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The Manifolds of Scalar Background Fields in Z_N Orbifolds¹

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ABSTRACT

We determine the Kähler and quaternionic manifolds of the untwisted multiplets of superstrings compactified on $Z_N \subset SU(3)$ symmetric orbifolds. In particular we show that, when moduli and family replicas are considered, there is a nontrivial interference between the massless modes related to $(1,1)$ and $(2,1)$ harmonic forms: the manifolds naturally depend on the net chiral family content and on the family-antifamily paired multiplets.

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It is well known [1,2] that $(2,2)$ superconformal field theories are related to Calabi-Yau manifolds [3] in which the gauge and spin connections are identified.

In these theories the moduli [4], i.e. the "flat directions" of the effective scalar potential, are in one to one correspondence with the massless families (and antifamilies). More precisely, to the moduli that are $(1,1)$ harmonic forms (deformations of the Kähler class) correspond $h_{(1,1)}$ antifamilies, while to the moduli that are $(2,1)$ harmonic forms (deformations of the complex structure) are associated $h_{(2,1)}$ families [5].

The aim of this paper is to study some properties of the Kähler metric of the scalar background fields in compactifications that are limiting cases of Calabi-Yau manifolds. A suitable limit in which to study properties of $(2,2)$ superconformal field theories is given by the so called orbifold limit [6]. Indeed, although orbifolds are singular varieties, string propagation on them is non-singular, so that it is perfectly consistent to study the resulting string theory.

It is easy to classify all orbifolds which possess an irreducible abelian holonomy group contained in $SU(3)$. They are the spaces [6] T_6/Z_N , for appropriate N and $Z_N \subset SU(3)$. If $Z_N \subset SU(3)$ but not in $SU(2)$, these orbifolds yield $N=1$ space-time supersymmetry in four dimensions. If $Z_N \subset SU(2)$ they give rise to $N=2$ space-time supersymmetry.

In this paper we will give the complete list of the manifolds for the scalar untwisted field configurations which correspond to such orbifolds. These results extend a previous study [7] done for the T_6/Z_3 orbifold, and generalize to the family replica results that have been obtained for the (untwisted) moduli sector.²

Our starting point is to make the connection between the $N=1$ ten dimensional supergravity compactified on T_6 and the $N=4$, four dimensional supergravity which is known to possess a $SO(6,6+n)/SO(6) \times SO(6+n)$ coset structure [8], where n is the number of degrees of freedom associated to the ten dimensional gauge fields.

²It should be emphasized that, while the result for the metric of the moduli is exact, at least to each order in α' in the classical string, the result for the metric of the family replicas is only true up to corrections which depend on the gauge couplings.

Let us now consider the scalar background fields on T_6 , given by

$$g_{\alpha\beta}, b_{\alpha\beta}, A_\alpha^I, \phi \text{ and } b_{\mu\nu} \quad (1)$$

$$\alpha, \beta = 1, \dots, 6 \quad \mu, \nu = 1, \dots, 4 \quad I \in \text{gauge group.}$$

We introduce a vielbein variable for the internal metric $\Phi_{\alpha i j}$ such that

$$\Phi_{\alpha}^{ij} \equiv (\Phi_{\alpha ij})^* = 1/2\epsilon^{ijkl}\Phi_{\alpha kl}, \quad \Phi_{\alpha ij}\Phi_{\beta}^{ij} = g_{\alpha\beta}, \quad (2)$$

$$i, j = 1, \dots, 4 \quad \text{in the 4 of } SU(4).$$

We have written the 'flat' internal indices as antisymmetric of $SU(4)$ for later convenience.

In terms of these variables we can define the coset representatives of $SO(6, 6+n)/SO(6) \times SO(6+n)$ as follows³,

$$\begin{aligned} \tilde{L}_{ij}^{\alpha} &= 1/2(\phi^{-3/8}\Phi_{ij}^{\alpha} + \phi^{3/8}\Phi_{\alpha ij} - 3\sqrt{2}\phi^{-3/8}B_{\alpha\tau}\Phi_{ij}^{\tau} + \phi^{-3/8}A_{\alpha}^I A_{\tau}^I \Phi_{ij}^{\tau}), \quad (3) \\ L_{ij}^{\alpha} &= 1/2(\phi^{-3/8}\Phi_{ij}^{\alpha} - \phi^{3/8}\Phi_{\alpha ij} + 3\sqrt{2}\phi^{-3/8}B_{\alpha\tau}\Phi_{ij}^{\tau} - \phi^{-3/8}A_{\alpha}^I A_{\tau}^I \Phi_{ij}^{\tau}), \quad (4) \\ L_{ij}^I &= \phi^{-3/8}A_{\alpha}^I \Phi_{ij}^{\alpha}. \quad (5) \end{aligned}$$

The $\tilde{L}_{ij}^{\alpha}$ are compensator fields, L_{ij}^{α} are the physical fields for the moduli sector and the L_{ij}^I are the scalars coming from the gauge fields. One can easily check that

$$\begin{aligned} \tilde{L}_{ij}^{\alpha}\tilde{L}_{\alpha}^{kl} - L_{ij}^{\alpha}L_{\alpha}^{kl} - L_{ij}^I L_I^{kl} &= \delta_i^k \delta_j^l - \delta_i^l \delta_j^k, \\ \tilde{L}^{\alpha ij} &\equiv (\tilde{L}_{ij}^{\alpha})^* = 1/2\epsilon^{ijkl}\tilde{L}_{kl}^{\alpha}, \\ L^{\alpha ij} &\equiv (L_{ij}^{\alpha})^* = 1/2\epsilon^{ijkl}L_{kl}^{\alpha}, \\ L^{I ij} &\equiv (L_{ij}^I)^* = 1/2\epsilon^{ijkl}L_{kl}^I. \end{aligned} \quad (6)$$

Equation (6) ensures that the L fields are indeed the coset representatives of $SO(6, 6+n)/SO(6) \times SO(6+n)$. Note that in $N=4$ there is an additional $SU(1, 1)/U(1)$ factor parametrized by the variable [9]

$$S = \Phi\phi^{-3/4} + 3\sqrt{2}iD, \quad (7)$$

³For a derivation of eqs. (3-5) see also M. Porrati and F. Zwirner, in preparation.

where D is the dual of the four-dimensional antisymmetric tensor $b_{\mu\nu}$ and $\Phi = \det \Phi_{\alpha ij}$.

Eqs.(3-5) are very convenient for finding the manifolds of scalar background fields for any T_6/Z_N orbifold. Of course, from the string point of view, the method gives rise only to the untwisted states of orbifold compactifications, i.e. those states that can be studied in the field theory limit.

Let us consider now all orbifolds with a Z_N holonomy group contained in $SU(3)$.

There are thirteen possibilities [6], nine of them with $Z_N \subset SU(3)$ but not in $SU(2)$, four of them with $Z_N \subset SU(2)$. The first nine give rise to heterotic strings with $N=1$ space time supersymmetry (or $N=2$ for type II superstrings), the last four give rise to heterotic strings with $N=2$ space-time supersymmetry ($N=4$ in type II superstrings).

We first consider the nine cases with $N=1$ space-time supersymmetry. The moduli sector of these theories was studied in ref. [10,11]. In ref. [12,13] it was shown that this sector, in the heterotic string, has the general form

$$\mathcal{K}_{N=1}^{\text{heterotic}} = \mathcal{K}_{N=2}^{IIA} \times \mathcal{K}_{N=2}^{IIB} \times \frac{SU(1,1)}{U(1)}, \quad (8)$$

where $\mathcal{K}_{N=2}^{IIA}$ and $\mathcal{K}_{N=2}^{IIB}$ are the Kähler manifolds of the $N=2$ vector multiplets in type IIA and IIB superstrings. These manifolds have complex dimension $h_{(1,1)}$ and $h_{(2,1)}$, the number of harmonic $(1,1)$ and $(2,1)$ forms on T_6/Z_N . Formula (8) is best explained by considering the supergravity fields which correspond to $h_{(1,1)}$ and $h_{(2,1)}$. They are⁴

$$(g_{i\bar{j}}, b_{i\bar{j}}) \quad (9)$$

and

$$g_{i,j} \quad (10)$$

respectively.

In the heterotic string these states generate two classes of chiral multiplets. In type II strings [14] these states are paired: with $A_{\mu\bar{j}}, A_{\bar{j}k}$, in IIA

⁴These fields give only the correct counting of states. The appropriate variables corresponding to a standard supergravity form contain interference terms, as it should already be obvious from eqs. (3-5).

and with $A_{\mu\nu\bar{3}}, A_{\mu\bar{3}\bar{3}}, A_{\mu\bar{3}\bar{3}}$ in IIB. Therefore we have the following multiplets in type IIA theory

$$(g_{i\bar{j}}, b_{i\bar{j}}, A_{\mu\bar{3}\bar{3}}), (A_{\mu\bar{3}\bar{3}}, g_{i\bar{j}}), (\phi, b_{\mu\nu}, A_{i\bar{j}k}) \quad (11)$$

while in type IIB we have

$$(g_{i\bar{j}}, b_{i\bar{j}}^c, A_{\mu\bar{3}\bar{3}}), (g_{i\bar{j}}, A_{\mu\bar{3}\bar{3}}), (\phi^c, b_{\mu\nu}^c). \quad (12)$$

Here $\phi^c, b_{\mu\nu}^c$ and $A_{\mu\nu\bar{3}}$ are the additional type IIB-supergravity bosonic fields. The graviphoton corresponds to the extra vector given by A_μ for type IIA, and $A_{\mu\bar{3}k}$ for type IIB.

The Kähler manifolds related to the vector multiplets in eq.(11) and (12) were reported in ref. [13].

We consider now how the general formula given by eq.(8) is modified when, in the heterotic string, we attach the (untwisted) family replicas to the moduli space.

For these orbifold compactifications we know that there is a family replica for each modulus [2,5], more precisely, there are $h_{(2,1)}$ families in the 27 of E_6 and $h_{(1,1)}$ 27 antifamilies. The net number of chiral families is given by $|h_{(1,1)} - h_{(2,1)}|$ while the number of family-antifamily coupled replicas is $h \equiv \min(h_{(1,1)}, h_{(2,1)})$.

Note that for the four orbifolds T_6/Z_N with $Z_N \subset SU(2)$ we have $h_{(1,1)} = h_{(2,1)}$, and E_6 is extended to E_7 .

Five of the orbifolds with $Z_N \subset SU(3)$, but not in $SU(2)$, have $h_{(2,1)} = 0$ and four have $h_{(2,1)} = 1$ [11,13]. Let us consider the two cases separately.

The five cases with $h_{(2,1)} = 0$ have 9, 5 and 3 (anti)families respectively. The nine antifamily case corresponds to $Z_N = Z_3$ and it is special since that Z_3 is the center of $SU(3)$. The moduli space of this T_6/Z_3 orbifold correspond to the exceptional [15,16] N=2 manifold $SU(3,3)/SU(3) \times SU(3) \times U(1)$. The moduli spaces for the 5 and 3 antifamilies cases are $SU(2,2)/SU(2) \times SU(2) \times U(1) \otimes SU(1,1)/U(1)$ and $[SU(1,1)/U(1)]^3$, respectively. The inclusion of (anti)families in these three different cases were all studied in ref. [7]. In ref. [7] indeed, the effective theories corresponding to Z_3, Z_5 and Z_{12} were constructed; the two additional orbifolds Z_7 and Z_8 give the same answer, for the untwisted massless sector, as Z_{12} . In the orbifold limit there is an extra

unbroken gauge group, other than E_6 , namely $SU(3)$ for Z_3 , $SU(2) \times U(1)$ for Z_5 and $U(1)^2$ for Z_7, Z_8 and Z_{12} .

These cases can be readily obtained by looking to eqs. (3-6) and retaining the appropriate Z_N invariant fields. The action of the generic Z_N truncation on the 4 of $SU(4)$ is given by the matrix

$$\Sigma_{ij} = \text{diag}(\alpha, \beta, \gamma, 1), \quad \alpha\beta\gamma = 1. \quad (13)$$

The restriction $\alpha, \beta, \gamma \neq \pm 1$ yields the five above described $h_{(2,1)} = 0$, N=1 models; when one of the entries of the matrix Σ_{ij} equals -1 we get a $h_{(2,1)} = 1$ model, while, e.g. $\alpha = 1$ obviously gives a N=2 model.

Three distinct cases, as we have said, are to be considered:

1. $\alpha = \beta = \gamma = e^{2\pi i/3}$, Z_3 9 (anti)families;
2. $\alpha = \beta \neq \gamma$, Z_5 5 (anti)families;
3. $\alpha \neq \beta \neq \gamma$, Z_7, Z_8, Z_{12} 3 (anti)families.

The fields of equations (3-6) surviving the truncation are found by writing the indices of the 6 of $SO(6)$ in the $(4 \times 4)_A$ of $SU(4)$. Taken into account the identities stemming from the duality constraints given in eq.(6), we find that the remaining fields are

$$\tilde{L}_{4j}^{4i}, L_{4j}^{4i}, L_{4j}^{(i,27)}, \quad i, j = 1, 2, 3, \quad (14)$$

for case 1,

$$\tilde{L}_{4j}^{4i}, L_{4j}^{4i}, L_{4j}^{(i,27)}, \quad i, j = 1, 2,$$

$$\tilde{L}_{4i}^{4i}, L_{4i}^{4i}, L_{4i}^{(i,27)}, \quad i = 3, \quad (15)$$

for case 2,

$$\tilde{L}_{4i}^{4i}, L_{4i}^{4i}, L_{4i}^{(i,27)}, \quad i = 1, 2, 3, \quad (16)$$

for case 3.

^aNotice that the untwisted sector alone has a family content that is in general anomalous with respect to the enhanced orbifold gauge-symmetry. To get a consistent anomaly free theory, it is necessary to include the twisted states.

In eqs. (14-16) we have split the adjoint of E_6 as

$$(248) \rightarrow (1, 78) + (8, 1) + (3, 27) + (\bar{3}, \bar{27})$$

under $SU(3) \times E_6$, and identified the $SU(3)$ gauge and holonomy indices [3, 7, 9]. The $N=4$ constraints reduce to the following set:

$$\begin{aligned} \tilde{L}_{4i}^{4i}(\tilde{L}_{4j}^{4i})^* - L_{4i}^{4i}(L_{4j}^{4i})^* - L_{4i}^{(i,27)}(L_{4j}^{(i,27)})^* &= \delta_i^j, \quad i, j, l = 1, 2, 3, \\ &\text{case 1;} \\ \tilde{L}_{4i}^{4i}(\tilde{L}_{4j}^{4i})^* - L_{4i}^{4i}(L_{4j}^{4i})^* - L_{4i}^{(i,27)}(L_{4j}^{(i,27)})^* &= \delta_i^j, \quad i, j, l = 1, 2, \\ \tilde{L}_{4i}^{4i}(\tilde{L}_{4j}^{4i})^* - L_{4i}^{4i}(L_{4j}^{4i})^* - L_{4i}^{(i,27)}(L_{4j}^{(i,27)})^* &= 1, \quad i = 3, \\ &\text{case 2;} \\ \tilde{L}_{4i}^{4i}(\tilde{L}_{4j}^{4i})^* - L_{4i}^{4i}(L_{4j}^{4i})^* - L_{4i}^{(i,27)}(L_{4j}^{(i,27)})^* &= 1, \quad i = 1, 2, 3, \\ &\text{case 3.} \end{aligned} \quad (17)$$

The constraints worked out in eq.(17) give the complete structure of the manifolds for the scalar fields in all three inequivalent cases; specifically, those manifolds are

$$\begin{aligned} &\frac{SU(3, 3 + 3n)}{U(1) \times SU(3) \times SU(3 + 3n)}, \\ &\text{case 1,} \\ &\frac{SU(2, 2 + 2n)}{U(1) \times SU(2) \times SU(2 + 2n)} \times \frac{SU(1, 1 + n)}{U(1) \times SU(1 + n)}, \\ &\text{case 2,} \\ &\left[\frac{SU(1, 1 + n)}{U(1) \times SU(1 + n)} \right]^3, \quad n = 27 \\ &\text{case 3.} \end{aligned} \quad (18)$$

The interesting cases are the Z_N orbifolds for which $h_{(2,1)} \neq 0$. There are four of them, but only two distinct cases (always at the effective lagrangian level), namely Z_4 whose spectrum contains one family and five antifamilies, and Z_6, Z_8, Z_{12} (with three antifamilies and one family).

The moduli spaces for these orbifolds were reported in ref. [11] and shown to be admissible $N=2$ Kähler manifolds in ref. [13]. We note here that the

$h_{(2,1)}$ manifold $SU(1,1)/U(1)$ corresponds to a $N=2$ minimal model, i.e. to a quadratic holomorphic function [16]. On the other hand, it has been shown on general grounds in ref. [17] that to the $h_{(1,1)}$ manifolds correspond F- functions⁶ which are cubic, due to the Peccei-Quinn symmetry associated to the b_{ij} modes. We will now study the general coset reduction on orbifolds in which families are also present.

We will first consider the Z_4 case. There the moduli sector is given by [11, 13]

$$\mathcal{K}_{\text{moduli}} = \mathcal{K}_{h_{(1,1)}}^{IIA} \times \mathcal{K}_{h_{(2,1)}}^{IIB}, \quad (19)$$

with⁷

$$\begin{aligned} \mathcal{K}_{h_{(1,1)}}^{IIA} &= \frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SU(1,1)}{U(1)} = \\ &= \frac{SU(2) \times SU(2) \times U(1)}{SU(2,2)} \times \frac{SU(1,1)}{U(1)}, \\ \mathcal{K}_{h_{(2,1)}}^{IIB} &= \frac{SU(1,1)}{U(1)} \Big|_{\text{minimal}}. \end{aligned} \quad (20)$$

If we include families and antifamilies of E_6 , we get additional states; in order to find them, as well as to understand the coset decomposition of our Z_4 truncation we use again eq.(13), this time with $\alpha = \beta = i$, $\gamma = -1$. the following fields survive:

$$\begin{aligned} \tilde{L}_{4j}^{4i}, L_{4j}^{4i}, L_{4j}^{(i,27)}, \quad i, j = 1, 2, \\ \tilde{L}_{43}^{43} \equiv a, \tilde{L}_{43}^{12} \equiv b, L_{43}^{43} \equiv L^0, L_{43}^{12} \equiv L^1, \\ L_{43}^{(3,27)} \equiv L^{(27)}, L_{43}^{(\bar{3},\bar{27})} \equiv L^{(\bar{27})}, \end{aligned} \quad (21)$$

and the constraints of eq.(6) reduce to

$$\begin{aligned} \tilde{L}_{4i}^{4i}(\tilde{L}_{4j}^{4i})^* - L_{4i}^{4i}(L_{4j}^{4i})^* - L_{4i}^{(i,27)}(L_{4j}^{(i,27)})^* &= \delta_i^j, \\ i, j, l &= 1, 2; \quad (22) \\ |a|^2 + |b|^2 - |L^0|^2 - |L^1|^2 - L^{(27)}(L^{(27)})^* - L^{(\bar{27})}(L^{(\bar{27})})^* &= 1, \quad (23) \\ ab - L^0 L^1 - L^{(27)} L^{(\bar{27})} - L^{(\bar{27})} L^{(27)} &= 0. \quad (24) \end{aligned}$$

⁶See ref. [16] and references therein for an explanation of the $N=2$ tensor-calculus formalism.

⁷Here we omit the extra $SU(1,1)/U(1)$ containing ϕ and the dual of $b_{\mu\nu}$.

Equation (22) is the standard constraint for a $SU(2, n)/SU(2) \times SU(n) \times U(1)$ manifold that we have already encountered, while eqs.(23,24) yield a $SO(2, n)/SO(2) \times SO(n)$ manifold. It follows thereby that the manifold $\mathcal{K}$ is no longer factorized in $h_{(1,1)}$ and $h_{(2,1)}$; instead, it factorizes in chiral and non-chiral states as

$$\begin{aligned}\mathcal{K}_{N=1}^{\text{heterotic}} &= \mathcal{K}^C \times \mathcal{K}^{NC}, \\ \mathcal{K}^C &= \frac{SU(2, 2(1+n))}{SU(2) \times SU(2(1+n)) \times U(1)}, \\ \mathcal{K}^{NC} &= \frac{SO(2, 2(1+n))}{SO(2) \times SO(2(1+n))}, \quad n = 27.\end{aligned}\quad (25)$$

This factorization property simplifies the integration over the paired states, in the case they get a large mass. The manifold of the effective theory would then be given by $\mathcal{K}^C$.

We now study the other cases, i.e. Z_6, Z_8 and Z_{12} .

In these cases the moduli space is [11,13]

$$\begin{aligned}\mathcal{K}_{\text{moduli}, h_{(1,1)}} &= \frac{SO(2, 2)}{SO(2) \times SO(2)} \times \frac{SU(1, 1)}{U(1)} = \left[\frac{SU(1, 1)}{U(1)} \right]^3, \\ \mathcal{K}_{\text{moduli}, h_{(2,1)}} &= \frac{SU(1, 1)}{U(1)} \Big|_{\text{min}}.\end{aligned}\quad (26)$$

The family sector can be studied along the same lines of the Z_4 case, the only difference being that now in eq.(13) $\alpha \neq \beta, \gamma = -1$. The fields invariant under Σ are now

$$\begin{aligned}\tilde{L}_{4i}^{4i}, L_{4i}^{4i}, L_{4i}^{(i, 27)}, \quad i = 1, 2, \\ \tilde{L}_{43} \equiv a, \tilde{L}_{43}^{12} \equiv b, L_{43}^{43} \equiv L^0, L_{43}^{12} \equiv L^1, \\ L_{43}^{(3, 27)} \equiv L^{(27)}, L_{43}^{(\bar{3}, \bar{27})} \equiv L^{(\bar{27})}.\end{aligned}\quad (27)$$

The constraints (23,24), yielding the $SO(2, n)/SO(2) \times SO(n)$ manifold remain therefore unchanged, while eq.(22) reduces to

$$\tilde{L}_{4i}^{4i}(\tilde{L}_{4i}^{4i})^* - L_{4i}^{4i}(L_{4i}^{4i})^* - L_{4i}^{(i, 27)}(L_{4i}^{(i, 27)})^* = 1, \quad i = 1, 2. \quad (28)$$

By eq.(28) and the above considerations, it follows that

$$\begin{aligned}\mathcal{K}^C &= \left[\frac{SU(1, 1+n)}{SU(1+n) \times U(1)} \right]^2, \\ \mathcal{K}^{NC} &= \frac{SO(2, 2(1+n))}{SO(2) \times SO(2(1+n))}, \quad n = 27.\end{aligned}\quad (29)$$

Finally we consider the last sequence of orbifolds, namely the four cases with $Z_N \subset SU(2)$. These orbifolds have $N=2$ space time supersymmetry, and the gauge group always includes E_7 .

There are two distinct cases, namely $Z_N = Z_2$ and $Z_N = Z_3, Z_4, Z_6$. For $Z_N = Z_2$ the additional gauge symmetry is $SU(2)$, while for the other cases it is $U(1)$.

The moduli space of these two cases takes the form [11,13]

$$\mathcal{M}_{\text{moduli}}^{N=2} = \mathcal{K} \times \mathcal{Q}, \quad (30)$$

where $\mathcal{K}$ must also contain a $SU(1, 1)/U(1)$ space, containing the dilaton and antisymmetric tensor. This is compatible with $N=2$ supersymmetry only if $\mathcal{K}$ is given by [16]

$$\mathcal{K} = \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, n)}{SO(2) \times SO(n)}. \quad (31)$$

For $Z_N = Z_2$, $h_{(2,1)} = h_{(1,1)} = 5$, so that the real dimension of the $\mathcal{M}$ manifold is twenty. These twenty modes correspond to a quaternionic manifold, which we have denoted by $\mathcal{Q}$, of quaternionic dimension four, and to a Kähler manifold $\mathcal{K}$ of complex dimension two. This is easily seen [7] by noticing that there are four vectors, $g_{\mu a}$, $b_{\mu a}$, $a = 1, 2$, that are Z_2 singlets. One of them is the graviphoton, one is the vector partner of the dilaton and antisymmetric tensor and the other two give two additional vector multiplets. Specifically, the complete moduli space in the Z_2 orbifold case, is given by

$$\mathcal{M} = \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2)}{SO(2) \times SO(2)} \times \frac{SO(4, 4)}{SO(4) \times SO(4)}. \quad (32)$$

When families and vector multiplets are added, we have to look to the Z_2 invariants in the coset reduction. Since the matrix Σ defining the action

of Z_2 on the 4 of $SU(4)$ corresponds to $\alpha = \beta = -1$, $\gamma = 1$, it is useful to split the indices of $SU(4)$ into $SU(2) \times SU(2)'$ indices, where $SU(2)'$ is inert under the truncation. The adjoint of E_8 decomposes under $SU(2) \times E_7$ as $(1, 133) + (3, 1) + (2, 56)$, and we define the truncation on the gauge $SU(2)$ as on the holonomy $SU(2)$. We are thereby led to the following set of Z_2 invariant fields

$$\begin{aligned} \tilde{L}_{ij}^{ai}, \tilde{L}_{ij}^{ab} &\equiv a, \tilde{L}_{ab}^{ab} \equiv b, \\ \tilde{L}_{ij}^{ai}, \tilde{L}_{ij}^{ab} &\equiv L_{ij}^a, \tilde{L}_{ab}^{ab} \equiv L_{ab}^1, \\ L_{ij}^{(2,56)}, L_{ab}^{(1,133)} &\equiv L_{ij}^{(1,133)}, L_{ab}^{(3,1)} \equiv L_{ab}^{(3,1)} \quad i, j \in SU(2), a, b \in SU(2)'. \end{aligned} \quad (33)$$

Reducing the $N=4$ constraints accordingly, the following set of equations is obtained

$$\begin{aligned} \tilde{L}_{ij}^{ai}(\tilde{L}_{ij}^{aj})^* - L_{ab}^a(L_{ij}^{ab})^* - L_{ab}^{(2,56)}(L_{ij}^{(2,56)})^* &= \delta_{ai}^{aj}, \\ i, j, l \in SU(2), \quad a, b, c \in SU(2)', & \quad (34) \\ |a|^2 + |b|^2 - |L^{012}|^2 - |L^{112}|^2 - L_{(1,133)}^{(1,133)}(L^{(1,133)})^* - L_{(3,1)}^{(3,1)}(L^{(3,1)})^* &= 1; \quad (35) \\ ab - L^0 L^1 - L_{(1,133)}^{(1,133)} L_{(3,1)}^{(3,1)} - L_{(3,1)}^{(3,1)} L_{(3,1)}^{(3,1)} &= 0. \quad (36) \end{aligned}$$

The constraints given in eq.(35), (36) are analogous to those appearing in (23), (24), while eq.(34) gives a quaternionic manifold $SO(4, n)/SO(4) \times SO(n)$ [18]. Therefore, the manifold of scalar matter fields is

$$\mathcal{M} = \mathcal{K} \times \mathcal{Q}$$

with

$$\begin{aligned} \mathcal{K} &= \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2 + n_V)}{SO(2) \times SO(2, 2 + n_V)}, \quad n_V = 133 + 3 + 248 \\ \mathcal{Q} &= \frac{SO(4, 4 + 2n_H)}{SO(4) \times SO(4, 4 + 2n_H)}, \quad n_H = 56. \end{aligned} \quad (37)$$

In the three remaining cases we get $h_{(1,1)} = h_{(2,1)} = 3$.

The manifold of scalar gauge-singlet fields is fourteen-dimensional. Since the number of vector fields is the same as before because $6 \rightarrow 2(1, 1) + (2, 2)$ under $SU(2) \times SU(2)'$, we have for the moduli two vector multiplets and two

hypermultiplets (in addition to the vector multiplet coming from ϕ and $b_{\mu\nu}$), the Kähler manifold is the same as before. The quaternionic manifold turns out to be

$$\frac{SO(2, 4)}{SO(2) \times SO(4)} = \frac{SU(2, 2)}{SU(2) \times SU(2) \times U(1)}.$$

We are not going to repeat in detail the analysis of the coset reduction for these truncations, since it is very similar to the Z_2 case. Suffice it to say that all three cases Z_3, Z_4, Z_6 are obtained by choosing, as usual, $\alpha \neq \beta$, and that the quaternionic manifold becomes

$$\mathcal{Q} = \frac{SU(2, 2 + n_H)}{SU(2) \times SU(2 + n_H) \times U(1)}, \quad n_H = 56; \quad (38)$$

while the Kähler manifold remains the same as for Z_2 .

Note that these $N=2$ and $N=1$ manifolds (with the exception of Z_2) are different from those that are obtained by the fermionic construction of ref. [19]. There $N=1$ space time supersymmetry corresponds to a $Z_2 \times Z_2 \in SU(3)$ while $N=2$ unbroken supersymmetry can be obtained with our Z_2 truncation. The untwisted sector of those theories gives $SO(2, n)/SO(2) \times SO(n)$ and $SO(4, n)/SO(4) \times SO(n)$ manifolds for vector multiplets and hypermultiplets respectively.

In conclusion, we have constructed all manifolds for the untwisted sector of arbitrary orbifolds with $Z_N \subset SU(3)$. We have studied the coupling of the moduli sector to the family replicas.

From our analysis it turns out that the manifold of scalar background fields corresponding to (1,1) and (2,1) harmonic forms is factorized in the moduli sector, but no longer when gauge-nonsinglets are included. In the last case it rather factorizes in two spaces related to the net number of chiral families and to the number of (non-chiral) pairs of families and antifamilies.

The previous results are somehow different from what one would have naively expected by looking at the kinetic terms presented in ref. [20]. The point is that eqs.(3-5) mix the (2,1) and (1,1) harmonic forms related to the families and antifamilies.

For the family sector, in order to obtain the full effective action, we have to work out also the superpotential term. This can be obtained through the gravitino mass matrix of the $N=4$ theory by Z_N reduction [8,18,21,22]. In

the case of $N=1$ space-time supersymmetry, with gauge group E_6 , this mass term is given by

$$m_{3/2} = L_i^a L_j^b L_k^c \epsilon^{ijk} \epsilon_{abc} d_{IJK}. \quad (39)$$

Here we have decomposed the indices of the 6 of $SU(4)$ into $3+\bar{3}$, and denoted by d_{IJK} the $(27)^3$ coupling of E_6 . The superpotential is readily obtained once we notice that the (unconstrained) fields y in which the superpotential is analytic are always given by a rescaling of the L fields:

$$L_i^{aI} = Y^{1/2} y_i^{aI}, \quad Y = \text{rescaling factor}. \quad (40)$$

In the Z_4 model, for instance, one finds no coupling for the 27 of E_6 (the family multiplet) and the following superpotential for the five $\bar{27}$ antifamilies

$$g(y_i^{aI}, y_j^{bJ}) = \epsilon^{ij} \epsilon_{ab} d_{IJK} y_i^{aI} y_j^{bJ} y_k^{cK}, \quad i, j, a, b = 1, 2. \quad (41)$$

Therefore the superpotential, unlike the Kähler potential, is diagonal in the $(1,1)$ and $(2,1)$ forms [20].

We have not addressed the question of what happens when the twisted sector is added. There are indications [22] that in this case the scalar field manifold is a non-symmetric, non-homogeneous space.

The coset space structure of the untwisted sector is merely a consequence of the Z_N reduction of the ten-dimensional supergravity lagrangian. However, some of the features studied in this paper seem to be typical of any orbifold-type compactifications, and may apply as well to the more general spaces which contain all massless scalars.

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