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NO-GHOST THEOREM AND $OSP(1,1|2)$ IN GAUGE THEORIES

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Abstract

We explain how generators of the $Osp(1,1|2)$ extended BRST algebra can be constructed in a general gauge theory and in an arbitrary gauge, order by order in a systematic expansion. As an example we consider both axial and Coulomb gauge Yang-Mills theories. We relate the construction of physical states to the representation theory of quantum $Osp(1,1|2)$ and explain why the identification of physical states with $Osp(1,1|2)$ singlet states emerges. This yields a particularly transparent demonstration of the no-ghost theorem.

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BRST symmetry [1] was originally introduced in Yang-Mills theory as a useful technicality for proving renormalizability. It has now matured into a powerful tool [2] that facilitates the understanding of reparametrization invariant theories such as strings and point particles [3-7]. Recently BRST symmetry has emerged in the context of topological quantum field theories where it might have truly far-reaching consequences [8].

The original BRST symmetry of a gauge fixed action was extended into an $\text{Osp}(1,1|2)$ symmetry in [3-5]. This generalized BRST symmetry ensures the cancellation of unphysical gauge degrees of freedom by the Parisi-Sourlas mechanism [9,10] and identifies physical states with the $\text{Osp}(1,1|2)$ singlet states. From this formulation a particularly simple path integral approach to gauge independence, unitarity and ghost decoupling has been developed [5].

Explicitly, the $\text{Osp}(1,1|2)$ generators have only been constructed for (super)string theories [4,5] even though general arguments [5] ensure that in principle these generators could be constructed in any gauge theory and with any choice of gauge fixing. But since the construction in [4,5] was based on certain properties specific to Virasoro generators and transverse state projectors, it has not been clear how it could be extended to other gauge theories, including the Yang-Mills theory.

In the present letter we shall explain how the $\text{Osp}(1,1|2)$ generators can be constructed in gauge theories other than string theories. We demonstrate that states in the extended Hilbert space are naturally classified according to the representations of the $\text{Osp}(1,1|2)$ Lie algebra, and explain why physical states determine singlet representations of $\text{Osp}(1,1|2)$. This yields a particularly transparent operator formulation of the no-ghost theorem in the general case. Our construction can be viewed as an $\text{Osp}(1,1|2)$ re-interpretation of the quartet mechanism [11]. It is complementary to the discussion in [12], where similar issues have recently been addressed from a different point of view.

Consider a constrained Hamiltonian system in a $2N$ -dimensional¹ phase space Γ , with local canonical coordinates q^i, p_i ; $i = 1, \dots, N$ and Poisson brackets $\{p_i, q^j\} = -\delta_i^j$. The physical phase space Γ^* is a $2(N-M)$ dimensional submanifold of Γ with local canonical coordinates q^*, p_* , characterized by M independent conditions

$$\varphi_a(q, p) = 0 \quad a = 1, \dots, M \quad (1)$$

¹We shall use formulation adequate for a system with a finite degree of freedom. Transition to field theory is made the usual way.

and we shall assume that the functions φ_a are of first class,

$$\{\varphi_a, \varphi_b\} = U_{ab}^c \varphi_c \quad (2)$$

For the purpose of presentation we shall use formulation appropriate to constant U_{ab}^c , even though our results are valid also for non-constant U_{ab}^c . In applications the φ_a are then generators of time-independent gauge transformations; for example in Yang-Mills theories φ_a are Gauss law generators.

From (1) we can locally solve for the M unphysical components p_a of the momentum variables, $\varphi_a(q, p) = 0 \Rightarrow p_a = p_a(q^i, p_*)$. In order to eliminate the unphysical position variables q^a we introduce another set of M functionals ψ^a (gauge fixing conditions) such that

$$\{\varphi_a, \psi^b\} = \Omega_a^b ; \quad \det|\Omega_a^b| \neq 0 \quad (3)$$

For simplicity we select

$$\{\psi^a, \psi^b\} = 0 \quad (4)$$

Hence we can identify ψ^a as the unphysical components q^a of the position variables, and as a consequence the physical phase space Γ^* is specified by

$$\varphi_a = 0 \quad \& \quad \psi^a = 0 \quad (5)$$

The antisymmetric matrix Ω_a^b in (3) determines the symplectic structure in the unphysical sector of the phase space Γ . Since (3) can be interpreted as the gauge variation of ψ^a , we can expect that Ω_a^b should include a term explicitly related to a gauge rotation of ψ^a . In order to expose this we introduce a non-singular matrix σ_a^b on Γ such that $\{\psi^a, \sigma_b^c\} = 0$ (so that (4) remains valid), and redefine

$$\psi^a \rightarrow \sigma_b^a \psi^b \quad (6)$$

Instead of (3) we then have

$$\{\varphi_a, \psi^b\} = \omega_a^b + T_{ac}^b \psi^c \quad (7)$$

in addition of (4), with

$$T_{ac}^b = \{\varphi_a, \sigma_d^b\}(\sigma^{-1})_c^d \quad \& \quad \omega_a^b = \sigma_c^b \Omega_a^c \quad (8)$$

and Jacobi identities

$$U_{ab}^e T_{ed}^c + T_{ae}^c T_{bd}^e - T_{be}^c T_{ad}^e = \{\varphi_a, T_{bd}^c\} - \{\varphi_b, T_{ad}^c\} \quad (9.a)$$

$$U_{ab}^c \omega_c^d = T_{bc}^d \omega_a^c - T_{ac}^d \omega_b^c \quad (9.b)$$

Indeed, in applications constraint algebras usually appear in the functional form of (4), (7).

If in (7) we introduce the further change of variables

$$\psi^a \rightarrow (\omega^{-1})_b^a \psi^b \quad (10)$$

we get

$$\{\varphi_a, \psi^b\} = \delta_a^b + \bar{T}_{ac}^b \psi^c \quad (11)$$

where

$$\bar{T}_{ac}^b = (\omega^{-1})_c^b T_{ae}^c \omega_d^e - (\omega^{-1})_c^b \{\varphi_a, \omega_d^c\} \sim \omega^{-1} T \omega + \omega^{-1} d\omega \quad (12)$$

and we conclude that (10) and (12) can be viewed as a gauge transformation of the symplectic group, with respect to the symplectic structure in the unphysical sector of the phase space and with $T_{ab}^c \equiv (T_a)_b^c$ the pertinent connection *i.e.* symplectic gauge field. Notice that the constraint algebra (7) transforms covariantly under these symplectic gauge transformations.

With (11), the Jacobi identity (9.b) tells us that if U_{ab}^c is constant we can expect $\bar{T}_{ab}^c$ also to be constant. In (4), (11) we then have the constraint algebra discussed in [4,5]. In particular, the previous discussion establishes that the general constraint algebra (7) and the constraint algebra (11) discussed in [4,5] are related to each other by the symplectic gauge transformation (10), (12).

For (4), (11) the $\text{Osp}(1,1|2)$ extension of the BRST algebra has been explicitly constructed in [4,5]. It is obtained by closing the algebra of the following quartet

$$R^{\theta\theta} = 2\eta^a \bar{\eta}_a \quad \& \quad R^{\bar{\theta}\bar{\theta}} = 2\mathcal{P}_a \bar{\mathcal{P}}^a \quad (13.a)$$

$$R^{-\theta} = \eta^a \varphi_a + \frac{1}{2} U_{ab}^c \eta^a \eta^b \mathcal{P}_c + \eta^a \bar{T}(\bar{\mathcal{P}}^b \bar{\eta}_c - \pi^b \lambda_c) + \bar{\eta}_a \pi^a \quad (13.b)$$

$$R^{+\bar{\theta}} = \mathcal{P}_a \bar{\psi}^a + \bar{\mathcal{P}}^a \lambda_a \quad (13.c)$$

with $\pi^a, \lambda_a, \eta^a, \mathcal{P}_a$ and $\bar{\mathcal{P}}^a, \bar{\eta}_a$ the extended phase space coordinates [2], and the generator $R^{-\theta}$ related to the conventional BRST operator by a (pseudo)unitary conjugation,

$$\Omega = \eta^a \varphi_a + \frac{1}{2} U_{ab}^c \eta^a \eta^b \mathcal{P}_c + \bar{\eta}_a \pi^a = e^{-\Phi} R^{-\theta} e^{\Phi} = R^{-\theta} + \{R^{-\theta}, \Phi\} + \dots \quad (14)$$

$$\Phi = \bar{T}_{ab}^c \eta^a \bar{\mathcal{P}}^b \lambda_c \quad (15)$$

Since (7) differs from (11) only by the symplectic gauge transformation (10), (12) we get the $\text{Osp}(1,1|2)$ generators corresponding to (7) by performing the inverse gauge transformation in (13). For a constant ω_a^b this amounts to a trivial substitution of (10), (12) into (13). However, for a non-constant ω_a^b an explicit inversion of the gauge transformation (10), (12) becomes difficult in practice since in general it is not easy to explicitly construct the inverse matrix $(\omega^{-1})_a^b$. Usually it becomes necessary to utilize perturbative techniques to invert ω_a^b iteratively, for example order by order in some coupling constant

$$(\omega^{-1})_a^b = c^{(o)}_a^b + \sum_{n>0} g^n c^{(n)}_a^b \quad (16)$$

In such expansions the leading contribution $c^{(o)}_a^b$ is generically a constant. Consequently in the leading order the $\text{Osp}(1,1|2)$ generators are obtained by a direct substitution of $(\omega^{-1}) \sim c^{(o)}$ in (10), (12), (13). The higher order corrections to the $\text{Osp}(1,1|2)$ generators can then be systematically constructed order by order in g .

In [4,5] a realization of (13) and the $\text{Osp}(1,1|2)$ generators were constructed explicitly for (super)string theories. Here we shall show how these generators can be constructed in Yang-Mills theory with φ_a the Gauss law operator,

$$\varphi_a \sim G^a(\vec{x}) = D_i^{ab} E^{bi}(\vec{x}) = (\delta^{ab} \partial_i + g f^{acb} A_i^c(\vec{x})) E^{bi}(\vec{x}) \quad (17)$$

$$\{G^a(\vec{x}), G^b(\vec{y})\} = g f^{abc} G^c(\vec{x}) \delta(\vec{x} - \vec{y}) \quad \Rightarrow \quad U_{ab}^c \sim g f^{abc} \delta(\vec{x} - \vec{z}) \delta(\vec{y} - \vec{z}) \quad (18)$$

Consider first $\psi^a \sim n^i A_i^a$ i.e. axial gauge with n^i an arbitrary (not necessarily constant) vector. Since

$$\{G^a(\vec{x}), n^i A_i^b(\vec{y})\} = -\delta^{ab} \partial_i [n^i \delta(\vec{x} - \vec{y})] - g f^{abc} \delta(\vec{x} - \vec{z}) \delta(\vec{y} - \vec{z}) n^i A_i^c(\vec{z}) \quad (19)$$

we have an algebra of the form (7) with

$$T_{ac}^b \sim -g f^{abc} \delta(\vec{x} - \vec{z}) \delta(\vec{y} - \vec{z}) \quad \& \quad \omega_a^b \sim -\delta^{ab} \partial_i [n^i \delta(\vec{x} - \vec{y})] \quad (20)$$

where ω_a^b is constant *i.e.* independent of the phase space variables. If we select n^i such that ω_a^b is invertible (*e.g.* n^i constant unit vector) we then obtain the pertinent generators (13) by a direct substitution. As a consequence the functional form of the axial gauge $\text{Osp}(1,1|2)$ generators is identical to that of the generators presented in [4,5] in the case of (super)string theories.

In the general case ω_a^b in (7) is not a constant, implying that the $\text{Osp}(1,1|2)$ generators cannot be constructed in a closed form. Instead, expansions such as (16) must be introduced. For example in the Coulomb gauge $\psi^a \sim \partial^i A_i^a$ we have

$$\{G^a(\vec{x}), \partial^i A_i^b(\vec{y})\} = -D_i^{ab} \partial^i \delta(\vec{x} - \vec{y}) + g f^{abc} \delta(\vec{x} - \vec{z}) \delta(\vec{y} - \vec{z}) \partial^i A_i^c(\vec{z}) \quad (21)$$

an algebra of the form (7) with

$$T_{ac}^b \sim g f^{abc} \delta(\vec{x} - \vec{z}) \delta(\vec{y} - \vec{z}) \quad \& \quad \omega_a^b \sim -D_i^{ab} \partial^i \delta(\vec{x} - \vec{y}) \quad (22)$$

Since ω_a^b now depends on the fields, (13) cannot be directly applied. In order to construct the $\text{Osp}(1,1|2)$ generators we introduce a perturbative expansion of the form (16),

$$(\omega^{-1})_a^b \sim \frac{1}{4\pi} \frac{\delta^{ab}}{|\vec{x} - \vec{y}|} + g \int \frac{dz}{(4\pi)^2} f^{acb} \frac{A_k^c(\vec{z})}{|\vec{x} - \vec{z}|} \partial^k \frac{1}{|\vec{z} - \vec{y}|} + \mathcal{O}(g^2) \quad (23)$$

Here the leading contribution is constant *i.e.* independent of the fields, implying that (13) yields the first order approximation to the $\text{Osp}(1,1|2)$ generators. Substituting (23) into (10), (12) corrections can then be systematically constructed, order by order in the Yang-Mills coupling constant g . However, unlike in the case of the axial gauge, in the Coulomb gauge the $\text{Osp}(1,1|2)$ generators cannot be represented in a closed form. Some of these generators involve ghost polynomials to all orders.

We shall now proceed to clarify the meaning of the $\text{Osp}(1,1|2)$ generators on the operator approach to quantization. (In the path integral approach see [5].) In particular, we shall explicitly demonstrate the identification of physical states with the $\text{Osp}(1,1|2)$ singlets

$$R^{\alpha\beta} |\phi\rangle = 0 \quad (24)$$

and explain how this implies the no-ghost theorem.

We first recall [5] that the Darboux theorem ensures the local existence (in a neighbourhood in the phase space) of a (pseudo)unitary conjugation of the $\text{Osp}(1,1|2)$ generators (13),

$$R^{\alpha\beta} \rightarrow \bar{R}^{\alpha\beta} = e^{-\Phi} R^{\alpha\beta} e^{\Phi} = R^{\alpha\beta} + \{R^{\alpha\beta}, \Phi\} + \dots \quad (25)$$

where $\bar{R}^{\alpha\beta}$ are generators of the $\text{Osp}(1,1|2)$ algebra based on constraint functionals $\varphi_a \rightarrow G_a$, $\psi^a \rightarrow F^a$ with the Heisenberg algebra

$$\{G_a, G_b\} = \{F^a, F^b\} = 0 \quad \& \quad \{G_a, F^b\} = -\delta_a^b \quad (26)$$

Explicitly,

$$\Phi = -F^a \bar{T}_{ab}^c (G_c F^b + \lambda_c \pi^b - \mathcal{P}_c \eta^b - \bar{\eta}_c \bar{\mathcal{P}}^b) \quad (27)$$

and

$$\bar{R}_a^{\theta\theta} = 2\eta^a \bar{\eta}_a \quad \& \quad \bar{R}_a^{\bar{\theta}\bar{\theta}} = 2\mathcal{P}_a \bar{\mathcal{P}}^a \quad \& \quad \bar{R}_a^{\theta\bar{\theta}} = \bar{R}_a^{\bar{\theta}\theta} = \bar{\mathcal{P}}^a \bar{\eta}_a + \mathcal{P}_a \eta^a \quad (28.a)$$

$$\bar{R}_a^{-\theta} = -\bar{R}_a^{\theta-} = \eta^a G_a + \bar{\eta}_a \pi^a \quad \& \quad \bar{R}_a^{-\bar{\theta}} = -\bar{R}_a^{\bar{\theta}-} = \bar{\mathcal{P}}^a G_a - \mathcal{P}_a \pi^a \quad (28.b)$$

$$\bar{R}_a^{+\theta} = -\bar{R}_a^{\theta+} = \eta^a \lambda_a - \bar{\eta}_a F^a \quad \& \quad \bar{R}_a^{+\bar{\theta}} = -\bar{R}_a^{\bar{\theta}+} = \bar{\mathcal{P}}^a \lambda_a + \mathcal{P}_a F^a \quad (28.c)$$

$$\bar{R}_a^{+-} = -\bar{R}_a^{-+} = -F^a G_a - \pi^a \lambda_a \quad (28.d)$$

In the general case of (7) the $\text{Osp}(1,1|2)$ generators are related to (13) by the symplectic gauge transformation (10), (12). Alternatively, we can also obtain these generators by introducing a conjugation of the form (25), with the generating functional (27) constructed order by order in perturbation theory using (16), (23) in (10), (12). With $c^{(o)b}_a$ in (16) a constant, the leading contribution in the expansion of the generating functional is obtained from (27) by a direct substitution. As a consequence the following discussion is directly applicable to the general case of (7), order by order in the expansion (16), (23).

In the quantum theory the conjugation (25) determines a Bogoliubov transformation *i.e.* it maps creation and annihilation operators in the old variables onto creation and annihilation operators in the new variables. Hence it also maps Fock states onto Fock states, representations of the $\text{Osp}(1,1|2)$ algebra onto representations of the $\text{Osp}(1,1|2)$ algebra, and in particular $\text{Osp}(1,1|2)$ singlet states onto $\text{Osp}(1,1|2)$ singlet states,

$$|\phi\rangle \rightarrow e^{-\Phi} |\phi\rangle = |\tilde{\phi}\rangle \quad (29)$$

Notice that the Fock states $|\tilde{\phi}\rangle$ are only defined locally as sections of a Hilbert bundle over the polarization of the phase space, with (28) realized as linear automorphisms of the bundle. This Hilbert bundle and the corresponding representation (28) of the $\text{Osp}(1,1|2)$ algebra are then related to the original representation corresponding to (7) and (11) and the pertinent Fock states by the Bogoliubov transformation (25). As a consequence, provided there are no quantum anomalies [5] in (25) or obstructions in

patching the sections $|\tilde{\phi}\rangle$ of the Hilbert bundle onto globally defined Fock states, it is sufficient to consider only the representation theory of (28). The representation theory of the $\text{Osp}(1,1|2)$ generators corresponding to (11) and (7) can then be obtained from that of (28) simply by inverting the pertinent conjugations (25).

We shall demand that both the quantum BRST operator $\bar{R}^{-\theta}$ and the quantum anti-BRST operator $\bar{R}^{-\bar{\theta}}$ are (formally) Hermitean. In general the constraint operators $\hat{G}_a$ and the gauge fixing operators $\hat{F}^a$ and $\hat{\pi}^a$, $\hat{\lambda}_a$ are (formally) Hermitean, hence the ghost operators $\hat{\eta}^a, \hat{\mathcal{P}}_a, \hat{\bar{\mathcal{P}}}^a, \hat{\bar{\eta}}_a$ are also (formally) Hermitean. This implies that $\bar{R}^{+\theta}$, $\bar{R}^{+\bar{\theta}}$ and $\bar{R}^{-+}$ are (formally) Hermitean while $\bar{R}^{\theta\theta}$, $\bar{R}^{\bar{\theta}\bar{\theta}}$ and $\bar{R}^{\theta\bar{\theta}}$ are (formally) anti-Hermitean.

From the Poisson brackets we obtain the following non-trivial commutators,

$$[\hat{G}^a, \hat{F}_b] = [\hat{\pi}^a, \hat{\lambda}_b] = -i\delta_a^b \quad (30)$$

and anticommutators

$$\{\hat{\eta}^a, \hat{\mathcal{P}}_b\} = \{\hat{\bar{\mathcal{P}}}^a, \hat{\bar{\eta}}_b\} = -\delta_b^a \quad (31)$$

Thus, only the anti-Hermitean operators $\hat{N} \equiv \bar{R}^{\theta\bar{\theta}}$ and $\hat{Q} \equiv -i\bar{R}^{-+}$ have a normal ordering ambiguity. These normal orderings can be uniquely determined by demanding that the quantum $\text{Osp}(1,1|2)$ algebra closes,

$$\hat{N} = \bar{R}^{\theta\bar{\theta}} = \frac{1}{2}[\hat{\mathcal{P}}_a, \hat{\eta}^a] - \frac{1}{2}[\hat{\bar{\eta}}_a, \hat{\bar{\mathcal{P}}}^a] \quad (32)$$

$$\hat{Q} = i\bar{R}^{-+} = -\frac{i}{2}\{\hat{G}^a, \hat{F}_a\} - \frac{i}{2}\{\hat{\pi}^a, \hat{\lambda}_a\} \quad (33)$$

Since $\hat{N}$ and $\hat{Q}$ commute, we can select a basis in which both are diagonal. The representations of $\text{Osp}(1,1|2)$ can then be labelled by N , the largest eigenvalue of $\hat{N}$ in the representation, by the eigenvalues² n of $\hat{N}$ and by the eigenvalues r of $\hat{Q}$. We denote the states in the representation N by $|r, n; N\rangle$, and we shall now determine the values of N , n and r that are relevant to the quantization of constrained systems.

Consider first the pair $\hat{\eta}^a$ and $\hat{\mathcal{P}}_a$ for some fixed a . These operators satisfy the two-dimensional Clifford algebra, and in order to construct a representation we introduce two (highest weight) states $|\eta\rangle$ and $|\mathcal{P}\rangle$ such that

$$\hat{\eta}|\eta\rangle = \hat{\mathcal{P}}|\mathcal{P}\rangle = 0 \quad (34)$$

²Since $R^{\theta\theta}$, $R^{\bar{\theta}\bar{\theta}}$ and $R^{\theta\bar{\theta}}$ determine the Lie algebra of $\text{Sp}(2)$, N and n label unitarily inequivalent representations of the $\text{Sp}(2)$ Lie algebra.

The $\hat{N}$ quantum numbers of these states are

$$\hat{N}|\eta\rangle = \frac{1}{2}|\eta\rangle \Rightarrow |\eta\rangle = |+\frac{1}{2}\rangle_\eta \quad (35)$$

$$\hat{N}|\mathcal{P}\rangle = -\frac{1}{2}|\mathcal{P}\rangle \Rightarrow |\mathcal{P}\rangle = |-\frac{1}{2}\rangle_{\mathcal{P}} \quad (36)$$

and $\hat{\mathcal{P}}$ acts as a creation operator on $|+\frac{1}{2}\rangle_\eta$, while $\hat{\eta}$ acts as a creation operator on $|-\frac{1}{2}\rangle_{\mathcal{P}}$,

$$\hat{\mathcal{P}}|+\frac{1}{2}\rangle_\eta = |-\frac{1}{2}\rangle_\eta \quad \& \quad \hat{\eta}|-\frac{1}{2}\rangle_{\mathcal{P}} = |+\frac{1}{2}\rangle_{\mathcal{P}} \quad (37)$$

Observe that even though $\hat{N}$ is anti-Hermitean, its eigenvalues are real. This is possible since (35)-(37) are zero norm states. In the following it will be useful to view the η -states as duals of the $\mathcal{P}$ -states, and it turns out to be consistent to assume that the non-vanishing inner products are

$$\eta \langle +\frac{1}{2} | -\frac{1}{2} \rangle_{\mathcal{P}} = c_{+-} \quad \& \quad \eta \langle -\frac{1}{2} | +\frac{1}{2} \rangle_{\mathcal{P}} = c_{-+} \quad (38)$$

with c_{+-} and c_{-+} some non-vanishing numbers.

The pertinent representations of $\hat{G}$ and $\hat{F}$ can be constructed by first considering the eigenvalue equation for $\hat{Q}$,

$$\hat{Q}|\mu\rangle = -\frac{i}{2}(\hat{G}\hat{F} + \hat{F}\hat{G})|\mu\rangle = \mu|\mu\rangle \quad (39)$$

We introduce Schrödinger polarization in F *i.e.* we realize the action of $\hat{F}$ multiplicatively on the wave functions $\Psi_\mu(F) = \langle F|\mu\rangle$ and that of $\hat{G}$ as a derivative. The solutions of (39) are then

$$\Psi_\mu(F) = \langle F|\mu\rangle \propto F^{\mu+\frac{1}{2}} \quad (40)$$

Notice that here μ can be an arbitrary complex number even though $\hat{Q}$ is formally anti Hermitean: standard arguments do not apply since the action of $\hat{Q}$ has been realized in a vector space with a non-separable basis and without a finite inner product.

In order to extract a separable Hilbert space from (40), we need to select a countable subset of (40) and construct the dual space and the ensuing inner product structure. Since there are infinitely many unitarily inequivalent ways to select a countable subset of (40) it is necessary to introduce further conditions:

Obviously, the separable Hilbert space that we wish to construct should be isomorphic with the Hilbert space in the Dirac quantized theory, where physical states are those

annihilated by the gauge generator $\varphi_a \rightarrow \hat{G}$,

$$\hat{G}|phys\rangle = 0 \quad (41)$$

In order to select the pertinent subset of (40) and to construct the dual basis and inner product, following [13] we then consider the complete set of basis vectors for a quantum mechanical harmonic oscillator with arbitrary frequency ω ,

$$\psi_n(q) = \left[\frac{\sqrt{\omega}}{\sqrt{\pi} 2^n n!} \right]^{\frac{1}{2}} e^{-\omega q^2} H_n(\sqrt{\omega} q) \quad (42)$$

Here H_n are Hermite polynomials, and these states are known to be orthonormal with respect to the standard inner product on a real line with the dual basis identified with the basis.

From (42) we construct a new basis as follows: Instead of (42) consider

$$\tilde{\psi}_n(q) = \left[\frac{1}{\sqrt{2\pi\omega^n 2^n n!}} \right]^{\frac{1}{2}} H_n(\sqrt{\omega} q) \quad (43)$$

With the standard inner product on the real line, the dual basis is then

$$\tilde{\psi}_n^\dagger(q) = \left[\left[\frac{2}{\pi} \right]^{\frac{1}{2}} \frac{\omega^{n+1}}{n!} \right]^{\frac{1}{2}} e^{-\omega q^2} H_n(\sqrt{\omega} q) = (-1)^n \left[\frac{\sqrt{2\pi}}{n!} \right]^{\frac{1}{2}} \frac{d^n}{dq^n} \left[\left[\frac{\omega}{\pi} \right]^{\frac{1}{2}} e^{-\omega q^2} \right] \quad (44)$$

Obviously, (43) and (44) are dual to each other independently of ω . Hence we can let $\omega \rightarrow \infty$, and in this limit (43) yields

$$\tilde{\psi}_n(q) \xrightarrow{\omega \rightarrow \infty} \frac{1}{\sqrt{2\pi n!}} q^n \quad (45)$$

while (44) becomes

$$\tilde{\psi}_n^\dagger(q) \xrightarrow{\omega \rightarrow \infty} (-1)^n \left[\frac{\sqrt{2\pi}}{n!} \right]^{\frac{1}{2}} \frac{d^n}{dq^n} \delta(q) \quad (46)$$

The basis (45) can be generated from the constant (*i.e.* $n = 0$) function using $\hat{q}$ as a creation operator. The operator $\hat{p}$ is then an annihilation operator and in particular, $\hat{p}$ annihilates the ground state $n = 0$. Thus the $n = 0$ state can be viewed as a highest weight state.

Similarly, we conclude for the dual basis (46) that $\hat{p}$ is a creation operator and $\hat{q}$ an annihilation operator. In particular, the highest weight state $n = 0$ in (46) is annihilated by $\hat{q}$.

From the previous discussion we abstract the desired representation of the operators $\hat{G}$ and $\hat{F}$ as follows: We introduce two states $|G\rangle$ and $|F\rangle$ such that

$$\hat{G}|G\rangle = \hat{F}|F\rangle = 0 \quad (47)$$

and two vector spaces, the first consisting of states $|r\rangle_G$ obtained by acting on $|G\rangle$ by $\hat{F}$ as a creation operator, and the second consisting of states $|r\rangle_F$ obtained by acting on $|F\rangle$ by $\hat{G}$ as a creation operator. Since

$$\hat{Q}|G\rangle = -\frac{1}{2}|G\rangle \Rightarrow |G\rangle = |-\frac{1}{2}\rangle_G \quad (48.a)$$

$$\hat{Q}|F\rangle = +\frac{1}{2}|F\rangle \Rightarrow |F\rangle = |+\frac{1}{2}\rangle_F \quad (48.b)$$

the states $|r\rangle_G$ are eigenstates of $\hat{Q}$ with eigenvalues $r = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$ while the states $|r\rangle_F$ are eigenstates of $\hat{Q}$ with eigenvalues $r = +\frac{1}{2}, +\frac{3}{2}, +\frac{5}{2}, \dots$. We identify the states $|r\rangle_G$ and $|r\rangle_F$ as dual to each other, with non-vanishing inner products

$${}_F\langle r|s\rangle_G = c_r\delta(r+s) \quad (49)$$

We are interested in the BRST operator $\hat{\Omega} = \hat{\eta}\hat{G}$ in (28.b) and its action in the vector space spanned by the direct product states $|r\rangle_G \otimes |\pm\frac{1}{2}\rangle_P$. Since $\hat{\Omega}$ raises the $\hat{Q}$ quantum number of a state by one and lowers the $\hat{N}$ quantum number by one, we conclude that all states of the form $|r, +\frac{1}{2}\rangle = |r\rangle_G \otimes |+\frac{1}{2}\rangle_\eta$, $r = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$ are not only closed in the $\hat{\Omega}$ cohomology, but in fact exact since they are obtained from states $|r-1, -\frac{1}{2}\rangle$ by acting on these states with $\hat{\Omega}$.

There is only one state in the representation which is closed but not exact. This state is $|-\frac{1}{2}, -\frac{1}{2}\rangle = |-\frac{1}{2}\rangle_G \otimes |-\frac{1}{2}\rangle_\eta$ and as a consequence the $\hat{\Omega}$ cohomology is simply

$$\hat{\Omega}|\phi\rangle = 0 \Rightarrow |\phi\rangle = |-\frac{1}{2}, -\frac{1}{2}\rangle \mod |r, +\frac{1}{2}\rangle \quad (50)$$

Since $\hat{G}|-\frac{1}{2}\rangle_G = 0$, there is an obvious isomorphism between $|-\frac{1}{2}, -\frac{1}{2}\rangle$ and the physical state (41) in the Dirac quantization. The cohomology in (50) defines the (quantum) BRST gauge freedom of this state, and in order to eliminate this freedom and to uniquely specify the representative $|-\frac{1}{2}, -\frac{1}{2}\rangle$ as the representative of the equivalence class (50), we introduce the additional conditions (24). At the present level the relevant condition is that $|\phi\rangle$ is annihilated by $R^{+\bar{\theta}} \sim \hat{\mathcal{P}}\hat{F}$. This then uniquely specifies the state

$|\frac{1}{2}, -\frac{1}{2}\rangle$ in the equivalence class (50) and establishes the isomorphism between the present approach and the Dirac approach.

Including the remaining operators in (28) with a fixed³ we find that for each a there is a unique state $|\tilde{\phi}\rangle$ that satisfies (24) *i.e.* is an $\text{Osp}(1,1|2)$ singlet state,

$$|\tilde{\phi}\rangle = |\frac{1}{2}\rangle_G \otimes |\frac{1}{2}\rangle_P \otimes |\frac{1}{2}\rangle_\lambda \otimes |\frac{1}{2}\rangle_{\bar{\eta}} \quad (51)$$

A further generalization to the direct sum representation, obtained by summing over all a in (28), yields that in the tensor product over a there is only one $\text{Osp}(1,1|2)$ singlet state, *i.e.* the unique solution to (24) is

$$|\tilde{\phi}\rangle = \otimes_a \left\{ |\frac{1}{2}\rangle_{G_a} \otimes |\frac{1}{2}\rangle_{P_a} \otimes |\frac{1}{2}\rangle_{\lambda_a} \otimes |\frac{1}{2}\rangle_{\bar{\eta}_a} \right\} \quad (52)$$

See also [14]. In particular, for each a only the highest weight, *i.e.* Fock vacuum states in the unphysical sector, appear in (52). This implies that with $\mathcal{H}_o$ the Hilbert space of the reduced phase space quantization, the physical Hilbert space $\mathcal{H}_{phys}$ over the extended phase space has the structure

$$\mathcal{H}_{phys} \sim \mathcal{H}_o \otimes |\tilde{\phi}\rangle \quad (53)$$

locally over the polarization of this phase space.

The physical Hilbert space of the original extended phase space variables is related to (53) by the locally defined Bogoliubov transformations (25), (29). Since a Bogoliubov transformation maps a Fock vacuum onto a Fock vacuum, we conclude that in terms of the original phase space variables the physical Hilbert space necessarily has the same structure (53). Provided there is no anomaly in the Bogoliubov transformation [5] and no obstructions in patching the locally defined Fock states over the polarization of the phase space, we then have the no-ghost theorem: Only the Fock vacua of the unphysical variables are consistent with the physical state condition (24), ensuring that unphysical degrees of freedom indeed cannot be excited.

For gauge theories with globally defined $\text{Osp}(1,1|2)$ generators, for example Yang-Mills theory in the axial gauge, the previous arguments are sufficient to ensure the no-ghost theorem non-perturbatively. However, in gauges where the $\text{Osp}(1,1|2)$ generators are determined perturbatively using expansions of the form (16) as in Coulomb gauge Yang-Mills theory, the previous arguments are only sufficient to ensure the no-ghost

³This is necessary for covariance.

theorem perturbatively. In order to extend the no-ghost theorem into a non-perturbative statement it is necessary to patch the locally defined Coulomb gauge theory into a globally defined theory [15].

In conclusion, we have outlined how the explicit construction of the $Osp(1,1|2)$ generators presented in [4,5] can be generalized to arbitrary constraint algebras of the form (7), and indicated how this construction can be implemented in the case of Yang-Mills theory. We have clarified the operator interpretation of $Osp(1,1|2)$ invariance, and in particular explained the identification of physical states with the singlet representation of $Osp(1,1|2)$. We have also presented a detailed discussion of the no-ghost theorem and its connection with the $Osp(1,1|2)$ singlet states in the general framework.

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