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Lasers, microwave tubes, and accelerators: a unified point of view

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ABSTRACT

Many practical devices of technical interest rely for their operation on the coherent interaction between harmonic electromagnetic fields and ensembles of charged particles, constrained to move in orbits or trajectories by electric or magnetic fields. These have in many cases been studied by different communities of physicists and engineers, and different points of view have emerged. At one extreme, a purely quantum mechanical optical approach is appropriate, at the other, classical engineering concepts developed originally for lumped circuits may be more suitable. Some relatively recent developments, such as the gyrotron and free electron laser have features in common with lasers, microwave tubes, and particle accelerators.

Some of these devices will be examined in an informal way from several different view points, with emphasis on a physical rather than mathematical description. Similarities and essential differences will be noted.

1. INTRODUCTION

At first sight, the very large synchrotrons and linear accelerators operating at high energy physics laboratories throughout the world would seem to have little in common with powerful modern lasers producing light in the optical or infra-red portion of the spectrum. Accelerators are essentially classical devices, in which the acceleration is provided by conventional high frequency or microwave generators. Most of the lasers that we shall hear about at this conference, on the other hand, generate radiation by atomic or molecular processes which need quantum mechanics for their correct description. Nevertheless, these devices have much in common with each other and with microwave tubes, and there is a continual gradation from one to the other; 'intermediate' examples are the cyclotron maser and the free electron laser (FEL). It is interesting that contemporary with the first quantum mechanical description of the FEL by Madey in 1972¹ was a suggestion by Palmer² that the inverse mechanism of interaction between fields and particles could be used in an accelerator. Later it was recognized that the basic principle had also been suggested as the basis for a microwave tube in 1947 by Gorn³, and that in 1960 this had been incorporated into the 'Ubitron', a short wave microwave generator⁴. Even more directly, the cyclotron maser concept can be considered from the viewpoints both of accelerator and maser theory; this has been idealized in practice as the gyrotron.

It is often of value to be able to look at phenomena from different points of view, and to work out schemes which show how a whole range of devices are related. This helps when assessing new suggestions, and facilitates inter-disciplinary discussions. When the free electron laser was first being discussed, for example, laser physicists, accelerator physicists, plasma physicists and accelerator physicists all saw features familiar to their own particular discipline, and produced their own description of 'how it works'. This caused a certain amount of confusion initially, and although the 'best' description is one that most readily allows calculations to be made for the purpose of design and analysis of operation, the existence of other descriptions enables it to be seen in better perspective.

Lasers, microwave tubes, and lasers all operate by coherent interaction of electromagnetic radiation with an ensemble of charges, which are constrained by electric or magnetic fields to move in suitable orbits or trajectories. The charges may be electrons in atoms, or electrons or ions in shaped magnetic fields, or in focusing systems.

As remarked earlier, the very direct classical description appropriate to accelerators and microwave tubes looks very different from the quantum description of lasers. It is, of course, possible to describe accelerators in quantum mechanical terms, but this is always unnecessarily complicated in making practical calculations. We know, for example, that when a particle is deflected by a bending magnet or focusing system, spacelike virtual photons are exchanged between the magnet and the charge; this statement is not, however, a starting point for useful calculations, and a useful 'physical picture' is not easy to work out. In the case of a laser, of course, a quantum mechanical viewpoint is necessary; the behaviour cannot be calculated classically. Nevertheless the essentials of laser action can be described classically in various ways, though a quantitative calculation of what happens in a real situation cannot be made. A description which will be made use of in the present

paper has been given by Gaponov, Petelin and Yulpatov⁵. A further paper, by Gover and Yariv, to which the present author is also indebted for ideas, discusses the quantum mechanical description of a number of devices that are normally described classically.

Lasers and microwave tubes generate coherent radiation, accelerators consume it. Historically, the production of coherent radiation, (used here to include alternating fields, which may be considered as 'virtual' radiation) began about 150 years ago with the first alternators. An interesting diagram present by Glass⁷ shows the frequency attained from 1830 to the present time. The development shows the classic S-shaped curve, with exponential increase between 1900 and 1952, from an arc transmitter at 10^4 Hz to a ruby laser at 10^{15} Hz.

Fig. 1. Schematic view of a number of practical devices where electromagnetic energy is converted into particle energy or vice versa. G denotes gun, T collector or target. Power in or out is denoted by $\tilde{W}$ or W , depending on whether it is constant or varies harmonically. $\tilde{W}_{in}$ is zero for an oscillator, and $\tilde{W}_{out}$ is zero for an accelerator.

Fig. 2. Curves showing normalized energy γ versus momentum $\beta\gamma$ for a relativistic charge and a photon. They also show frequency against wave number, for the associated de Broglie or electromagnetic waves.

Before discussing the physical processes in more detail we attempt a simple schematic diagram to include as many devices as possible. Fig. 1 is appropriate to microwave tubes, conventional high energy accelerators, (synchrotrons, linear accelerators and cyclotrons) and also to the free electron laser and cyclotron maser. This covers the middle range of frequencies. At very low frequencies, represented by 50 cycle alternators for example, the input is mechanical power rather than an electron beam. At very high frequency it could be light from flashlamps, though electron beams are also used. In the ammonia maser the electron beam is replaced by a beam of ammonia atoms.

2. INTERACTION OF FREE PARTICLE AND PLANE WAVE; PHOTON VIEWPOINT

Some features of the interaction of charges with radiation will now be discussed both from the quantum mechanical and classical points of view. In this and the following sections we look at the condition for interaction to be possible; later we consider the mechanism of energy interchange. We use the standard relativistic notation; for particles: velocity $= \beta c$, momentum $= \beta\gamma m_0 c$, total energy $\gamma m_0 c^2$, with rest mass $m_0 c^2$. The Einstein relation in normalized form may be written

$$\gamma^2 = \beta^2 \gamma^2 + 1 \quad (1)$$

so that

$$\gamma = (1 - \beta^2)^{-1/2} \\ \approx 1 + \frac{1}{2} \beta^2 \quad (\text{non-relativistic}) \quad (2)$$

and

$$\beta = (1 - \gamma^{-2})^{1/2}$$

$$\approx 1 - \frac{1}{2\gamma^2} \quad (\text{extreme relativistic}) \quad (3)$$

For the associated de Broglie waves, the relations $\gamma m_0 c^2 = i\hbar(\partial/\partial t)$ and $\beta \gamma m_0 c = -i\hbar(\partial/\partial x)$ may be used in eqn. 1 to give the Klein-Gordon equation for particles in free space,

$$c^2 \nabla^2 + \frac{\partial^2}{\partial t^2} = \frac{c^2}{\lambda_c^2}, \quad (4)$$

where λ_c is the Compton wavelength $\hbar/m_0 c$. (This equation does not take account of spin, but is adequate for our purpose). For a harmonic wave of form $\exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$ eqn. 4 gives the hyperbolic dispersion relation for a wave with wave normal along the z-axis

$$\omega^2 - c^2 k^2 = c^2 / \lambda_c^2, \quad (5a)$$

where k is written for k_z . This is illustrated in Fig. 2. Also shown is the dispersion relation for photons. Since $m_0 = 0$ and $\lambda_c = \infty$ it reduces to two straight lines

$$\omega = \pm ck \quad (5b)$$

It is convenient to use the normalized energy γ and momentum $\beta\gamma$, and the scales in Fig. 2 are in the same units for γ and $\beta\gamma$. The particle velocity is $\partial E/\partial p$, equal to $c\partial\gamma/\partial(\beta\gamma) = \beta c$. When $v = c$ the lines have unit slope. Note also that the particle velocity is equal to the group velocity of the de Broglie wave, $\partial\omega/\partial k$.

If an electron emits a photon, its position on the curve moves from point B to point A; to do this it must lose more (normalized) momentum than energy. Since the photon always has equal energy and momentum, the electron cannot therefore emit or absorb a single photon. There are, nevertheless, two ways in which photon emission can be achieved. First the photon can be 'slowed down' by coupling it to matter. This can most simply be done by allowing it to propagate through a dielectric. If this has refractive index n , then the phase and group velocity of the wave are both reduced by a factor n , so that $\beta = 1/n$. Such a wave can interact in a resonant manner with a particle moving at velocity equal to or greater than c/n . The phase velocity of a wave moving at an angle θ to the particle direction is $c/n\cos\theta$, so that resonant interaction occurs when

$$\cos\theta = \frac{1}{\beta n} \quad (6)$$

This is the well-known Cherenkov condition, discussed further below. It is illustrated in Fig. 3.

Fig. 3. Diagram illustrating the Cherenkov condition. The photon is 'slowed down' as shown by the dotted curve. Interaction is now possible if $\beta > 1/n$.

Fig. 4. Diagram illustrating Compton effect. An energetic photon is emitted forwards, and a weak backward photon is absorbed.

Fig. 5. The relation in eqns. 8, 9 and 10 may be obtained geometrically from Fig. 4. The cases of a backscattered photon or an undulator are shown.

The second type of interaction, known as the Compton effect, requires the presence of two photons of different frequency. This is illustrated in Fig. 4, for the case when the emitted photon goes forward, in the same direction as the charge, whereas the absorbed photon is travelling in the opposite direction. The energy lost by the charge is equal to the difference of the energies of the photons. The momentum lost, however, is equal to the sum of their momenta. The reason for this is that the lower energy photon is absorbed, rather than emitted, and also it is travelling in the opposite direction. The ratio energy/momentum lost by the particle is equal to β , the slope of the hyperbola, so that,

denoting the emitted and absorbed photon energies by subscripts 1 and 2

$$\beta = \frac{h(\omega_1 - \omega_2)}{ch(k_1 + k_2)} = \frac{k_1 - k_2}{k_1 + k_2} \quad (7)$$

The photon energies are small compared with that of the charge, so that β does not vary much during the interaction. Re-arranging this equation

$$\frac{k_2}{k_1} = \frac{\lambda_1}{\lambda_2} = \frac{1-\beta}{1+\beta} \quad (8)$$

For a relativistic electron with $\gamma \gg 1$, $\beta = 1 - 1/2\gamma^2$, so that

$$\frac{\omega_2}{\omega_1} = \frac{\lambda_1}{\lambda_2} \approx \frac{1}{4\gamma^2} \quad (9)$$

This is a familiar result, often obtained by arguments involving a Lorentz transformation. The same result can be obtained geometrically from Fig. 4, as illustrated in Fig. 5. It is also seen from this figure that if we replace the absorbed photon by a static magnetic undulator, with finite wavelength but zero frequency, this can be represented by a photon with momentum but no energy. From the geometry of the figure we have in this case, (denoting the undulator wavelength by λ_u)

$$\frac{\lambda_1}{\lambda_u} \approx \frac{1}{2\gamma^2} \quad (10)$$

This type of interaction, in which motion of an electron through an undulator emits photons by stimulated emission, forms the basis of the free electron laser, to be discussed later.

So far Cherenkov and Compton interactions have been considered. Both of these apply to individual particles. In order to satisfy energy and momentum balance Cherenkov radiation requires a photon modified by the presence of matter, whereas the Compton effect relies on two photons. If many particles are present in a single plane wave they can deplete or augment its energy, but collective interaction between the particles themselves have not been included. In the absence of the plane wave the particles would not mutually interact. If the particle density is such that this mutual interaction cannot be neglected, as would be the case for example if the plasma frequency were not small compared to other relevant frequencies, then the situation becomes more complicated. The internal degrees of freedom of the ensemble of particles, which might for example be a uniform plasma, now enter into the problem. This is the regime of the Raman effect, which has many manifestations. Some of these will be referred to later; here we merely quote Marcuse⁸, "The literature on Raman scattering is large and bewildering, because of the many different ways in which the effect can be described".

No mention has yet been made of absorption or emission of photons by charges bound in orbits or oscillating in potential wells arising from the presence of other charges. Under these conditions it is angular momentum and energy that must be conserved. The angular momentum can only change in units of $\hbar$, and the permitted changes in energy depends on the structure of the particular system. The classical description of such systems, particularly with regard to the cyclotron and cyclotron maser, will be given in section 4. An important characteristic of practical systems is that the classical oscillator is non-linear. In general the frequency decreases with energy, and this implies that the energy levels become more closely spaced as the energy increases.

3. CLASSICAL DESCRIPTION OF THE INTERACTION BETWEEN WAVES AND CHARGES a) IN FREE SPACE

In this section the interaction of a plane wave and a charge in free space will be examined from the classical point of view. Since the wave always moves faster than the charge, the latter experiences an oscillatory force, and, for high frequencies and moderate field strengths, does not acquire a high energy. A relativistic particle moving almost as fast as the wave in the same direction as the wave experiences a transverse electric field for a relatively long time, but the effect of this field is reduced by a factor $(1-\beta) \approx 1/2\gamma^2$ by the oppositely directed $\mathbf{v} \times \mathbf{B}$ force in the magnetic field as illustrated in Fig. 6. It is easily shown that in a wave of given strength the maximum values of $\dot{x}$ and x are proportional to $1/\gamma$ and γ respectively. This type of motion will be considered later in connection with the inverse FEL mechanism.

Although perhaps not strictly relevant to the subject of this talk it is of interest to consider the motion of a particle which is initially at rest.

Fig. 6. Interaction of plane wave with relativistic particle moving in the same direction.

Fig. 7. Motion of a particle in a plane electromagnetic wave, a) particle initially at rest, b) particle with initial velocity in a direction opposite to the wave sufficient to cancel the forward velocity.

The motion is shown in Fig. 7. A particle placed at point 1 with $v = 0$ at a time and place where the field has a maximum value moves as shown in (a). The forward motion comes from the $v \times B$ force. After a quarter cycle the particle is at 2, and the field changes sign; after a further quarter cycle it is brought to rest at 3. After the next half cycle the motion repeats with the y -velocity reversed. No net momentum in the z -direction is transferred. Another possible orbit is the 'figure of eight', where there is no net forward motion. This consists of the previous motion together with a steady component in the $-z$ direction as shown in (b).

If the electric field is very large, (or the frequency small) the particle can obtain a relativistic transverse energy in the first quarter cycle. The condition for this is that the cyclotron frequency ω_c in the peak magnetic field should exceed ω , the frequency of the wave. The energy attained still decreases to zero after half a cycle, but for large ω_c/ω it is given by $\gamma \approx (\omega_c/\omega)^2$. This can happen in laser fields or, more strongly, in pulsars. Because the pulsar has a dipole field, which decreases as $1/r$, however $\gamma \approx (\omega_c/\omega)^{2/3}$, but this is still very large. In the Crab Nebular pulsar proton energies of 10-100 TeV can be expected⁹.

Forward acceleration does occur in a plane wave if there is dissipation of energy by radiation. It is simple to calculate the forward force. Without dissipation it is of the form $\langle \cos \omega t \sin \omega t \rangle$, which is zero¹⁰. Dissipation causes a phase shift in one of the components, so that the term in the bracket $\langle \rangle$ is finite. In a non-relativistic calculation, including the classical radiation reaction force $F_x = \frac{2q^2}{3c^3} \ddot{x}$, the accelerating force is (in MKS units, used throughout)

$$\langle F_z \rangle = \frac{1}{3} (4\pi\epsilon_0) r_c^2 E_y^2 \quad (11)$$

where r_c is the classical particle radius,

$$r_c = q^2 / 4\pi\epsilon_0 m_0 c^2 \quad (12)$$

This force is responsible for 'radiation pressure', of importance in astrophysics. If a single photon is emitted, the process may be identified with spontaneous Compton effect. It is a 'second order' force, with the following properties. First, it is proportional to q^2 and E^2 , and second, the wave and particle are not synchronous. It is interesting to note that this is analogous to the non-synchronous force in an induction motor, where the starting torque is zero if the resistivity is zero.

There is a further way of obtaining second-order acceleration, even in the absence of radiation, by making E_y a function of z . By beating together two waves of nearly equal frequency a large scale harmonic variation at the difference frequency is formed. It can be shown that this gives rise to a force in the z -direction given by

$$\langle F_z \rangle = \frac{1}{4} \frac{q^2}{m\omega^2} \text{grad} \langle E^2 \rangle \quad (13)$$

This mechanism is suggested for producing plasma waves in the 'beat wave accelerator'; it was also used in earlier attempts to use r.f. confinement for fusion, (about 1960).

4. CLASSICAL INTERACTION BETWEEN A SINGLE WAVE AND A CHARGES b) IN THE PRESENCE OF MATTER

A particle moving in the same direction as a plane wave sees a field perpendicular to its direction of motion, as described in the previous section. If the wave normal makes an angle θ with the particle direction there is a component of field in the same direction as the particle velocity. Such a wave is illustrated in Fig. 8. The field components, with polarization such that the magnetic field is in the x-direction, and harmonic dependence $\exp i(\omega t - \mathbf{k} \cdot \mathbf{x})$ may be written

$$\begin{aligned} E_y &= -E_0 \cos\theta \exp i[-(z\cos\theta + y\sin\theta)/\lambda + \omega t] \\ E_z &= E_0 \sin\theta \exp i[-(z\cos\theta + y\sin\theta)/\lambda + \omega t] \\ Z_0 H_x &= E_0 \exp i[-(z\cos\theta + y\sin\theta)/\lambda + \omega t] \end{aligned} \quad (14)$$

The value of k_z is reduced to $k_z \cos\theta$, however, and this means that the phase velocity in the z direction is increased to $c/\cos\theta$, so that, since $\cos\theta < 1$, resonant interaction in which the particle velocity equals the phase velocity of the wave is not possible;

$$v_z = \beta_z c = c/\cos\theta > c \quad (15)$$

Fig. 8. Plane wave at an angle θ to the z-axis. The field components are given in eqn. 14.

Fig. 9. Plane wave at an angle θ_1 in a dielectric slab with refractive index n . If $n \cos\theta_1 > 1$ the wave is totally internally reflected, giving rise to the external evanescent fields shown. The form of these fields is given in eqn. 17.

The wave and particle velocities can, however, be made equal by slowing down the wave in a medium of refractive index n , so that the phase velocity in the z-direction is less than c . The condition for this is that $c/n\cos\theta_1$ should be less than c , or $n\cos\theta_1 > 1$. Interaction occurs when $\beta = 1/n\cos\theta_1$; this is the inverse Cherenkov effect, which can in principle be used for particle acceleration. It is interesting that the first suggestion for laser acceleration, by Shimoda¹¹ in 1962, was for a gas filled waveguide using this mechanism. Unfortunately the refractive index n must be large if E_z is to be large, and this leads to problems with breakdown and scattering. A detailed analysis has been given by Fontana and Pantell¹²; the method is not of practical value.

We now consider a dielectric with a planar surface perpendicular to the y-axis. The condition that $n\cos\theta_1 > 1$ is just that for total internal reflection at the surface, by Snell's law. Fields outside the dielectric, shown in Fig. 9, decay exponentially away from the surface. Since $\cos\theta_0 > 1$, $\sin\theta_0 = (1 - \cos^2\theta_0)^{1/2}$ is pure imaginary. Setting $\cos\theta = c/v = 1/\beta$, we find that

$$\sin^2\theta = -1/\beta^2\gamma^2 \quad (16)$$

and the field components in eqn. 14 may be written

$$\begin{aligned} E_y &= \frac{-E_0}{\beta} \exp\left(\frac{-y}{\beta\gamma\lambda_0}\right) \exp i\left(\omega t - \frac{z}{\beta\lambda_0}\right) \\ E_z &= \frac{-iE_0}{\beta\gamma} \exp\left(\frac{-y}{\beta\gamma\lambda_0}\right) \exp i\left(\omega t - \frac{z}{\beta\lambda_0}\right) \\ Z_0 H_x &= E_0 \exp\left(\frac{-y}{\beta\gamma\lambda_0}\right) \exp i\left(\omega t - \frac{z}{\beta\lambda_0}\right) \end{aligned} \quad (17)$$

where λ_0 is the free space wavelength.

Outside the dielectric the field component E_z is $\pi/2$ out of phase with the other two components. All components fall to a value $1/e$ in a distance $\beta\gamma\lambda$ from the surface. The field E_z can interact in a resonant manner with a particle of energy $\gamma m_0 c^2$. This is again the Cherenkov effect, and this kind of interaction, in which the particles move in free space, is used in linear accelerators and travelling wave tubes (TWT). Quantum mechanically the field can be thought of in terms of virtual photons, continually emitted from and re-absorbed by the surface. As γ increases $\cos\theta_0$ decreases towards unity. The photons become more nearly real, and by the uncertainty principle, can move further from the surface before returning.

Since the fields E_z and H_x differ in phase by $\pi/2$, the impedance in the y-direction is purely reactive. Looking outward from the surface it is capacitive, with value $Z_0 \sin\theta_0 = -iZ_0/\beta\gamma$, where $Z_0 = (\mu_0/\epsilon_0)^{1/2}$, the impedance of free space. Since the impedance looking towards the surface is inductive, it is possible to support an evanescent wave on a surface such as that shown in Fig. 10 which consists of parallel plates terminated by a conducting sheet. This can be considered as a form of grating built up from strip transmission lines of length D. The impedance per square, $iZ_0 \tan\omega D/c$, is inductive if the length of the transmission lines is less than $\lambda/4$. The matching condition is

$$i \tan(\omega D/c) = (1 - \cos^2\theta)^{1/2} Z_0 \quad (18)$$

whence

$$\cos\theta = \sec(\omega D/c) \quad (19)$$

Fig. 10. Inductive surface capable of supporting evanescent wave.

Fig. 11. Inductive surface 'rolled up' to form disc loaded waveguide. This can support a wave with phase velocity equal to c with a component of electric field in the z-direction.

Note that the plates can be spaced as closely as required, so that this interaction is not to be identified with the Smith-Purcell effect¹³. These effects are closely related, however, as will be indicated later.

Unfortunately a problem arises if it is required to accelerate relativistic particles with this type of wave. The ratio of the field components (eqn. 17) is

$$|E_z| : |E_y| : |Z_0 H_x| = |\sin\theta| : \cos\theta : 1 = \frac{1}{\beta\gamma} : \frac{1}{\beta} : 1 \quad (20)$$

As $\beta\gamma$ increases towards relativistic energies the decay length $\beta\gamma\lambda$ becomes larger, the wave becomes more nearly a plane wave, and $|E_z|/|E_y| + 1/\gamma \rightarrow 0$. It is clearly desirable in a practical device that this ratio should not become small.

There are two ways of overcoming this difficulty. The first consists of combining two waves, at (complex) angles θ and $-\theta$. This can be done by propagating a wave between two surfaces of the type shown in Fig. 10. If the relative phases are chosen correctly this produces a waveguide mode in which, on the symmetry plane, the transverse fields cancel, to leave only E_z . Away from this plane it may readily be shown, by adding the fields in eqn. 14 to a similar set with $-\theta$ for θ and expanding the resulting cosh and sinh functions for small argument, that $E_z > E_y$ within about λ of the symmetry plane.

If instead of two waves, an axially symmetrical manifold of waves with wave normals in a cone making a (complex) angle θ to the z axis is considered, the hyperbolic functions become I_1 and I_0 Bessel function, and the familiar disc-loaded waveguide configuration illustrated in Fig. 11 is obtained.

Several other related structures capable of supporting slow waves are illustrated in Fig. 12. First, the disc structure could be replaced by a metal backed dielectric, in which total internal reflection is occurring, a scheme considered in the early days of linacs. Second, the system can be turned 'inside out', to form dielectric waveguide, a series of discs on a rod, or, in a two dimensional version, an array of closely spaced dipoles such as is used a guide used to enhance the gain of television antennas. Closely related to these structures, but without axial symmetry, is the helix used in travelling wave tubes.

Fig. 12. Several structures capable of supporting slow waves.

- a) Disc loaded waveguide
- b) Dielectric loaded guide
- c) Array of rods or discs
- d) Dielectric rod.

So far we have considered single waves on infinite systems, with unique frequency and unique wavelength. We now examine infinite periodic systems, where many wavelengths may be associated with a single frequency. A familiar device for which this might be considered an idealization is the particle accelerator, in which acceleration is achieved by successive passage through a series of cavities. These may be in a straight line, as in a linear accelerator, or round an orbit as in a synchrotron.

To obtain continuous acceleration, particles must pass through all the cavities during the half cycle with sign such that on average the field is in the direction to increase the particle energy. For cavities oscillating in phase this implies intervals between cavities of an integral number of cycles, and, a relation between cavity spacing D , frequency f , and particle velocity βc

$$\beta c = fD/n \quad (21)$$

where n is an integer.

Setting $\beta = 1/\cos\theta$, and $c = f\lambda_0$, where λ_0 is the free-space wavelength, this can be written

$$\cos\theta = \frac{n\lambda_0}{D} \quad (22)$$

This is the familiar formula for the directions in which light is scattered from diffraction grating at normal incidence, and represents a line spectrum that is obtained by taking a Fourier transform of the periodic field along the axis. For a diffraction grating $\cos\theta$ can be less than unity, representing radiated waves, or greater than unity, representing evanescent waves moving along the surface. These waves may be coupled together. This is the case in 'Smith-Purcell' radiation¹³, generated when electrons move close to the surface of a grating with sufficiently widely spaced lines. The electrons couple to the evanescent mode, which is in turn coupled to a radiating field. For an accelerator, only waves with $\cos\theta > 1$ are useful. Antenna and particle accelerators are illustrated in Fig. 13.

Fig. 13. The first diagram is a hypothetical infinite 'slot' antenna array, fed by generators in phase. By Huyghens' principle radiation is reinforced in directions such that $n\lambda = D\cos\theta$, but cancels elsewhere. Without the generators, it could represent a diffraction grating. The second diagram represents a portion of a cavity linear accelerator. Making the (unrealistic) idealization that both are infinite and periodic, the bottom diagram represents the spectrum of waves as a function of θ . In the antenna both radiated and evanescent waves are present, but in the linear accelerator only waves with $\cos\theta > 1$ are interesting.

In this section wave configurations suitable for interacting with individual particles by the Cherenkov effect have been described. These are characterized by a phase velocity less than that of light, and a component of E in the direction of propagation. The particles can be within a dielectric, or near its boundary. Alternatively they can be near a surface carrying a surface wave, or within suitable disc or dielectric loaded waveguides or arrays of cavities. For particles in free space, energy can be considered as being transferred by virtual photons which must remain close to the surface. These are spacelike photons with normalized momentum greater than energy, $\beta\gamma > \gamma$, so that $\beta > 1$.

5. CLASSICAL DESCRIPTION OF TWO-WAVE (COMPTON) INTERACTION

Interaction of a particle with two waves in the absence of matter was considered briefly in section 2, and illustrated in Fig. 4. We now look at this from a classical point of view. Returning to the remarks in the first paragraph of section 3, we consider a relativistic particle moving in the same direction as a plane wave with frequency ω_1 . It experiences a transverse force

$$F_1 = qE_1(1-\beta)\cos(1-\beta)\omega_1 t \quad (23)$$

If there is now another wave in the opposite direction with frequency ω_2 this exerts a force

$$F_2 = qE_2(1+\beta)\cos(1+\beta)\omega_2 t \quad (24)$$

The particle sees the same frequency from both of these waves when

$$\frac{\omega_1}{\omega_2} = \frac{1+\beta}{1-\beta} \approx 4\gamma^2 \text{ when } \beta \approx 1, \quad (25)$$

a result already obtained in eqn. 9. For the same field strength the second wave exerts a force which is greater by a factor $(1+\beta)/(1-\beta)$. The second wave can be considered as modulating the particle orbit transversely as shown in Fig. 14. The transverse electric field of the first wave then exerts a force which has a small component along the

Fig. 14. Diagram illustrating two wave interaction, (Compton effect). The second wave can be a photon travelling in the opposite direction to the particle, or a static undulator. We consider the latter case. After travelling a distance $\lambda/(1-\beta)$ the forward wave has moved one wave length ahead of the particle. During this time the particle has 'seen' one cycle of the transverse force. At the same time it has made one cycle of transverse oscillation, driven by the second wave. This produces a force with a net component along the trajectory, with sign and direction depending on the phase of the particle.

trajectory, as illustrated in the figure. The sign and magnitude of the force depend harmonically on the position of the particle in the z -direction. The force on the particle can be considered to arise from a ponderomotive potential, which travels at the same velocity of the particle at resonance, and has amplitude proportional to the product of the amplitudes of both waves¹⁴. This velocity, from eqn. 25, is

$$\beta c = \frac{\omega_1 - \omega_2}{k_1 + k_2}, \quad (26)$$

as in eqn. 7. If the low frequency wave is replaced by a static undulator, then $\omega_2 = 0$. The dynamics of the motion in the ponderomotive potential will be discussed in section 6. This type of interaction is now familiar in the free electron laser. Proposals for using the inverse mechanism in particle accelerators have been made, but the scheme presents a number of problems, and is not likely to be useful¹⁵.

6. CLASSICAL INTERACTION OF PARTICLE WITH ROTATING WAVE: ACCELERATOR AND LASER REGIMES

Classically a particle confined to an orbit by a transverse magnetic or radial electric field can gain energy and angular momentum from, or lose them to, a multipole field rotating

at the same frequency. (We note here that a plane wave can be resolved into two dipole fields rotating in opposite directions). If the angular rotation frequency were independent of energy, continuous energy gain or loss would be possible. This is not in general the case, however. In a uniform magnetic field, for example, the angular frequency ω is equal to $eB/\gamma m_0$, and therefore decreases with energy. In electrostatic fields ω decreases even if relativistic effects are ignored with energy. Strict isochronism is only found in isochronous cyclotrons, where the mean magnetic field is designed to increase with radius in such a way that B/γ remains constant. In all other systems, driven by a wave with constant frequency, the radius of the particle orbit oscillates between two limits, one of which may be zero.

This may be regarded also as a form of two wave interaction, in which one of the waves has a finite frequency and angular mode (or quantum) number, and the other is a static field. The 'wave' is bound to charges and currents, and must be thought of in terms of virtual photons.

As an example we consider the specific case of an electron constrained to move in a circular orbit. This could be in a Coulomb electric field, as in an atom or a uniform magnetic field, as in a cyclotron or cyclotron maser. The rotation frequency is a function of radius; this is illustrated for a uniform magnetic field in Fig. 15, with r plotted against ω_c , in the form of a 'resonance curve'. Let us consider a particle at radius r_0 rotating with frequency ω_0 . This will be strictly resonant only with a rotating wave of frequency ω_0 . Whether the particle gains or loses energy depends on its phase with respect to that of the wave.

Fig. 15. Radius of particle orbit in a uniform magnetic field as a function of its angular velocity. This may be regarded as a 'resonance curve' for a non-linear oscillator.

A discussion of the energy interchange mechanism for this interaction, and those discussed earlier, is given in the next section.

7. ENERGY INTERCHANGE BETWEEN PARTICLE BEAMS AND WAVES

In the previous section three mechanisms for resonant interaction between a particle and wave were described in classical terms. The first of these is the Cherenkov effect, in which a wave, with component of electric field in the direction of propagation, is slowed down by the presence of matter to a phase velocity less than that of light. Such a wave can interact in a resonant manner with a non-relativistic particle. The 'matter' may take the form of a dielectric, a grating structure, a helix or an array of cavities. The particle can travel in free space, but for efficient coupling must be within λ of material bodies.

The second method of interaction requires two waves, or a single plane wave plus magnetic undulator. The two waves combine to produce a harmonic 'ponderomotive potential' travelling with velocity $(\omega_1 - \omega_2)/(k_1 + k_2)$. Although the electric field is perpendicular to the mean direction of motion of the particle, the particle has an alternating transverse velocity such that there is a resonant component of field along the trajectory. This is the mechanism of the free electron laser and the ubitron microwave tube, illustrated in Fig. 14. The third example concerns resonant interaction between a rotating wave and particle constrained to a circular (or near-circular) orbit by a transverse magnetic field or a radial electric field. The simplest example is that of a uniform magnetic field, used in the cyclotron accelerator and idealized cyclotron maser.

So far we have discussed only conditions for synchronism, we shall now discuss the question of energy interchange between the wave and particle.

Perhaps the simplest example to consider first is a travelling wave in a linear accelerator, for example a disc loaded structure illustrated in Fig. 11. Suppose first that a particle is injected at the same velocity as the wave; it will be accelerated or decelerated according to its phase. If accelerated, it will move forward until it reaches a decelerating phase, it will then slow down and oscillate about the phase where the field is zero. Such a particle is described as 'trapped'. In order to obtain continuous acceleration the wavelength of the wave needs to be continuously increased along the waveguide as the particle velocity increases.

If the particle is injected slightly faster than the wave, it may oscillate in phase, but if the wave is too weak, it will move continuously forward. If the wave velocity increases along the accelerator, the particle may be trapped, or it may start to move forward and then reverse. Analogies with motion under gravity on a corrugated surface are obvious. These features are illustrated in Fig. 16.

Fig 16. Schematic potential surfaces for particles moving in harmonic waves with electric field in the direction of motion. a) Uniform motion of wave, and b) with acceleration. Phase ranges of oscillation for trapped particles are shown by the arrows.

Fig. 17. 'Biased pendulum' analogy for motion of accelerated particles. The pendulum is shown at rest at the phase stable angle θ_s .

Analogous behaviour is found in the ponderomotive potential associated with the two-wave interaction used in the FEL. It is also found in the orbital motion in a uniform magnetic field described in the last section. If a particle is rotating with frequency ω_0 at a radius r_0 , and a rotating field of frequency ω_0 is suddenly switched on, oscillations in phase occur and the particle becomes trapped. In this case the radius varies also. In order to obtain continuous acceleration the frequency must be continuously decreased. This is done in the synchro-cyclotron accelerator. In the synchrotron accelerator both the magnetic field and the frequency are increased with time in such a way that the orbit radius remains constant. The accelerating field is provided by resonant cavities. In this case, as explained earlier, the cavity fields can be Fourier analysed into a set of rotating waves, one of which is synchronous with the particle.

The basic concept of accelerator operation has already been sketched. It can be described with reference to a single particle. It is interesting to note that the 'tapered FEL', where particles are trapped and decelerated is a device in the same class. Perhaps it should be designated a decelerator; this point will be discussed later.

Before discussing the characteristics of laser action, which depends on an ensemble of charges interacting with the wave, we note that the particle motion in all the devices that we have discussed obey equations of the same form. In the case of a constant wavelength constant frequency system this is the simple pendulum equation. When acceleration is present the pendulum is biased, as shown in Fig. 17.

The pendulum equation for these two cases can be written quite generally as

$$\ddot{\phi} + \Omega^2 \sin\phi = 0 \quad (\text{unbiased}) \quad (27a)$$

$$\ddot{\phi} + \Omega^2 (\sin\phi - \sin\phi_s) = 0 \quad (\text{biased}) \quad (27b)$$

where ϕ_s is the 'stable phase', about which the particles (or pendulum) oscillate. It is zero for an unbiased pendulum. The first integral of these equations is readily found to be (where subscript 0 denotes initial conditions)

$$\dot{\phi} - \dot{\phi}_0^2 = 2\Omega^2 (\cos\phi - \cos\phi_0) \quad (\text{unbiased}) \quad (28a)$$

$$\dot{\phi} - \dot{\phi}_0^2 = \frac{2\Omega^2}{\cos\phi_s} [\cos\phi - \cos\phi_0 - (\phi_1 - \phi) \sin\phi_s] \quad (\text{biased}) \quad (28b)$$

The phase diagrams of $\dot{\phi}$ against ϕ are shown for the unbiased pendulum and a typical biased pendulum in Fig. 18. To be useful, the quantities ϕ and Ω must be related to the parameters of the particular system under consideration. The details of how this is done may be found in

texts on accelerators and the FEL^{14,16}. It is not difficult to show that for small oscillations (where Ω is not a function of amplitude):

ϕ is proportional to the difference between the particle energy and the energy of a particle for which $\phi = 0$.

Ω is proportional to the accelerating field, and to the guiding field (or undulator field in the case of the FEL). For a two wave Compton interaction it is proportional to the product of the field amplitudes of the two photons.

Fig. 18. Phase diagrams for particles in waves, or pendulum.

(It is important to note that in practical cases Ω may be a function of amplitude so that the above equations do not apply; the physical process, however, is essentially the same). The constant of proportionality $\Delta\gamma/\phi$ relating energy increment to ϕ is proportional to the effective mass m^* of the particle. For a non-relativistic linear accelerator $m^* = m_0$, the rest mass. In the relativistic regime $m^* = \gamma^3 m_0$, the longitudinal relativistic mass. In orbital machines this mass can be negative, depending on the energy and the type of focusing system. This arises because an accelerated particle, if not strongly bound in the radial direction travels on a longer path, so that and its angular velocity, and hence ϕ , decreases. For the cyclotron, $m^* = -\gamma m_0/\beta^2$, which becomes essentially infinite in the non-relativistic region. (This means that phase does not change with energy, which is how a low energy classical cyclotron works. For a true isochronous cyclotron with $B(r)/\gamma = \text{constant}$, m^* is strictly infinite).

Having discussed phase motion of a particle in a harmonically varying field, and emphasized that this is essentially the same for linear or orbital motion, we now discuss laser action.

8. LASER ACTION

Accelerators can function if only a single particle is to be accelerated, but lasers and microwave tubes essentially require interaction with an ensemble of particles. Initially we consider non-interacting particles, and also a laser or microwave field for which the amplitude varies slowly. We do not restrict the discussion to short wavelengths, it is equally applicable, for example, a travelling wave tube operating in the independent particle regime. The TWT normally operates in the collective or Raman regime, to be discussed later, but a ballistic treatment has been given by Webber¹⁷.

This analysis applies to any of the systems so far discussed. We consider motion, as described by eqns. 27a and 28a only, where the stable phase is zero. Wave frequencies and wavelengths are independent of time and distance, and the strengths of fields determining orbits do not vary with time. To be specific, we can focus on a FEL or idealized cyclotron maser, (remembering, however, that the effective particle masses have different sign).

We start by assuming a longitudinally uniform beam of monoenergetic particles, moving through an undulator, or a gas of orbiting monoenergetic relativistic particles in a magnetic field. (These assumptions are not all required, but they preserve simplicity). We then suddenly apply a low amplitude travelling or rotating wave. If this is exactly synchronous with the particle, half the particles are accelerated and half decelerated, depending on their phase. By symmetry the net interchange of energy between wave and particles is zero. If the wave is not quite synchronous, however, the acceleration and deceleration are not quite symmetrical; there is a net energy interchange from wave to particles or vice versa. The particles are represented initially by points on a uniform straight line with constant ϕ_0 , as shown in Fig. 19. They then move round the curves shown,

losing or gaining energy depending on ϕ_0 . The energy interchange as a function of time can be found by finding the change in ϕ as a function of t and ϕ_0 , the initial value of ϕ_0 , and integrating over ϕ_0

$$\langle \phi(t) - \phi_0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} (\phi(t) - \phi_0) d\phi_0$$

Fig. 19. Diagram to illustrate conditions when a wave is suddenly imposed on a monoenergetic beam of particles. The particles are represented by the horizontal line, and they begin to move round the curves as indicated. Note that under 'small signal' conditions Ω is much larger, and the theory only applies if ϕ_0/Ω is small, and the horizontal line in this case is far beyond the separatrix.

Fig. 20. Form of small signal gain curve, showing energy interchange at a small time t for different values of ϕ_0 . The abscissa $x = \frac{1}{2} \phi_0 t$.

Equation 28a cannot be directly integrated. When the wave is weak, so that Ω is small compared with ϕ_0 , and the particles are initially well outside the separatrix, it is possible to find a solution by iteration¹⁶. (Note that this is not the situation illustrated in Fig. 19. The line would be well above the top of the diagram). This leads to the result, for $\phi_0 t \ll 1$,

$$\langle \phi(t) - \phi_0 \rangle = \frac{\Omega^4}{\phi_0^3} (\cos \phi_0 t - 1 + \frac{1}{2} \phi_0 t \sin \phi_0 t) \quad (30)$$

which can be written in the alternative form

$$\langle \phi(t) - \phi_0 \rangle = \frac{\Omega^4 t^3}{16} \frac{d}{dx} \left(\frac{\sin x}{x} \right)^2 \quad \text{where } x = \frac{1}{2} \phi_0 t \quad (31)$$

This is proportional to the differential of the spontaneous emission curve for the undulator, a fact that is evident from a quantum viewpoint, but perhaps surprising classically. With change of sign it is the familiar 'small signal gain curve'. For positive ϕ_0 , where the particles initially move faster than the wave, energy is transferred from the particles to the wave; for negative ϕ_0 , the interchange is in the opposite direction. These remarks apply to a system with 'positive mass' particles; in the negative mass regime, as with the cyclotron maser, the flow is in the opposite direction. The form of the gain curve, given by the function $-(d/dx)(\sin x/x)^2$, is plotted in Fig. 20. This is only valid in the small signal regime, where $\Omega/\phi_0 \gg 1$, and the line representing the initial conditions in Fig. 19 is well beyond the separatrix.

In this analysis we considered a monoenergetic electron beam; it can readily be extended to any distribution in ϕ_0 and $\dot{\phi}_0$ by taking a suitable integral over these variables. An interesting application is the calculation of Landau damping in a plasma in which there is a continuous distribution in ϕ_0 , but initially a uniform distribution in $\dot{\phi}_0$. For a Maxwellian distribution in $\dot{\phi}_0$, $F(\phi_0)$ is a monotonically decreasing function, and the integral is proportional to $\partial F / \partial \phi_0$, which is proportional to $\partial F / \partial v$ over the limited range where the integrand is large. The standard formula is obtained, but the method lacks rigour and must be regarded as illustrative only¹⁶.

The description here is classical, but the essential physical mechanism can be thought of as applying to quantum systems, even though classical theory cannot, of course, be used to derive quantitative results. In a quantum mechanical description there is no phase information, and energies are quantized. Both, however, have the common feature that if a 'population inversion' occurs, in which energetic particles or states predominate, energy can be coherently transferred to the radiation field. Although not specifically demonstrated above using classical arguments, this result also depends on the frequency Ω decreasing as a function of energy⁵, as illustrated in Fig. 15. Quantum mechanically, of course, this is obvious since the density of states decreases with energy if Ω decreases with energy.

So far, interaction between an ensemble of non-interacting particles and a wave has been considered. Of course, in a practical device many more topics have to be considered. In order to calculate such parameters as gain, more information about specific systems is needed. If there is energy interchange between particles and wave, the amplitude of the latter changes with time, and this must be taken into account. The geometrical arrangement in beam type devices must be such that interaction occurs for a finite time only, and the saturation mechanism needs to be examined. In travelling wave tubes and FEL's finite time is ensured by finite length. In the practical manifestation of the cyclotron maser, not only is there angular velocity, as in a cyclotron, but there is also velocity along the axis, so that electrons interact with the field for a finite time. In conventional lasers this finite time arises from the finite lifetime of the transition responsible for the required radiation. The detailed theory of the various devices is specialized and different.

Conventional lasers take many forms, but, though they have many features in common, they cannot simply be compared with the beam devices just considered. A proper classical analogy of a specific device would, in general, involve more than one oscillator, coupled to each other and to the radiation field. A two level system, however, can be described by a single oscillator. This has been analysed in classical terms in detail by Borenstein and Lamb¹⁸. The analysis is readily applied to the ammonia beam maser, where, as with the electron beam lasers, the interaction time with the high frequency field is determined by the length and beam velocity.

As indicated earlier, many devices operate in the Raman regime. Particle accelerators do not, but travelling wave tubes, gyrotrons, and high current free electron lasers operating at millimetre wavelengths in general do operate in this regime. The essential features of the interaction are the same, but the energy interchange is with plasmons (or photons associated with energy gained or lost by internal degrees of freedom in the case of molecules or atoms). The normal classical description for beam devices is in terms of negative and positive energy plasma waves on the beam, which for positive mass are slower and faster respectively than the electron velocity, and vice versa for negative mass. This viewpoint is discussed in Ref. 16, for example. Detailed calculations are often difficult and specialized.

9. CONCLUDING REMARKS

Some characteristics of the interaction between particles and coherent electromagnetic fields have been discussed. An attempt will now be made to review the way in which various practical devices might be classified. First, we note that a single plane wave in free space only interacts weakly with a single particle, whether initially stationary or moving with high velocity.

The presence of matter, or another plane wave of different frequency, is required before resonant interaction becomes possible. Cherenkov interaction can occur if the particle moves in or near a dielectric with sufficiently high dielectric constant, or near a suitably designed periodic conducting (or dielectric) surface. Stimulated Compton interaction can occur if two waves simultaneously interact with the particle. The essential feature of the Compton interaction is the introduction of a second frequency. This can also be achieved by letting the particle move through a static undulator. Alternatively, the particle can be held in a potential well in a static electric field, where it may make an orbit, or oscillate transversely. Yet again, a uniform magnetic field can give rise to a circular orbit or helical trajectory.

If sufficient particles are present to act collectively, then resonant interaction occurs not with the particle, but with a collective mode of the ensemble. In the case of a particle beam this would be a plasma wave on the beam, with phase velocity different from the particle velocity. Quantum mechanically this is described as a plasmon, and the interaction is known as the Raman effect.

These interactions are used in lasers, microwave tubes and particle accelerators. We now enquire into the similarities and differences between these devices. Can they be clearly distinguished? Can any device in which a radiation field is enhanced by interaction with charges be termed a laser? In what sense is an accelerator an inverse laser? (A PASAR perhaps, particle acceleration by stimulated absorption of radiation).

Several comments may be made. First, obviously the restriction to optical wavelengths for a laser is an arbitrary one; X-ray lasers and FEL's operate either side of optical frequencies. Second, lasers tend to operate with cavities with very high mode numbers; the charges interact directly with essentially real photons, whereas in microwave tubes and particle accelerators the interaction is via virtual photons. This may not be considered to be a fundamental difference. With regard to the interaction itself, both in lasers and conventional microwave tubes, the frequency (and wavelength in the case of linear devices

such as the FEL and TWT) of the perturbing field are both independent of time and position. In accelerators, on the other hand, the wavelength (in linear accelerators) or frequency (in synchrotrons) vary during the accelerating cycle except in the extreme relativistic regime where β is essentially unity. In fixed frequency cyclotrons the frequency is constant but the orbit radius increases continuously during the acceleration. Whether the frequency and wavelength are fixed or not does not perhaps constitute a fundamental difference, since both FEL's and TWT's, with tapered wiggler and tapered phase velocity of the wave respectively, have been used in the trapped particle regime to enhance efficiency. Perhaps these devices should be designated 'linear decelerators'.

In the light of these arguments, it would appear that there is no sharp dividing line between these various devices. It is accepted that the discussion in this paper is simplified, and may be naive compared with other discussions of this sort¹⁹. The scope can obviously be widened to include discussion of other phenomena, for example, to draw analogies between technical periodic structures and natural ones.

It is evident that there are many points of view on these topics, and the question remains as to how valuable a further synthesis might be. It is the author's view that an intuitive synthesis of this type, perhaps different for different individuals depending on their background, is helpful for inventing and assessing new concepts.

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