}_k) \right\} = 0 \quad (5.73)$$

provided ϕ^j satisfies Eq. (5.71) and changes by a constant when its argument is taken around any isometric circle.

To demand that ϕ^j be single-valued would require it to be an automorphic function not only with respect to the group generated by the multi-loop cycling transformations, P_n^j , but also by a rotation about the centres of the isometric circles, and in general, such functions with the desired pole structure do not exist.

The desired ϕ^j is given explicitly in Ref. 13. In what follows we have changed the normalization of our oscillators for this and other rotational points (see the Appendix). Recalling Eq. (5.20), we find that Eq. (5.73) becomes

$$\begin{aligned}
 V \{ & \frac{a_{-n}^{u(j)}}{\sqrt{n}} + \sum_{m=1}^{\infty} E_{nm}^j a_m^{u(j)} \sqrt{m} + \sum_{\substack{k=1 \\ k \neq j}}^N \{ \\
 & \sum_{p=1}^{\infty} \frac{1}{\sqrt{p}} (\Gamma(V^j)^{-1} V^k)_{np} a_p^{u(k)} \\
 & + \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} E_{nm} \sqrt{m} [(\Gamma(V^j)^{-1} V^k)_{mp} a_p^{u(k)} \\
 & + \frac{1}{\sqrt{n}} [\Gamma(V^j)^{-1} V^k(0)]^n a_0^{u(k)} + \sum_{m=1}^{\infty} E_{nm}^j ((\Gamma(V^j)^{-1} V^k(0))^m a_0^{u(k)} \} \\
 & = 0
 \end{aligned} \tag{5.74}$$

The vertex is then found to be

$$\begin{aligned}
 V = & \langle 0 | \dots \langle 0 | \exp - \{ \sum_{i,j=1}^N \{ \frac{1}{2} (a^i | \bar{E}^i (V^i)^{-1} V^j | a^j) \\
 & + (a^i | \bar{E}^i | (V^i)^{-1} V^j(0)) a_0^j + \frac{1}{2} a_0^i \mathcal{K}_{00}^{ij} a_0^j \} \} \\
 & \tag{5.75}
 \end{aligned}$$

• V^{tree}

where

$$\bar{E}_{nm}^j = \sqrt{n} E_{nm}^j \sqrt{m}$$

By V^{tree} , we mean the expression which occurs for the corresponding tree graph, once one replaces the tree external line cycling transformation by those appropriate for the loop. This factorization comes about since the third and fifth terms of Eq. (5.75) are of the same type as occurs for the tree graph which is discussed in the beginning of this section.. We have yet to determine $\mathcal{K}_{00}^{ij}$ which requires the Q^μ overlap. This is done in Ref. 13 where the reader will also find more details of the above derivation including the concrete expression for E_{nm}^j .

Equation (5.75) is a very general result, holding for the open bosonic string scatterings, for arbitrary number and type of physical external states. It will hold for all planar and non-planar graphs and one may expect its generic form to generalize to all string theories.

Coset Space Interpretation of Moduli Space

There has recently been a discussion of moduli space as a coset space⁵³. This construction has been used in the Grassmannian approach⁹ and further discussed in the last paper given in Ref. 9. We now demonstrate how a coset space structure is automatically encoded into the axioms, given at the beginning of this section. The following is unpublished work which was found in discussions with A. Neveu.

Let us consider first the case of the multi-loop tadpole. Rather than label the vertex by the multipliers and fixed points, it is more convenient to use the P_n themselves as labels, i.e., $V(P_1, P_2, \dots, P_j, \dots, P_M)$. The transformation of the change of the moduli of Eq. (5.23) and (5.25) can be written as

$$V(P_1 \dots P_n \dots P_M) \mathcal{H} = V(\mathcal{H} P_1 \mathcal{H}^{-1}, \dots, \mathcal{H} P_n \mathcal{H}^{-1}, \dots, \mathcal{H} P_M \mathcal{H}^{-1}) \cdot C(P_n, \mathcal{H}) \quad (5.76)$$

In this equation we have explicitly shown the factor C which results from normal-ordering. In general, it was not explicitly shown above, but was understood to be present and computed when required, such as for the measure. It must obey the group requirement

$$C(P_n \mathcal{H}_1) C(\mathcal{H}_1 P_n \mathcal{H}_2^{-1}, \mathcal{H}_2) = C(P_n, \mathcal{H}_2 \mathcal{H}_1) \quad (5.77)$$

The infinitesimal transformation $\mathcal{H}(z) = z + \varepsilon f(z)$ generates a vector field $f(z)(\partial/\partial z)$ or a -1 differential and must satisfy Eq. (5.27) which we repeat for convenience

$$P_n(z) + \varepsilon f(z) \frac{dP_n}{dz} = \hat{P}_n(z) + \varepsilon f(P_n z) \quad (5.78)$$

where $\hat{P}_n = \mathcal{H} P_n \mathcal{H}^{-1}$ is the desired change in the moduli. We recall that $f(z)$ is single-valued and analytic everywhere except at $z = 0$ where it can have poles.

Let us now consider those infinitesimal transformations which leave the moduli inert (i.e., $P_n = \hat{P}_n$). In this case $P_n = \hat{P}_n$ and Eq. (5.78) tells us that f is a modular form of weight -1 with respect to the Schottky group. We recall a modular form of weight k satisfies the equation

$$f(Tz) = \left(\frac{dT}{dz}\right)^{-k} f(z). \quad (5.79)$$

for all elements T in the group G .

However, this is just what is expected for a vector-field as then $f(z)(\partial/\partial z)$ is invariant. Consequently, this vector-field is defined on the Riemann surface itself. The form of such functions is determined by the Riemann-Roch theorem to have a pole at least of order of $3g-3+1$ for $g > 2$ and so of the form

$$f(z) = \sum_{m=0}^{\infty} \frac{a_{m-3g+3-1}}{z^{3g-3+m+1}} + \sum_{n=0}^{\infty} a_n z^n. \quad (5.80)$$

for $g > 2$. Such a vector field can be constructed out of the prime form, σ -function and θ -functions on the Riemann surface⁵⁴. The combination of three functions one takes is unique once one specifies the poles which occur. Given any infinitesimal transformation, we can write Eq. (5.76) as

$$V(P_n) + \epsilon \oint_{z=0} \frac{dz}{z} L(z) \frac{f(z)}{z} = V(\hat{P}_n) + \epsilon \int d(P_n, \mathcal{K}) \quad (5.81)$$

where $C(P_n, \mathcal{K}) = 1 + \epsilon d(P_n, \mathcal{K})$. Thus a transformation that does not change the moduli dependence of the vertex involves the action of $L_{-3g+3-2}$, and $L_{-3(g-3)-3}, \dots$, etc., as well as L_n , $n > -1$.

Let us now consider an $f(z)$ which induces a change in the moduli. Such a function cannot be analytic at $z = 0$ and since $P_n \neq \hat{P}_n$, it does not behave as a vector under the action of P_n , and so is not defined on the Riemann surface. We may remove any poles of order greater than $3g-3+1$ since these correspond to transformations that leave the vertex inert. As such f is of the form

$$f(z) = \sum_{n=1}^{3g-3} \frac{a_{-n}}{z^{-n}} + \sum_{m=0}^{\infty} a_m z^m \quad (5.82)$$

and so involves $L_{-2}, \dots, L_{-3g+3}$ as well as $L_n, n > -1$. The precise poles one gets at $z = 0$ is determined by which moduli transformation one makes.

Lastly we have f 's which are analytic at $z = 0$. These involve $L_n, n > -1$. On physical states $L_n - \delta_{n0}$, $n > 0$ vanish and so in some sense these transformations are irrelevant. In fact, if we consider the equivalence class of vertices which are the same on physical states, that is

$$V_1 \sim V_2 \quad \text{if} \quad V_1 = V_2 \prod_{r=1}^3 \exp \sum_{n=0}^{\infty} (L_n^{(r)} - \delta_{n0}) a_n^{(r)} \quad (5.83)$$

Then such functions which do not contain constants induce changes that leave the equivalence class inert. The functions which contain constant functions include the action of L_{-1} and it is these that change the Koba-Nielsen points. This is apparent from Eq. (5.24) when we recall that T_i are elements of $SL(2, \mathbb{R})$.

The reader will recognize the classes of functions considered in Ref. 53, namely functions analytic at $z = 0$, functions not analytic at $z \neq 0$ and not extendable and finally functions not analytic at $z = 0$ but extendable. The set of transformations $\mathcal{H}$ which leaves the moduli inert (i.e., $\mathcal{H} P_n \mathcal{H}^{-1} = P_n, n = 1, \dots, M$) form a group which we call B_M . Then the vertex is, in effect, defined at least locally on the coset space V/B_m where V is the Virasoro group.

To illustrate the above discussion, let us consider the case of $N = 1, M = 1$ which is the tadpole vertex which we considered earlier. Here we have one moduli, ω and the moduli-changing transformations were given in terms of the Weierstrass function which has a simple pole at $z = 0$ and so involves L_{-2} . The moduli-preserving transformations were periodic and so involved poles of order two (i.e., the Weierstrass $\wp$ function) and higher order. These then involved $L_{-3}, L_{-4}, \dots$, etc. The action of the functions which involve L_{-1} and are analytic, is best seen when we have more than three legs so that not all the Koba-Nielsen variables are chosen to be fixed values.

The factor d of Eq. (5.81) is easily calculable by taking the zero momentum vacuum $|0,0\rangle$ on all legs and one finds that

$$\{ d(p_n, \mathcal{H}) + U \sum_{n=2}^{\infty} a_n L_{-n-1} \prod_{v=1}^N |0, 0\rangle_v \} = 0 \quad (5.84)$$

This may be evaluated by recalling the expression for V of Eq. (5.75) and as a result

$$d(p_n, \mathcal{H}) = -\frac{1}{2} \sum_{m=0}^{n-1} \sum_{n=1}^{\infty} a_{-(m)} \sqrt{n-m} E_{n-m, m} \sqrt{m}$$

The corresponding coset space construction for more than one external leg may be extracted from the equations given at the beginning of Section 5.

It is interesting to note that the coset space that the vertex is defined on varies from loop to loop. In fact, B_M gets smaller and smaller in the limit of infinite loops and the coset space becomes the Virasoro group. As the number of loops increases to infinity, one also has a vertex invariant under an infinite number of $SL(2, \mathbb{R})$ cycling transformations. In this sense, a non-perturbative effect is controlled by an object which is defined on the Virasoro group, that is, $V(g)$, where g is an element of the Virasoro group. Neglecting the phase term, it will transform under a conformal transformation $\mathcal{H}$ by $V(g)\mathcal{H} = V(\mathcal{H}g\mathcal{H}^{-1})$.

The extension of the co-ordinate space of string theory to include anticommuting co-ordinates has led to considerable simplifications of the formalism of string theory and in particular for the gauge covariant string. The discussion so far has not included these anticommuting oscillators; however, it is straightforward to include them. The vertex which has the anticommuting oscillators incorporated will obey the overlaps for Q^μ etc., discussed above, but also overlaps for conformal operators constructed from the ghost oscillators. In fact, this has already been carried out for the three vertex, where, using these anticommuting oscillators, the additions to V^{csv} were found¹¹. For this $V_T^{(3)}$ the zero norm physical states decouple, as it was shown to satisfy^{11, 17}

$$V_T^{(3)} \sum_{r=1}^3 Q^{(r)} = 0.$$

where $Q^{(r)}$ is the BRST charge on leg r . It was pointed out that the full vertex $V_T^{(3)}$ would form the basic building block for a new dual model which would lead to the correct measure without the use of internal

Brink-Olive projectors⁴. This work has been extended by the Copenhagen group⁵⁵ who has found the anticommuting additions to the N string vertex.

It is certainly true that the formulation with anticommuting oscillators and the BRST charge Q possesses considerable elegance and it could be of advantage to use it in the approach to string theory advocated here. In particular, in the determination of the measure, it may simplify matters. Beyond one loop, as the above discussion suggests, the change in the moduli will in general involve the action of not only L_{-2} but also L_{-3} , L_{-4} , etc., on the vertex. While these can be interpreted as zero norm physical states, the procedure is not so straightforward as in the one-loop case. Although, there are zero norm physical states which involve these L_{-n} 's, they cannot be written in terms of L_{-n} 's acting on states which are annihilated by L_n $n \geq 1$; in general additional operators, such as K_n , are required. If we work with the anticommuting oscillators, all zero norm states are of the form $Q |\Lambda\rangle$ and the determination of the measure will involve the moduli-changing equation and the enforcement of

$$\left\{ \int dz_i d\sigma_v \bar{f}(z_i, \sigma_v) V_T^{(n)} \right\} \sum_{n=1}^N \varphi^{(n)} = 0$$

on the full vertex V_T^N .

ACKNOWLEDGEMENTS

I would like to thank C. Gomez, M. Freeman, D. Olive and A. Neveu for discussions.

APPENDIX

PROJECTIVE TRANSFORMATIONS

A projective transformation, sometimes also called a linear or mobius transformation, acts on the complex plane $\mathbb{C}$ by

$$z \rightarrow T(z) = \frac{az+b}{cz+d} \quad (\text{A.1})$$

It only depends on three parameters since the transformation is unaffected by multiplying all of the four parameters a, b, c and d by a constant. The parameters may be real or complex. It is easily verified that the product of two such transformations is also of a transformation of this form and we refer to the set of all such transformations as well as their composition rule as the projective group. The group of all projective transformations is generated by a dilation, a translation and an inversion which respectively are given by

$$\begin{aligned} z \rightarrow \lambda z &= D(\lambda)z \\ z \rightarrow z + \mu &= T(\mu)z \\ z \rightarrow -\frac{1}{z} &= I(z) \end{aligned} \quad (\text{A.2})$$

Then, one finds that

$$T\left(\frac{a}{c}\right) \cdot D\left(\frac{ad-b}{c}\right) \cdot I \cdot T(d) D(c) z = \frac{az+b}{cz+d} \quad (\text{A.3})$$

We may write the parameters of the transformation of Eq. (A.1) as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (\text{A.4})$$

and it is easily verified that the parameter composition rule of two projective transformations is the same as the one that gives by multiplying the corresponding 2 by 2 matrices. However, the matrices 1 and -1 where 1 is the unit matrix.

have the same projective transformation. The transformations which correspond to a matrix with positive definite determinant, i.e., $(ad-bc) > 0$ form a subgroup. Given $ad-bc > 0$ we can obtain all transformations by using only those for which $ad-bc = 1$. For this subset the projective group is isomorphic to $SL(2, \mathbb{R})/\{1, -1\}$ if a, b, c, d are real and $SL(2, \mathbb{C})/\{1, -1\}$ if a, b, c, d are complex. We recall that $SO(2, 1)$ and $SO(3, 1)$ are isomorphic to $SU(1, 1)/\{1, -1\}$ and $SL(2, \mathbb{C})/\{1, -1\}$ respectively.

Any projective transformation on the points z_1, z_2, z_3, z_4 leaves invariant the quantity called the cross-ratio

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \quad (A.5)$$

This is most easily established by considering separately a translation, a dilation and an inversion.

A fixed point $\tilde{z}$ of a projective transformation is one such that

$$\tilde{z} = \frac{a\tilde{z} + b}{c\tilde{z} + d} \quad (A.6)$$

Clearly, if there are more than two fixed points then the transformation is the identity transformation. When there are two fixed points, say α and β , the transformation can be written in the form

$$\frac{T(z) - \alpha}{T(z) - \beta} = w \frac{(z - \alpha)}{z - \beta} \quad (A.7)$$

In fact, if we make the change of variable $\bar{z} = [z - \alpha][z - \beta]^{-1}$ then the transformation reads $\bar{z}' = w\bar{z}$. We refer to w as the multiplier and it is clear that any such projective transformation can be labelled by its two fixed points and its multiplier. The transformation has only one fixed point if $a+d = \pm 2$.

A projective transformation, T , can be used to take any three distinct points z_1, z_2, z_3 to any other three distinct points $\bar{z}_1, \bar{z}_2, \bar{z}_3$

and this transformation is then uniquely determined. To demonstrate this, let z_1, z_2, z_3 go to $\infty, 0, 1$. Then, the explicit transformation is given by

$$z = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} \quad (\text{A.8})$$

The inverse transformation will take $\infty, 0, 1$ to z_1, z_2, z_3 . Consequently, we can go from z_1, z_2, z_3 to $\infty, 0, 1$ and from $\infty, 0, 1$ to $\bar{z}_1, \bar{z}_2, \bar{z}_3$ by the composition of two transformations. It is unique, since, if there are two such transformations T_1 and T_2 , then $T_1^{-1}T_2$ has three fixed points and the only such transformation is the identity transformation. In view of this theorem, we may specify a projective transformation by stating how it moves three distinct points, and so use the notation

$$\left(\begin{array}{ccc} z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{array} \right) \text{ for } T. \quad (\text{A.9})$$

A real projective transformation preserves or reverses the cyclic-ordering of points along the real axis. According to Eq. (A.2), any projective transformation can be as a translation, inversion and dilation. Clearly translation preserves the cyclic order and it is easily seen that inversion does too. For dilation $z \rightarrow \lambda z$ the order is preserved or reversed according to whether λ is greater or less than zero, respectively. In these considerations, the points $+\infty$ and $-\infty$ are identified as one might expect by recalling the relation between the sphere and the complex plane plus a point at ∞ . As a result, whether a transformation preserves or reverses the cyclic order of points depends on whether it has an even number of dilations for which $\lambda < 0$ or not. Examining Eq. (A.3), we realize that if $ad-bc > 0$ then the cyclic order of points is always preserved. It also follows that the same statement holds for any contour in C related by a transformation to the real line.

As an example of the use of the invariance of the cross-ratio we now show that $T' = STS^{-1}$ has the same multiplier as T . Clearly the fixed points of T' are $S(\alpha)$ and $S(\beta)$. Then we see that

$$\frac{(T'(z) - S(\alpha))(z - S(\beta))}{(T'(z) - S(\beta))(z - S(\alpha))} = \omega = \frac{(T(S^{-1}z) - \alpha)(S^{-1}z - \beta)}{(T(S^{-1}z) - \beta)(S^{-1}z - \alpha)} = \omega \quad (\text{A.10})$$

The isometric circle associated with $T(z)$ of Eq. (A.1) is given by

$$|cz+d| = |ad-bc| \quad (A.11)$$

and this is the unique place where infinitesimal lengths are preserved by the action of $T(z)$. We can define the infinite dimensional matrix associated with this transformation by:

$$\frac{[T(z)]^n}{\sqrt{n!}} = \sum_{m=1}^{\infty} T_{nm} \frac{z^m}{\sqrt{m!}} + \frac{[T(0)]^n}{\sqrt{n!}} \quad (A.12)$$

If we define the infinite component column vector

$$|z\rangle = \left(\frac{z}{\sqrt{1}}, \frac{z^2}{\sqrt{2!}}, \dots, \frac{z^n}{\sqrt{n!}}, \dots \right) \quad (A.13)$$

then Eq. (A.2) can be written

$$|Tz\rangle = T|z\rangle + |T(0)\rangle \quad (A.14)$$

where the obvious matrix multiplication is assumed. The transformation Γ is defined by

$$\Gamma(z) = \frac{1}{z} \quad (A.15)$$

It will be more convenient when discussing vertices to use oscillators $a_n^\mu, a_n^{\mu+}, n > 1$ which commute to 1 rather than n . That is

$$[a_n^\mu, a_m^{\nu+}] = \eta^{\mu\nu} \delta_{n,m} ; n, m \geq 1 \quad (A.16)$$

In this notation

$$\phi^\mu(z) = \sum_{n=1}^{\infty} \left(-\frac{a_n^\mu z^{-n}}{\sqrt{n!}} + \frac{a_n^{\mu+} z^n}{\sqrt{n!}} \right) + a_0^\mu \ln z + \frac{\partial}{\partial a_0^\mu} \quad (A.17)$$

The shorthand

$$(a^i | T | a^j) \quad (A.18)$$

is taken to mean

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n^i T_{nm} a_m^j \quad (\text{A.19})$$

It can be shown that

$$(a^i | T | a^j) = (a^j | \hat{T} | a^i) \quad (\text{A.20})$$

where $\hat{T}$ is the infinite matrix

$$\hat{T} = \Gamma T^{-1} \Gamma \quad (\text{A.21})$$

A useful identity is given by

$$(z_1 | z_2) = - \ln (1 - z_1 z_2) \quad (\text{A.22})$$

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