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**On the Algebraic Characterization of Witten's
Topological Yang-Mills Theory**

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ABSTRACT

We interpret in terms of "basic" cohomology the recently proposed supersymmetric, supergauge invariant formulation of topological Yang-Mills theory. Our interpretation shows that this formulation leads to the correct observables.

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1. Introduction

In a recent series of articles [1], E. Witten investigates the expression of various topological invariants in terms of local field theory. The first examples of this sort we know of are due to A.S. Schwarz [2] who gives a field theory expression for the Ray-Singer analytic torsion [3], and are related to the quantization of differential forms [4]. The situations considered by E. Witten are of a more exotic type and lead to essentially non-linear field theories, to be treated in the weak coupling regime. In principle, to obtain the sort of results one expects, a rigorous treatment of the renormalized perturbation expansion ought to be sufficient for a rigorous mathematical construction. Here, we shall be concerned with gauge fields and the recently discovered Donaldson invariants [5].

The following construction owes much to seminars by I.M. Singer, L. Baulieu [6] and P. Braam [7]. However, since local field theory is to be used [8], we find it necessary to characterize the model by a complete set of Ward identities. We believe that ref.[6], as well as subsequent proposals [9] are incomplete in this respect. The solution is to be found in an article by T.H. Horne [10]. The purpose of this note is to explain why, in more geometrical terms.

2. The differential algebra

As stated in E. Witten's paper [1a] (eq. (2.41)) and emphasized in ref.[6], one wishes to gauge-fix a topological invariant, e.g.

$$S_{inv}(A) = \int_M \text{tr}(F(A) \wedge F(A)) \quad , \quad (1)$$

where $F(A)$ is the curvature of a connection A on a principal G -bundle $P(M, G)$, over a compact 4-manifold M , without boundary, and tr is an invariant polynomial over $\text{Lie } G$. The group G is assumed to be compact.

The action S_{inv} is, by essence, invariant under arbitrary variations of A :

$$\delta A = \psi \quad . \quad (2)$$

From now on, all fields are differential forms on M , taking values in $\text{ad}(\text{Lie } G)$. One insists on gauge-fixing S_{inv} , leaving the gauge freedom pending till the end (because of the known Gribov difficulty), localizing the system on the self-dual connections. The corresponding

gauge fixing term is:

$$S^{(1)} = \int_M \text{tr}(b \wedge b - b \wedge F^- - \bar{\psi} \wedge (D\psi)^-) \quad , \quad (3)$$

where F^- is the antiself-dual part of $F(A)$ - for some metric g on M -, b and $\bar{\psi}$ are antiself-dual two-forms and $(D\psi)^-$ is the antiself-dual part of $D\psi$, the covariant differential of the one-form ψ .

The new action $S_{inv} + S^{(1)}$ is invariant under the Slavnov symmetry:

$$\begin{aligned} s_1 A &= \psi \quad , \quad s_1 \psi = 0 \quad , \\ s_1 \bar{\psi} &= b \quad , \quad s_1 b = 0 \end{aligned} \quad (4)$$

and satisfies

$$S^{(1)} = s_1 \int_M \text{tr}(\bar{\psi} \wedge (b - F^-)) \quad . \quad (5)$$

It is still gauge invariant, i.e. invariant under:

$$\begin{aligned} \delta A &= D\omega \quad , \\ \delta \psi &= [\psi, \omega] \quad , \\ \delta \bar{\psi} &= -[\omega, \bar{\psi}] \quad , \\ \delta b &= [b, \omega] \quad , \end{aligned} \quad (7)$$

where $\omega \in \text{Lie } \mathcal{G}$, $\mathcal{G}$ being the gauge group of $P(M, G)$. This yields the nilpotent s_2 operation:

$$\begin{aligned} s_2 A &= \psi - D\omega \quad , \\ s_2 \psi &= [\psi, \omega] \quad , \\ s_2 \bar{\psi} &= -[\omega, \bar{\psi}] + b \quad , \\ s_2 b &= [b, \omega] \quad , \\ s_2 \omega &= -\frac{1}{2}[\omega, \omega] \quad . \end{aligned} \quad (8)$$

The action $S_{inv} + S^{(1)}$ is invariant under s_2 and does not depend on ω (the Faddeev-Popov ghost for $\text{Lie } \mathcal{G}$). Except for the $\int b \wedge b$ term, it is invariant under (φ is odd):

$$\begin{aligned} \delta b &= [\bar{\psi}, \varphi] \quad , \\ \delta \psi &= D\varphi \quad , \end{aligned} \quad (9)$$

which yields the nilpotent operation s (now with φ even):

$$\begin{aligned}
sA &= \psi - D\omega \quad , \\
s\psi &= [\psi, \omega] - D\varphi \quad , \\
s\bar{\psi} &= -[\omega, \bar{\psi}] + b \quad , \\
sb &= [b, \omega] - [\bar{\psi}, \varphi] \quad , \\
s\varphi &= [\varphi, \omega] \quad , \\
s\omega &= -\frac{1}{2}[\omega, \omega] + \varphi \quad .
\end{aligned} \tag{10}$$

It is easy to modify $S^{(1)}$ in such a way that it is invariant under (10). Following eq. (5) we find:

$$\hat{S}^{(1)} = s \int_M \text{tr}(\bar{\psi} \wedge (b - F^-)) = \int_M \text{tr}(b \wedge b - b \wedge F^- - \bar{\psi} \wedge (D\psi)^- + \bar{\psi} \wedge [\bar{\psi}, \varphi]) \tag{11}$$

Notice that in eq.(10), sb needs a φ dependent term in order for s to be nilpotent. The φ invariance can be gauge fixed in a gauge invariant way using the gauge function $D^*\psi$:

$$\hat{S}^{(2)} = \int_M \text{tr}(*\beta \wedge D^*\psi + *\bar{\varphi} \wedge (D^*D\varphi + [* \psi, \psi]) \tag{12}$$

Including $\hat{S}^{(2)}$ in the action, one gets the Slavnov symmetry defined by (10), together with:

$$\begin{aligned}
s\bar{\varphi} &= \beta + [\bar{\varphi}, \omega] \quad , \\
s\beta &= -[\omega, \beta] + [\varphi, \bar{\varphi}] \quad ,
\end{aligned} \tag{13}$$

such that we have the following expression:

$$\hat{S}^{(1)} + \hat{S}^{(2)} = s \int_M \text{tr}(*\bar{\varphi} \wedge D^*\psi + \bar{\psi} \wedge (b - F^-)) \tag{14}$$

A few remarks are in order:

i. $S_{inv} + \hat{S}^{(1)} + \hat{S}^{(2)}$ is not quite the most general ω independent, gauge invariant, renormalizable action invariant under s - actually of the form $S_{inv} + sS_{gf}$: one may add an extra term compatible with ghost number neutrality and renormalizability as in refs. [1,6,9,10] and of the form $s \int \text{tr}(\beta[\varphi, \bar{\varphi}])$. Both ω independence and gauge invariance are essential.

ii. Changing generators according to

$$\begin{aligned}
\psi' &= \psi - D\omega \quad , \\
b' &= b - [\omega, \bar{\psi}] \quad , \\
\varphi' &= \varphi - \frac{1}{2}[\omega, \omega] \quad , \\
\beta' &= \beta + [\bar{\varphi}, \omega] \quad ,
\end{aligned} \tag{15}$$

the s -operation assumes the form:

$$\begin{aligned}
sA &= \psi' \quad , \quad s\psi' = 0 \quad , \\
s\bar{\psi} &= b' \quad , \quad sb' = 0 \quad , \\
s\omega &= \varphi' \quad , \quad s\varphi' = 0 \quad , \\
s\bar{\varphi} &= \beta' \quad , \quad s\beta' = 0 \quad .
\end{aligned} \tag{16}$$

It therefore has vanishing cohomology as well as vanishing cohomology mod d . The desired cohomology [1] is, however, not the local cohomology of s mod d , but its restriction to ω independent, gauge invariant objects. What is involved is equivariant cohomology [7,11], or rather its original form, namely “basic” cohomology [12], which is exactly adapted to the present local field theory context, as we shall see.

iii. It is interesting to observe that s can be split into a sum of two anticommuting differentials and that the algebra can be cast in a supersymmetric form, which is *distinct* from that of ref. [10], if we insist that, as we will demonstrate, s generates the supersymmetry. However, the superfield content will be that of ref. [10] (without imposing the gauge condition $\omega = 0$). Details are given in the appendix.

3. The “basic” cohomology of s

The differential algebra defined by the structure equations (10) and (13) has the following property, which makes it a differential algebra with an action of the gauge Lie algebra; for $\lambda \in \text{Lie } \mathcal{G}$ define:

$$\begin{aligned}
\delta_\lambda \psi &= [\psi, \lambda] \quad , \quad \delta_\lambda A = D\lambda \quad , \\
\delta_\lambda b &= [b, \lambda] \quad , \quad \delta_\lambda \bar{\psi} = -[\lambda, \bar{\psi}] \quad , \\
\delta_\lambda \varphi &= [\varphi, \lambda] \quad , \quad \delta_\lambda \omega = -[\lambda, \omega] \quad , \\
\delta_\lambda \bar{\varphi} &= [\bar{\varphi}, \lambda] \quad , \quad \delta_\lambda \beta = -[\lambda, \beta] \quad .
\end{aligned} \tag{17}$$

Define also for $\lambda \in \text{Lie } \mathcal{G}$, ι_λ by

$$\begin{aligned}\iota_\lambda A = \iota_\lambda \psi = \iota_\lambda \bar{\psi} = \iota_\lambda b = \iota_\lambda \varphi = \iota_\lambda \bar{\varphi} = \iota_\lambda \beta = 0 \quad , \\ \iota_\lambda \omega = \lambda \quad .\end{aligned}\tag{18}$$

One can easily check that

$$\delta_\lambda = \iota_\lambda s + s \iota_\lambda \tag{19}$$

and one has the classical [12] commutation rules:

$$\begin{aligned}[\iota_\lambda, \iota_\mu]_+ &= 0 \quad , \\ [\delta_\lambda, \delta_\mu]_- &= \delta_{[\lambda, \mu]} \quad , \\ [\delta_\lambda, \iota_\mu]_- &= \iota_{[\lambda, \mu]} \quad , \\ [s, \iota_\lambda]_+ &= \delta_\lambda \quad , \\ [s, \delta_\lambda]_- &= 0 \quad .\end{aligned}\tag{20}$$

This makes $\{\delta_\lambda, \iota_\mu | \lambda, \mu \in \text{Lie } \mathcal{G}\}$ into a graded Lie algebra. Recall that $S_{tot} = S_{inv} + \hat{S}^{(1)} + \hat{S}^{(2)}$ fulfils:

$$s S_{tot} = \delta_\lambda S_{tot} = \iota_\mu S_{tot} = 0 \quad , \quad \lambda, \mu \in \text{Lie } \mathcal{G} \quad .\tag{21}$$

In technical terms S_{tot} is a “basic” [12] local functional for the differential structure (10),(13), with the Lie $\mathcal{G}$ action defined by (17),(18).

Now let us turn λ and μ into ghosts, in the usual fashion (λ odd and μ even) and define

$$W = \delta + \iota \quad ,\tag{22}$$

where δ and ι are obtained by (17),(18) on all fields except for A, ω , λ and μ for which

$$\begin{aligned}WA &= -D\lambda \quad , \\ W\omega &= -[\lambda, \omega] - \mu \quad , \\ W\lambda &= -\frac{1}{2}[\lambda, \lambda] \quad , \\ W\mu &= [\mu, \lambda] \quad .\end{aligned}\tag{23}$$

One easily shows that $W^2 = 0$. Adjoining λ and μ as new generators to our differential algebra, we still have a choice to define $s\lambda$ and $s\mu$. In particular, if we define s on λ and μ by:

$$s\lambda = \mu \quad , \quad s\mu = 0 \quad ,\tag{24}$$

we obtain

$$[s, W]_+ = 0 \quad . \quad (25)$$

The comparison with the supersymmetric formalism of ref. [10] is now straightforward. In terms of the primed variables defined in eq. (15), one may introduce the superfields:

$$\begin{aligned} \mathcal{A}_x &= A + \theta\psi' \quad , \\ \mathcal{A}_\theta &= \omega + \theta\varphi' \quad , \\ \bar{\Psi} &= \bar{\psi} + \theta b' \quad , \\ \bar{\Phi} &= \bar{\varphi} + \theta\beta' \quad . \end{aligned} \quad (26)$$

Then one has

$$s = \frac{\partial}{\partial\theta} \quad . \quad (27)$$

The supergauge transformation ghost

$$\Lambda = \lambda + \theta\mu \quad , \quad (28)$$

fulfils $W\Lambda = -\frac{1}{2}[\Lambda, \Lambda]$ and s still acts on Λ by $\partial/\partial\theta$. W acts on all fields by supergauge transformations, with $\mathcal{A}_x$, $\mathcal{A}_\theta$ a superconnection and $\bar{\Psi}$, $\bar{\Phi}$ transforming under the adjoint representation.

In terms of the unprimed variables the action and local cohomology mod d are characterized by ω independence and gauge invariance, as we have already remarked. In terms of the primed variables and the supersymmetric formulation of ref. [10], this is equivalent to supersymmetry (invariance under $\partial/\partial\theta$) and supergauge invariance. So, this equivalence proves in particular that the supersymmetric supergauge invariant cohomology is identified with the “basic” cohomology, which is known to be correct ([1,6,7]). We refer to ref. [10] for the s -invariant gauge fixing of W .

4. Concluding remarks

The algebraic set-up proposed in ref. [10] has been shown to describe the “basic” cohomology adapted to the characterization of a perturbative treatment [8] of the situation described by E. Witten [1] in terms of equivariant cohomology [7,11]. There are two heavy technical problems to be dealt with:

- i.* Perturbative renormalization theory for a field theory associated with an arbitrary compact manifold without boundary in a particular topological sector.
- ii.* The proper treatment of different vacua and the inclusion in the $s - W$ operation of global zero modes, that ought to make the theory not completely empty.

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Appendix

One may split s in eqs. (10,13) as

$$s = \sigma + w \quad , \quad (A.1)$$

with:

$$\begin{aligned} \sigma A &= \psi \quad , \quad \sigma \psi = 0 \quad , \\ \sigma \bar{\psi} &= b \quad , \quad \sigma b = 0 \quad , \\ \sigma \omega &= \varphi \quad , \quad \sigma \varphi = 0 \quad , \\ \sigma \bar{\varphi} &= \beta \quad , \quad \sigma \beta = 0 \quad . \end{aligned} \quad (A.2)$$

and

$$\begin{aligned} w\psi &= [\psi, \omega] - D\varphi \quad , \quad wA = -D\omega \quad , \\ wb &= [b, \omega] - [\bar{\psi}, \varphi] \quad , \quad w\bar{\psi} = -[\omega, \bar{\psi}] \quad , \\ w\varphi &= [\varphi, \omega] \quad , \quad w\omega = -\frac{1}{2}[\omega, \omega] \quad , \\ w\bar{\varphi} &= [\bar{\varphi}, \omega] \quad , \quad w\beta = -[\omega, \beta] + [\varphi, \bar{\varphi}] \quad . \end{aligned} \quad (A.3)$$

One can easily check that $\sigma^2 = w^2 = [\sigma, w]_+ = 0$. This structure suggests the use of a supersymmetric formalism. Let:

$$\begin{aligned} \mathbf{A} &= A + \theta\psi \quad , \\ \Omega &= \omega + \theta\varphi \quad , \\ \bar{\Psi} &= \bar{\psi} + \theta b \quad , \\ \bar{\Phi} &= \bar{\varphi} + \theta\beta \quad . \end{aligned} \quad (A.4)$$

Then, in terms of the superfields:

$$\sigma = \frac{\partial}{\partial \theta} \quad , \quad (A.5)$$

and w is a supergauge transformation:

$$\begin{aligned} w\mathbf{A} &= -\mathbf{D}(\mathbf{A})\Omega \quad , \\ w\Omega &= -\frac{1}{2}[\Omega, \Omega] \quad , \\ w\bar{\Psi} &= -[\Omega, \bar{\Psi}] \quad , \\ w\bar{\Phi} &= [\bar{\Phi}, \Omega] \quad , \end{aligned} \quad (A.6)$$

where the covariant differential $D(A)$ is given by

$$D(A)\Omega = d\Omega + [A, \Omega] \quad . \quad (A.7)$$

Eq. (A.6) defines a differential superalgebra with a super Lie algebra action in terms of:

$$\Lambda = \lambda + \theta\mu \quad , \quad \lambda, \mu \in \text{Lie } \mathcal{G} \quad . \quad (A.8)$$

We define δ_Λ according to

$$\begin{aligned} \delta_\Lambda A &= D(A)\Lambda \quad , \\ \delta_\Lambda \Omega &= [\Omega, \Lambda] \quad , \\ \delta_\Lambda &= [\bar{\Psi}, \Lambda] \quad , \\ \delta_\Lambda &= [\bar{\Phi}, \Lambda] \end{aligned} \quad (A.9)$$

and ι_Λ according to

$$\iota_\Lambda A = \iota_\Lambda \bar{\Psi} = \iota_\Lambda \bar{\Phi} = 0 \quad , \quad \iota_\Lambda \Omega = \Lambda \quad . \quad (A.10)$$

Then we have:

$$\delta_\Lambda = [w, \iota_\Lambda] \quad . \quad (A.11)$$

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