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$Z_N \times Z_M$ -ORBIFOLDS AND DISCRETE TORSION

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ABSTRACT

We extend previous work on Z_N -orbifolds to the general $Z_N \times Z_M$ abelian case for both $(2, 2)$ and $(0, 2)$ models. We classify the corresponding $(2, 2)$ compactifications and show that a number of models obtained by tensoring minimal $N = 2$ superconformal theories can be constructed as $Z_N \times Z_M$ -orbifolds. Furthermore, $Z_N \times Z_M$ -orbifolds allow the addition of discrete torsion which leads to new $(2, 2)$ compactifications not considered previously. Some of the latter have negative Euler characteristic and Betti numbers equal to those of some complete intersection Calabi-Yau (CICY) manifolds. This suggests the existence of a previously overlooked connection between CICY models and orbifolds.

Up to now, three different methods have been used to construct $(2, 2)$ compactifications of the heterotic string. These are compactifications on complete intersection Calabi-Yau (CICY) manifolds [1], $(2, 2)$ orbifold compactifications [2] and tensoring of minimal $N = 2$ superconformal theories [3]. Some Z_N -orbifolds have also been obtained in the covariant lattice approach [4]. In the orbifold formalism, the abelian Z_N case was studied in detail in ref. [5]. Connections among the three different constructions are still not well understood, particularly in the case of 6 compactified dimensions on which we focus our attention [6]. Inspection of the list [7,8] of known $N = 2$ models shows that some of them have the same massless spectrum and discrete symmetries as known CICY manifolds. Likewise, the $Z_6' \equiv Z_2 \times Z_3$ -orbifold has the same massless spectrum as the $1^4 2^2 4$ and $1^3 2^4 (-10)$ $N = 2$ models, a fact unnoticed before. On the other hand, no connection has been found between Z_N -orbifold models (which have positive Euler characteristic, $\chi > 0$) and CICY models (which have $\chi < 0$).

In this letter we extend previous work on Z_N -orbifolds to the general abelian $Z_N \times Z_M$ case. We list all the possible $(2, 2)$ models obtained in this way and compute their basic properties. We observe that a number of $N = 2$ models can be constructed as particular points in the moduli space of $Z_N \times Z_M$ -orbifolds. The extension to $Z_N \times Z_M$ -orbifolds is non-trivial since in this case there exists the possibility of adding discrete torsion as explained in [9]. This then leads to new $(2, 2)$ compactifications not considered in detail before. Interestingly enough, some of the models with discrete torsion have zero or negative Euler characteristic and Betti numbers matching those of previously constructed CICY manifolds [10]. All these results point towards the existence of connections among the three known approaches to $(2, 2)$ compactifications. At the end of this letter we discuss the possible implications of these results.

Let us now briefly discuss the construction of the most general abelian orbifold leaving four uncompactified dimensions and one unbroken supersymmetry. Since the discrete moding group G has to be inside an $SU(3)$ subgroup of $SO(6)$ it has a generic $Z_N \times Z_M$ structure. Furthermore, we know from the theory of abelian finite groups that M must be divisible by N . The Z_N and Z_M actions on the 6-dimensional torus lattice may be realized in principle by any of the thirteen “shifts” considered in ref. [2]. They correspond to all the automorphisms of a 6-dimensional lattice that leave one unbroken supersymmetry. It turns out that many N, M combinations in $Z_N \times Z_M$ are not possible because Z_N and Z_M cannot be simultaneously realized as automorphisms of a 6-dimensional lattice. For example, things of the type $Z_2 \times Z_8(Z_{12})$ are not allowed. No Z_2 , different from those already included in $Z_8(Z_{12})$, can be realized simultaneously with a $Z_8(Z_{12})$ automorphism. This can be checked by computing

$\text{Tr } g$ for all $g \in Z_N \times Z_M$. This quantity must be an integer if Z_N and Z_M is to act crystallographically on a 6-dimensional lattice. Likewise, $\det(1 - g)$ gives the number of fixed points of g and must be an integer. All the allowed $Z_N \times Z_M$ moddings as well as the corresponding shifts $\vec{a}$ (of order N) and $\vec{b}$ (of order M) are shown in the table. All moddings in the table can be realized as automorphisms of an $SU(3)^3$ lattice except for the $Z_2 \times Z_4$ and $Z_4 \times Z_4$ examples which might be realized on a cubic lattice. The table also includes the Z'_6 orbifold which is equivalent to $Z_2 \times Z_3$.

To compute the massless spectrum of these orbifolds we need to know the generalized GSO-projection on invariant states. The one-loop partition function has the general form

$$Z = \frac{1}{NM} \sum_{g,h} \varepsilon(g, h) Z(g, h) \quad (1)$$

where $g, h \in Z_N \times Z_M$. We will not give here the expression for $Z(g, h)$ which can be found in ref. [5] after making obvious replacements. We want however to emphasize the importance of the phases $\varepsilon(g, h)$ in (1). In a Z_N -orbifold these phases are fixed by one-loop modular invariance. This is because a $\tau \rightarrow -1/\tau$ transformation (τ is the modular parameter) connects the untwisted $(1, \theta^n)$ sectors to the twisted $(\theta^n, 1)$ sectors. A $\tau \rightarrow \tau + m$ transformation then relates $(\theta^n, 1)$ to (θ^n, θ^m) . Thus, the phase $\varepsilon(g, h)$ is completely determined (in the case of standard embedding $\varepsilon(g, h) = 1$ can be chosen). For a $Z_N \times Z_M$ orbifold with two cyclic generators θ ($\theta^N = 1$) and ω ($\omega^M = 1$) the sector (θ, ω) cannot be connected through modular transformations with the orbit containing the untwisted $(1, \theta^l \omega^s)$ sectors. Thus, in the one-loop partition function one can assign a non-trivial phase $\varepsilon(\theta, \omega)$ to the (θ, ω) sector. However, as explained in ref. [9], two-loop modular invariance and factorization impose restrictions on the phases. These restrictions follow from the following equations

$$\begin{aligned} \varepsilon(g, hf) &= \varepsilon(g, h)\varepsilon(g, f) \\ \varepsilon(g, h)\varepsilon(h, g) &= 1 \\ \varepsilon(g, g) &= 1 \end{aligned} \quad (2)$$

In the case of torus compactification, the arbitrariness of $\varepsilon(g, h)$ can be seen to correspond to the freedom of turning on a background antisymmetric field on the torus [9]. Thus, $\varepsilon(g, h)$ is called discrete torsion. For $Z_N \times Z_M$ -orbifolds, eqs. (2) imply that the phase $\varepsilon(\theta, \omega)$ must

be an N^{th} -root of unity (recall M is divisible by N). This is

$$\varepsilon(\theta, \omega) \equiv \varepsilon \quad , \quad \varepsilon^N = 1 \quad (3)$$

Furthermore, for general sectors the phases are given by

$$\varepsilon(\theta^k \omega^l, \theta^t \omega^s) = \varepsilon^{(ks-lt)} \quad (4)$$

The usual formula for the Euler characteristic in the Z_N case can be extended to $Z_N \times Z_M$ -orbifolds. We have

$$\chi = \frac{1}{MN} \sum_{k,l=0}^{N-1} \sum_{t,s=0}^{M-1} \varepsilon(\theta^k \omega^l, \theta^t \omega^s) \chi(\theta^k \omega^l, \theta^t \omega^s) \quad (5)$$

where $\chi(\theta^k \omega^l, \theta^t \omega^s)$ is the Euler characteristic of the subspace left simultaneously fixed by $\theta^k \omega^l$ and $\theta^t \omega^s$.

For each allowed value of ε we can then compute the number of generations $N_{gen} = \chi/2$. However, to determine the complete massless spectrum of the orbifold (number of E_6 $\underline{27}$'s and $\overline{\underline{27}}$ as well as the number of E_6 singlets) eq. (5) is not enough. To this purpose, it is necessary to first find the states satisfying the masslessness condition and then project out the non-invariant states under the orbifold action.

The generalized GSO-projector on $Z_N \times Z_M$ -orbifold invariant states is easily generalized from the results in ref. [5]. To fix notation, let us write the vertex operator of a right-moving scalar in the -1 -picture (twisted sectors) [11,12]

$$V_{-1} = e^{-\phi} e^{ikx} e^{i(r+v)H} \Lambda \quad (6)$$

Here r is a vector in the scalar $SO(10)$ lattice and H_i are the bosonized NSR fermions. v is the shift $v = ka + lb$ associated to the $\theta^k \omega^l$ element of $Z_N \times Z_M$ (in this notation, $a = (0, 0, a_1, a_2, a_3)$). Λ is a product of twist operators, one for each complex coordinate rotated by $\theta^k \omega^l$. The conformal dimension of Λ is given by

$$dim\Lambda = \sum_i \frac{1}{2} v_i (1 - v_i) \quad (7)$$

V_{-1} has conformal dimension one for $k^2 = 0$ provided that $\frac{1}{2}(r+v)^2 + dim\Lambda = \frac{1}{2}$ which is just the condition $m_R^2 = 0$.

For charged matter, the left-moving part of the vertex operator is given by

$$V_L = e^{ikx} e^{i(p+V)F} \Lambda \quad (8)$$

where F_I are the bosonized internal fermions and p belongs to the $E_8 \times E_8$ lattice. V is the embedding shift $V = kA + lB$ associated to the embedding of $\theta^k \omega^l$ in the gauge degrees of freedom. V_L has conformal dimension one for $k^2 = 0$ provided that $\frac{1}{2}(p+V)^2 + \dim \Lambda = 1$ which is the condition $m_L^2 = 0$. For charged oscillator states the mass formula also has a contribution from the oscillator modding.

We can now give an expression for the degeneracy factor (number of fields) corresponding to each state (labelled by r, p and possible oscillator indices) in a $\theta^k \omega^l$ twisted sector. Following the same reasoning as in ref. [5] we arrive at

$$D(\theta^k \omega^l) = \frac{1}{MN} \sum_{t=0}^{N-1} \sum_{s=0}^{M-1} \varepsilon^{(ks-lt)} \tilde{\chi}(\theta^k \omega^l, \theta^t \omega^s) \Delta(k, l; t, s) \quad (9)$$

where $\tilde{\chi}$ is the number of points left simultaneously fixed by $\theta^k \omega^l$ and $\theta^t \omega^s$. If $\theta^k \omega^l$ leaves unrotated some of the complex coordinates, $\tilde{\chi}$ must be calculated using only the sub-lattice which is rotated. $\Delta(k, l; t, s)$ is a state dependent phase given by

$$\Delta(k, l; t, s) = \exp\{2\pi i[(p + kA + lB)(tA + sB) - (r + ka + lb)(ta + sb) - t\rho_a - s\rho_b]\} \quad (10)$$

where $\rho_{a,b}$ appear in the case of oscillator modes and indicate how the oscillator is rotated under θ and ω respectively.

Strictly speaking, formula (10) is only valid for the standard embedding ((2, 2) compactifications). For (2, 0) compactifications, one finds an extra contribution to Δ given by

$$\exp\{-i\pi[(kA + lB)(tA + sB) - (ka + lb)(ta + sb)]\} \quad (11)$$

For a single Z_N we recover the result derived in ref. [5]. Compactifications with non-standard embedding are restricted by modular invariance. Looking at modular transformations $\tau \rightarrow \tau + N(\tau + M)$ of $Z(g, h)$ in (1) leads to the usual level matching conditions that can be written as

$$N'[(kA + lB)^2 - (ka + lb)^2] = 0 \mod 2 \quad , k = 0, \dots, N-1 \quad , l = 0, \dots, M-1 \quad (12)$$

where N' is the order of the $(ka + lb)$ shift.

From now on we will concentrate on the standard embedding $(2, 2)$ $Z_N \times Z_M$ -orbifold compactifications. The table shows all possible models with the different phases allowed by eq. (3) . ϵ and ϵ^* lead to the same results. Altogether we can construct nineteen different models. We have computed the number of massless E_6 $\underline{27}$'s (N_{27}) and $\overline{\underline{27}}$'s ($N_{\overline{27}}$) for all models using the degeneracy formula (9) . For $\epsilon \neq 1$, χ decreases as expected. Since $\epsilon(1, g) = 1$, the contribution of the untwisted sector to χ does not change under the addition of discrete torsion. This gives the lower bound of 3 for N_{27} in all models.

We have also computed the total number of massless E_6 singlets (N_{sing}) in some of the models but this task gets more complicated for examples with Z_6 factors because multiple oscillator states proliferate and the projection procedure is quite tedious. For every model, we have checked the existence of moduli or “blowing-up” fields. These are oscillator singlet states related to $\underline{27}$'s and $\overline{\underline{27}}$'s by the action of the left world-sheet supersymmetry current. As shown in ref. [13] , these fields have a potential flat to all orders and can thus acquire arbitrary vacuum expectation values.

Examining the table and the list of models obtained through CICY and $N = 2$ constructions we can make the following comments :

1. Examples 1,(2),3,5,9,11,14 and 16 may be identified with some of the $N = 2$ models in ref. [7,8]. The same is true for the $Z'_6 \equiv Z_2 \times Z_3$ model included for completeness. The table shows candidates for this identification. These are models with the same N_{27} and $N_{\overline{27}}$ as one of the $Z_N \times Z_M$ -orbifolds. The fact that the 1^9 $N = 2$ model is related to the $Z_3 \times Z_3$ -orbifold was already remarked in ref. [5]. The other example with $N_{\overline{27}} = 0$ in ref. [7] can be related to the $Z_1 \times Z_1$ -orbifold. As an extra check we have computed N_{sing} for models 1, 2, 3, 4 ($N_{sing}=246$), 5 ($N_{sing}=256$), 6 ($N_{sing}=224$) and 11 ($N_{sing}=282$). For models 1, 3 and 5, N_{sing} coincides with that of their $N = 2$ relatives if we consider the enhanced gauge symmetry versions (rank 22 for models 1, 3 and rank 21 for model 5). In these multicritical versions there is an extra massless singlet per extra $U(1)$, as is well known. For $Z_4 \times Z_4$, N_{sing} coincides with that of the 2^6 $N = 2$ model in the rank 18 version of the orbifold. A further similarity is given by the fact that $N = 2$ constructions have a Z_{k+2} symmetry for each level k component theory [3] . In the examples reported above, the corresponding orbifolds do possess this type of symmetries as part of their moding group.
2. There are three examples with $\chi < 0$ (models 2,4 and 13) and four with $\chi = 0$

(8,10,15,19). We have compared these models with the list of Hodge numbers of CICY manifolds of ref. [10]. We find that models 2,4 and 10 have the same N_{27} and $N_{\overline{27}}$ as three CICY manifolds. In particular, model 10 has the same Betti numbers as the unique CICY manifold with $\chi = 0$. This might be an indication that strings propagating on orbifolds in the presence of discrete torsion could be equivalent to strings propagating on a CICY manifold. A more detailed comparison between these models should be done in order to confirm this suggestion. If one could find orbifold versions of more CICY manifolds, all computations in the latter would become much easier since the orbifold formalism is by far simpler and the corresponding conformal field theory is well known. Notice also that if one may change the properties of an orbifold (N_{27} , $N_{\overline{27}}$) by just adding discrete torsion (as in models 3 and 4) there could also be CICY manifolds related in the same way. This seems reminiscent of the recent suggestion of the existence of a “universal moduli space of (2,2) compactifications” [14].

3. We obtain models with $|\chi| = 0, 12, 24, 48, 72, 84, 96, 120, 144, 168$ and 180, always a multiple of 12 (as also happens with Z_N -orbifolds). About 70% of the $N = 2$ models in [7,8] have Euler characteristic equal to the above numbers. This might indicate that many more $N = 2$ models admit a (non-abelian ?) orbifold construction. Some of them might also be (2,2) compactifications obtained from asymmetric orbifolds [4][15]. On the other hand, a number of $N = 2$ models seems difficult to obtain from orbifolds. That is the case of the 3^5 model with $|\chi| = 200$ which has Z_5 symmetries. Order 5 modings are not allowed for $N = 1$ supersymmetric orbifolds. Something similar happens with the $1\ 16^3$ model leading to the three generation Yau manifold [16,17].
4. As emphasized, no Z_N -orbifold (apart from $Z'_6 \equiv Z_2 \times Z_3$) appears in the list of known $N = 2$ models [7,8] and neither does model 7 in our table. Contrariwise, all $Z_N \times Z_M$ -orbifolds generated by shifts of the form $(a_1, 0, a_3)$, $(0, b_2, b_3)$ and having $\varepsilon = 1$, are contained in that list. These are models in which each twist generator separately leaves an unbroken $N = 2$ space-time supersymmetry, with the combination of both leaving only $N = 1$. Although the lists in [7,8] do not exhaust all possible $N = 2$ superconformal constructions, the fact that only a couple of the orbifold models with discrete torsion $\varepsilon \neq 1$ agree with a known $N = 2$ model could indicate that the partition function for the tensoring of $N = 2$ minimal models considered in [3] is not the most general one allowed by modular invariance. It could be possible to add some relative phases (discrete torsion in the orbifold language) and still obtain a modular invariant partition function. Work

along these lines is in progress.

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TABLE CAPTION. $Z_N \times Z_M$ -Orbifolds. Here $\alpha = \pm 2\pi/3$ and $\beta = \pm 2\pi/6$. $N = 2$ superconformal models with similar massless spectrum are shown in the last column.

#	Orbifold	a	b	ε	$\chi/2$	$N_{\overline{27}}$	N_{27}	$N=2$ cand.
0	$Z'_6 \equiv Z_2 \times Z_3$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(0, \frac{1}{3}, -\frac{1}{3})$	1	24	11	35	$1^4 2^2 4$ $1^3 2 4 (-10)$
1	$Z_2 \times Z_2$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(0, \frac{1}{2}, -\frac{1}{2})$	1	48	3	51	$1^2 2^2 4^2$ $1 2 4^2 (-10)$
2				-1	-48	51	3	$4^2 (-10)^2$
3	$Z_3 \times Z_3$	$(\frac{1}{3}, 0, -\frac{1}{3})$	$(0, \frac{1}{3}, -\frac{1}{3})$	1	84	0	84	$1^9; 1^7 4; 1^3 4^3$
4				$e^{i\alpha}$	-24	27	3	-
5	$Z_2 \times Z_4$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(0, \frac{1}{4}, -\frac{1}{4})$	1	60	1	61	$1 2^4 4$ $2^3 4 (-10)$
6				-1	12	9	21	-
7	$Z_2 \times Z'_6$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$	1	36	0	36	-
8				-1	0	15	15	-
9	$Z_2 \times Z_6$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(0, \frac{1}{6}, -\frac{1}{6})$	1	48	3	51	$1^2 2^2 4^2$ $4^2 (-10)^2$ $1 2 4^2 (-10)$
10				-1	0	19	19	-
11	$Z_4 \times Z_4$	$(\frac{1}{4}, 0, -\frac{1}{4})$	$(0, \frac{1}{4}, -\frac{1}{4})$	1	90	0	90	2^6
12				-1	42	0	42	-
13				$\pm i$	-6	12	6	-
14	$Z_3 \times Z_6$	$(\frac{1}{3}, 0, -\frac{1}{3})$	$(0, \frac{1}{6}, -\frac{1}{6})$	1	72	1	73	$1^5 4^2$
15				$e^{i\alpha}$	0	13	13	-
16	$Z_6 \times Z_6$	$(\frac{1}{6}, 0, -\frac{1}{6})$	$(0, \frac{1}{6}, -\frac{1}{6})$	1	84	0	84	$1^9; 1^7 4; 1^3 4^3$
17				-1	48	3	51	$1^2 2^2 4^2$ $1 2 4^2 (-10)$ $4^2 (-10)^2$
18				$e^{i\alpha}$	24	3	27	-
19				$e^{i\beta}$	0	9	9	-