CERN-TH.5188/88

A NOTE ON $SU(\infty)$ CLASSICAL YANG-MILLS THEORIES

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ABSTRACT

We find that the classical $SU(N)$ Yang-Mills theories, in the large N limit, can be described by a new type of gauge principle. The gauge potentials become c-number functions of two additional co-ordinates, which parametrize the surface of an internal sphere at every space-time point. The new gauge transformations are the area preserving co-ordinate transformations of the sphere.

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In an interesting work [1], Hoppe has studied the canonical quantization of a relativistic spherical membrane of Dirac type [2], in the light cone gauge [3]. He was able to show that this theory is exactly the same as the quantum mechanics of space constant $SU(N)$ Yang-Mills potentials [4] in the limit $N \rightarrow \infty$. The basic ingredients of his proof are first, that the $SU(N)$ Lie algebra, when $N \rightarrow \infty$, can be shown to be isomorphic to the infinite dimensional Lie algebra of area preserving (or symplectic) diffeomorphisms of the sphere S^2 which is the symmetry of the relativistic membrane after gauge fixing; second, the particular form of the membrane Hamiltonian in the light-cone gauge is the same as that of $SU(N)$ space constant Yang-Mills potentials (in the gauge $A_0 = 0$) for $N \rightarrow \infty$.

One would like to have an analogous geometrical interpretation for the full Yang-Mills theory, that is for space-time dependent potentials. In this work, although we do not find any direct relation of the Yang-Mills theory to the membrane theory or an extension of it, we do find a simple geometrical picture of the $SU(\infty)$ Yang-Mills theory by making use of the equivalence between the $SU(N)$, $N \rightarrow \infty$, Lie algebra and that of the "symplectic" or area preserving co-ordinate transformations of an "internal" sphere S^2 . Indeed we show that in the large N limit the classical $SU(N)$ fields ($N \times N$ matrices):

$$A_\mu(x) = A_\mu^a t^a \quad t^a \in SU(N), \quad a=1, \dots, N^2-1 \quad (1)$$

$\mu=0, 1, 2, 3$

are replaced by c-number functions of an "internal" sphere S^2 :

$$A_\mu(x, \vartheta, \varphi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} A_\mu^{\ell m}(x) Y_{\ell m}(\vartheta, \varphi), \quad (2)$$

where $Y_{\ell m}$ are the spherical harmonics on S^2 . The gauge transformations

$$\delta A_\mu = \partial_\mu w + [A_\mu, w], \quad w = w^a t^a \quad (3a)$$

and

$$\delta F_{\mu\nu} = [F_{\mu\nu}, w], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (3b)$$

are replaced by

$$\delta A_\mu(x, \vartheta, \varphi) = \partial_\mu w(x, \vartheta, \varphi) + \{A_\mu, w\} \quad (4a)$$

$$\delta F_{\mu\nu} = \{F_{\mu\nu}, w\}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}, \quad (4b)$$

where the Poisson bracket of two functions, f and g on S^2 is defined as

$$\{f, g\} = \frac{\partial f}{\partial \cos \theta} \frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi} \frac{\partial g}{\partial \cos \theta} \quad (5)$$

The commutators are replaced by Poisson brackets according to

$$\lim_{N \rightarrow \infty} N[A_\mu, A_\nu] = \{A_\mu, A_\nu\} \quad (6)$$

as we pass from Eq. (1) to (2).

The action of SU(N) Yang-Mills theory

$$S_N = \frac{1}{4g_N^2} \int d^4x \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \quad (7)$$

in the large N limit becomes

$$S_{N=\infty} = \frac{1}{16\pi g^2} \int_{S^2} d\Omega \int d^4x F_{\mu\nu}(x, \vartheta, \varphi) F^{\mu\nu}(x, \vartheta, \varphi) \quad (8)$$

where

$$F_{\mu\nu}(x, \vartheta, \varphi) = \partial_\mu A_\nu(x, \vartheta, \varphi) - \partial_\nu A_\mu(x, \vartheta, \varphi) + \{A_\mu, A_\nu\} \quad (9)$$

and g is the limit of $N^{-1} \cdot g_N$.

In this note we shall give a heuristic proof of this construction. We begin by reviewing the defining properties of the infinite dimensional Lie algebra of symplectic or area preserving diffeomorphisms of the sphere (SDiff(S^2)). The generators of symplectic transformations of a surface, locally have the form

$$L_f = \frac{\partial f}{\partial \sigma_2} \frac{\partial}{\partial \sigma_1} - \frac{\partial f}{\partial \sigma_1} \frac{\partial}{\partial \sigma_2} \quad (10)$$

and they satisfy the algebra

$$[L_f, L_g] = L_{\{f, g\}} \quad (11)$$

for f, g differentiable functions of S^2 , $f, g \in C^\infty(S^2)$. Here $\{f, g\}$ is the Poisson bracket defined as

$$\{f, g\} = \frac{\partial f}{\partial \sigma_1} \frac{\partial g}{\partial \sigma_2} - \frac{\partial f}{\partial \sigma_2} \frac{\partial g}{\partial \sigma_1} \quad (12)$$

in local co-ordinates σ_1, σ_2 of the surface such that the area element is

$$dA = d\sigma_1 d\sigma_2 \quad (13)$$

that is, in canonical or symplectic co-ordinates [5]. In the case of the sphere $\sigma_1 = \phi$, $\sigma_2 = \cos\theta$. For the sphere, we can form a basis of generators choosing $f = Y_{\ell,m}(\theta,\phi)$ to be the spherical harmonics [1,6].

$$L_{\ell,m} = \frac{\partial Y_{\ell,m}}{\partial \cos\theta} \frac{\partial}{\partial \phi} - \frac{\partial Y_{\ell,m}}{\partial \phi} \frac{\partial}{\partial \cos\theta} \quad (14)$$

Relation (11) implies that

$$[L_{\ell,m}, L_{\ell',m'}] = f_{\ell,m,\ell'}^{\ell''m''} L_{\ell'',m''} \quad (15)$$

where the structure constants $f_{\ell,m,\ell'}^{\ell''m''}$ are defined by expanding the Poisson brackets on the basis $Y_{\ell m}$:

$$\{Y_{\ell,m}, Y_{\ell',m'}\} = f_{\ell,m,\ell'}^{\ell''m''} Y_{\ell'',m''} \quad (16)$$

These structure constants have been calculated explicitly in Ref. [1]. To get a feeling of the geometrical meaning of these generators, we note that for $\ell = 1$, $m = 0, \pm 1$, these are the usual angular momentum generators (up to normalization), $L_z, L_{\pm}$. Also,

$$L_{\ell,m} Y_{\ell',m'} = -\{Y_{\ell,m}, Y_{\ell',m'}\} = -f_{\ell,m,\ell'}^{\ell''m''} Y_{\ell'',m''} \quad (17)$$

so

$$[L_{1,\pm 1}, L_{\ell,m}] \cong [\ell(\ell+1) - m(m\pm 1)]^{1/2} L_{\ell,m\pm 1} \quad (18)$$

$$[L_{1,0}, L_{\ell,m}] \cong m L_{\ell,m} \quad (19)$$

Equations (18) and (19) mean that the infinite set of generators $L_{\ell,m}$ is reduced to an infinite sum of irreducible sets of operators of definite angular momentum ℓ ,

$$L_{\ell,-\ell}, L_{\ell,-\ell+1}, \dots, L_{\ell,\ell} \quad \ell = 1, 2, \dots \quad (20)$$

with respect to the $SU(2)$ Lie algebra, $L_{1,0}, L_{1,\pm 1}$ of solid rotations of the sphere. We now choose an appropriate basis for $SU(N)$ [1]. The spherical harmonics $Y_{\ell m}(\theta,\phi)$ are harmonic homogeneous polynomials of degree ℓ in three Euclidean co-ordinates x_1, x_2, x_3

$$x_1 = \cos\varphi \sin\vartheta, \quad x_2 = \sin\varphi \sin\vartheta, \quad x_3 = \cos\vartheta \quad (21)$$

$$Y_{\ell,m}(\vartheta, \varphi) = \sum_{\substack{i_1, i_2, i_3 \\ k=1, \dots, \ell}} a_{i_1, \dots, i_\ell}^{(m)} x_{i_1} \dots x_{i_\ell}, \quad (22)$$

where $a_{i_1, \dots, i_\ell}^{(m)}$ is a symmetric and traceless tensor. For fixed ℓ there are $2\ell+1$ linearly independent tensors $a_{i_1, \dots, i_\ell}^{(m)}, m = -\ell, \dots, \ell$ [7].

Let S_1, S_2, S_3 be $N \times N$ Hermitian matrices which form an N -dim irreducible representation of the Lie algebra $SU(2)$

$$[S_i, S_j] = i \varepsilon_{ijk} S_k \quad (23)$$

In Ref. [1] it was shown that the matrix polynomials

$$T_{\ell,m}^{(N)} = \sum_{i_1, i_2, i_3} a_{i_1, \dots, i_\ell}^{(m)} S_{i_1} \dots S_{i_\ell} \quad (24)$$

for $\ell = 1, \dots, N-1, m = -\ell, \dots, \ell$ can be used to construct a basis of N^2-1 matrices for the fundamental representation of $SU(N)$, with corresponding structure constants $f^{(N)}$:

$$[T_{\ell,m}^{(N)}, T_{\ell',m'}^{(N)}] = i f_{\ell,m,\ell',m'}^{(N)} T_{\ell,m}^{(N)} \quad (25)$$

There is a normalization of the generators $T_{\ell,m}^{(N)}$ such that the limit

$$N \cdot f_{\ell,m,\ell',m'}^{(N)} \xrightarrow{N \rightarrow \infty} f_{\ell,m,\ell',m'}, \quad (26)$$

exists and coincides with the structure constants of $S\text{Diff}(S^2)$ as defined in Eq. (16) [1]. After these preliminaries we proceed to establish the relation of the infinite dimensional algebra (15), $S\text{Diff}(S^2)$, to the $SU(N)$ algebra as $N \rightarrow \infty$, by an argument which avoids the explicit computation of Ref. [1] for the structure constants $f^{(N)}$ and f . If we rescale the generators of $SU(2)$ by $1/N$,

$$S_i \rightarrow T_i = \frac{1}{N} S_i \quad (27)$$

they satisfy the algebra

$$[T_i, T_j] = i \frac{1}{N} \varepsilon_{ijk} T_k \quad (28)$$

and the Casimir element

$$T^2 = T_1^2 + T_2^2 + T_3^2 \simeq 1 + \frac{1}{N} \quad (29)$$

has a finite limit for $N \rightarrow \infty$. Under the norm

$$|x|^2 \equiv \text{Tr } x^2, \quad (30)$$

for $x \in \text{SU}(2)$, the generators T_i , $i = 1, 2, 3$ have definite limits as $N \rightarrow \infty$ which are three objects x_1, x_2, x_3 which commute [by (28)] and are constrained [by (29)] according to

$$x_1^2 + x_2^2 + x_3^2 = 1. \quad (31)$$

If we consider two polynomial functions of three commuting variables $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$, the corresponding matrix polynomials $f(T_1, T_2, T_3)$, $g(T_1, T_2, T_3)$ have commutation relations for large N which follow from (28).

$$\lim_{N \rightarrow \infty} \frac{N}{i} [f, g] = \varepsilon^{ijk} x^j \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^k}. \quad (32)$$

This is similar to the passage from quantum mechanics to classical mechanics. This can be generalized to all semi-simple Lie groups. Thus the large representations of $\text{SU}(2)$ give rise to a symplectic structure on the sphere. Large N limits of quantum models have been considered as classical theories with particular symplectic structures in Refs. [8] and [9]. If we parametrize x_i by polar co-ordinates (21), we see that the right-hand side of (32) is nothing else than the Poisson brackets of Ref. [5]. Consider now the basis $T_{\lambda m}^{(N)}$ of $\text{SU}(N)$ obtained by replacing in (24) the matrices S_i by the rescaled ones T_i . Then according to (27) we obtain

$$\lim_{N \rightarrow \infty} \frac{N}{i} [T_{\ell m}^{(N)}, T_{\ell' m'}^{(N)}] = \{Y_{\ell m}, Y_{\ell' m'}\} \quad (33)$$

If we replace the left-hand side of this equation by Eq. (25) and the right-hand side with Eq. (16) we obtain Eq. (26).

To arrive at the new type of $\text{SU}(\infty)$ gauge invariance described by Eqs. (1)-(9), we expand the classical Yang-Mills fields in the basis of the matrices $T_{\lambda m}^{(N)}$:

$$A_\mu(x) = A_\mu^{\ell m}(x) T_{\ell m}^{(N)}, \quad \ell = 1, \dots, N-1, m = -\ell, \dots, \ell. \quad (34)$$

Equation (33) implies that the commutator of the gauge fields as $N \rightarrow \infty$ has as a limit the Poisson bracket

$$N [A_\mu, A_\nu] \xrightarrow{N \rightarrow \infty} \{A_\mu(x, \theta, \varphi), A_\nu(x, \theta, \varphi)\} \quad (35)$$

The factor of N necessary for the existence of this limit can be obtained by rescaling the gauge field in the initial Yang-Mills theory by the substitution

$$A_\mu \rightarrow N A_\mu . \quad (36)$$

It is easy to see now that the large N limit of the classical Yang-Mills theory (1), (3a-b) and (7) is the "Poisson gauge theory" (2), (4a-b) and (8).

The new gauge theory presented here may be relevant for the study of the planar limit of $SU(N)$ Yang-Mills theory [10,11,12].

ACKNOWLEDGEMENTS

One of us (E.G.F.) would like to acknowledge useful discussions with N. Papanicolaou. This work is in part supported by NATO grant No.0009/87.

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