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HIDDEN QUANTUM SYMMETRIES IN RATIONAL CONFORMAL FIELD THEORIES

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ABSTRACT

We show that to any Rational Conformal Field Theory one can associate a new symmetry algebra (a quantum group) which neatly includes the fusion algebra and the polynomial equations. We study the general properties of this algebra and present some examples. We also show that for a very large class of Conformal Field Theories one can associate to each one of them a representation of the Hecke algebra, and a collection of knot and link invariants.

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I INTRODUCTION.

One of the open problems in String Theory [1] and Conformal Field Theory (CFT) is the classification of conformally invariant two dimensional field theories. There has been steady progress in the classification of Rational Conformal Field Theories (RCFT) (see for example [2-5]). These are theories with a finite number of primary fields. Recently , important steps were taken in the understanding of the classification problem using the fusion algebra [6] and the polynomial equations [7-9] which characterize a modular invariant RCFT. In this paper we want to show that to any RCFT one can associate a quantum symmetry (in the sense of quantum groups) which neatly encompasses both the fusion algebra and the polynomial equations. In particular , one of the representations of this quantum algebra can be interpreted as the open path version of the Verlinde operators [6]. From an abstract point of view, one starts with a set of fusion rules N_{ij}^k . Given these , we can define an algebra of operators V (the chiral vertices [9,10]) with sewing rules allowing us to construct conformal blocks with the requirement that the blocks provide representations of the braid and modular groups , and also satisfy the duality property of the conformal amplitudes : one of the basic dynamical properties of any CFT [2]. Once this basic structure is given , we can construct an algebra Q of symmetries which leads in particular to the polynomial equations. We have found some simple representations of the algebra Q , but more general representations are possible. In section II we review some of the basic properties of any CFT that will be useful to us later. In the second part of this section we define the algebra Q and some of its properties , we discuss the representation in terms of open Verlinde operators , and derive its action on characters. In section III we work out an explicit example using the Coulomb gas representation of the correlation functions for the Ising model. In section IV we show how , to very general CFT's one can associate a knot polynomial. We believe that this correspondence between CFT and knot polynomials will help in extending Witten's work [11] relating the three dimensional Chern-Simons theory to the Wess-Zumino-Witten models [3] in two dimensions. This will strengthen the intriguing connection between purely two dimensional theories and intrinsically three-dimensional objects : knots and links.

II GENERAL STRUCTURE OF A RCFT.

IIa. We begin by recalling some of the basic general properties of a (Rational) Conformal Field Theory. It is well known that conformal invariance plays a very important role in the study of critical phenomena and in the classification of classical solutions to string theory [1]. One of the characteristic features of a CFT is its infinite dimensional chiral algebra $\mathcal{O}$. This algebra depends on the model, but it always contains the identity and the energy-momentum tensor $T(z)$. For example, in the minimal models [2] $\mathcal{O}$ is just the Virasoro algebra, in the Wess-Zumino-Witten (WZW) models we have a Kac-Moody algebra [3,4,5], etc. In general, the left and right algebras are not necessarily the same, and we have in general a symmetry algebra $\mathcal{G} = \mathcal{G}_L \oplus \mathcal{G}_R$. In this paper we will consider for simplicity the case when both algebras are isomorphic, but it will become clear later on that our considerations also apply to the more general case of different left and right algebras. The symmetry of the theory under $\mathcal{G}$ implies that the Hilbert space of states can be expressed as a direct sum of irreducible representations of $\mathcal{G}$:

$$\mathcal{H} = \bigoplus_{i, \tau} \mathcal{H}_i \otimes \mathcal{H}_\tau \quad (2.1)$$

where $\mathcal{H}_i$ (resp. $\mathcal{H}_\tau$) is an irreducible representation of $\mathcal{G}_L$ (resp. $\mathcal{G}_R$). Each of the factors in the sum (2.1) is generated by a primary field (highest weight vector) of the operator algebra of the theory: $\phi_{i, \tau}(z, \bar{z})$. The other states (descendants) in $\mathcal{H}_i \otimes \mathcal{H}_\tau$ are obtained by acting on $\phi_{i, \tau}(z, \bar{z})$ with the creation operators of the algebra $\mathcal{O}$. A CFT is said to be rational if the number of representations appearing in (2.1) is finite.

Other basic properties of CFT's are locality, and associativity of the operator product expansion (OPE). Since the associativity property plays a very important role in any discussion on CFT, we present it in detail. If ϕ_i, ϕ_j, ϕ_k are primary operators in the theory, the structure constants of the operator algebra are given by the three point functions on the sphere:

$$\begin{aligned} & \langle \phi_i(z_i, \bar{z}_i) \phi_j(z_j, \bar{z}_j) \phi_k(z_k, \bar{z}_k) \rangle \\ &= c_{ijk} z_{ij}^{-\Delta_{ij}} z_{ik}^{-\Delta_{ik}} z_{jk}^{-\Delta_{jk}} \bar{z}_{ij}^{-\bar{\Delta}_{ij}} \bar{z}_{ik}^{-\bar{\Delta}_{ik}} \bar{z}_{jk}^{-\bar{\Delta}_{jk}} \end{aligned} \quad (2.2)$$

$$\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k, \text{ etc.}$$

and $\Delta_i, \Delta_j, \Delta_k$ are the conformal spins of the primary fields ϕ_i, ϕ_j , and ϕ_k respectively. The spectrum of dimensions $(\Delta_i, \bar{\Delta}_i)$ is one of the properties characterizing a CFT. Once the

three-point functions (2.2) are known, those for the descendant fields are consequences of the Ward identities of the algebra $\mathcal{U}$. To be more precise in (2.2), one would have to add an extra index counting all inequivalent couplings of the representations ϕ_i, ϕ_j, ϕ_k . This is equivalent to specifying the fusion algebra

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k \quad (2.3)$$

where N_{ij}^k is a set of integers counting the number of different couplings. The integral matrices $(N_i)_j^k = N_{ij}^k$ have played a prominent role in recent developments [6,7,8].

The associativity of the OPE can be expressed in terms of the crossing symmetry of the four point functions. Using the $SL(2, \mathbb{C})$ invariance of the amplitudes on the sphere, the associativity of the OPE is equivalent to the requirements :

$$G_{nm}^{\ell k}(x, \bar{x}) = G_{n\ell}^{mk}(1-x, 1-\bar{x}) = x^{-2\Delta_n} \bar{x}^{-2\bar{\Delta}_n} G_{nk}^{\ell m}\left(\frac{1}{x}, \frac{1}{\bar{x}}\right) \quad (2.4)$$

$$G_{nm}^{\ell k}(x, \bar{x}) = \langle k | \phi_\ell(1, 1) \phi_n(x, \bar{x}) | m \rangle$$

(see [2] for details) . Using the OPE one can write the correlation functions (2.4) as sums of holomorphic times antiholomorphic quantities : the conformal blocks :

$$G_{nm}^{\ell k}(x, \bar{x}) = \sum_p C_{nm}^p C_{k\ell p} \left| \mathcal{H}_p^{k\ell nm}(x) \right|^2 \quad (2.5)$$

$\mathcal{H}_p^{k\ell nm}(x)$ is completely determined by the properties of the theory with respect to the algebra. We can represent $\mathcal{H}_p^{ijkl}$ diagrammatically as in fig.1, where p is the intermediate state or operator appearing in the OPE of kl . The duality of the amplitudes (2.4) can be expressed as in fig.2. Geometrically the blocks $\mathcal{H}_p^{ijkl}(x)$ can be thought of as holomorphic sections of a (projectively) flat vector bundle over the moduli space of the sphere with four distinguished points. The dimensions of these vector bundles depends on the number of couplings at each vertex in fig.2, and it is given by $\sum_p N_{ij}^p N_{kl p}$. What we learn from the previous discussion is that the basic building blocks of the theory are the three point functions (fig.3) : $V_{0,ijk}$. To obtain local correlation functions, we need to find the monodromy invariant combinations of the left and right blocks (2.5). The metric producing the invariant is determined by the structure constants of the operator algebra. One could however turn things around. If we knew the conformal blocks from some independent construction, then the

structure constants of the OPE can be determined by finding monodromy invariants.

The duality invariance on the physical amplitudes as well as the associativity of the OPE imply that the blocks $\{ \mathcal{H}_p^{ijkl}(z) \}$ provide a representation of the braid group on four strands. More explicitly, if by analytic continuation we exchange (braid) j and k in fig.2, then (using the notation of ref. [7]) :

$$\begin{array}{c} j \\ | \\ i \text{---} \text{---} p \text{---} \text{---} l \\ | \\ k \end{array} = \sum_{p'} C_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \begin{array}{c} k \\ | \\ i \text{---} \text{---} p' \text{---} \text{---} l \\ | \\ j \end{array} \quad (2.6)$$

(and similarly for higher point blocks). In the language of quantum field theory (2.6) is equivalent to requiring s-u duality of the amplitudes. The associativity of the OPE is then the requirement of s-t duality, which for conformal blocks (again in the notation of [7]) becomes:

$$\begin{array}{c} j \\ | \\ i \text{---} \text{---} p \text{---} \text{---} l \\ | \\ k \end{array} = \sum_q N_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \begin{array}{c} j & k \\ \diagdown & \diagup \\ & q \\ | \\ i \text{---} \text{---} l \end{array} \quad (2.7)$$

It was shown in [7] that the matrices C and N satisfy some polynomial equations (the hexagon and pentagon identities respectively) which may be taken as the starting point for the classification of RCFT's.

Since braiding (i,j) in the left hand side of (2.7) is equivalent to performing half a Dehn twist around the (i,j) points, it is possible to represent the matrix N_{pq} in terms of the braiding matrix $C_{pp'}$ and the braiding of (i,j) and (k,l) . Using the OPE rules

$$\phi_i(z) \phi_j(w) = \sum_p C_{ij}^p (z-w)^{-\Delta_i - \Delta_j + \Delta_p} \phi_p + \text{descendants}$$

the result of braiding i and j is simply to multiply $\mathcal{H}_p^{ijkl}$ on the left-hand side of (2.7) by $\exp(i\pi(\Delta_p - \Delta_i - \Delta_j))$, and similarly for the braiding of (k,l) . To obtain the diagram on the right hand side of (2.7), we need two operations : First we do the braiding indicated in fig.4, and second we need to perform a cycling transformation which follows from analytic continuation. The net result of these operations is :

$$\begin{aligned}
N_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} &= \varepsilon_{ij}^p \varepsilon_{il}^{p'} e^{i\pi(\Delta_p + \Delta_{p'} - \Delta_j - \Delta_l)} C_{pp'} \begin{bmatrix} i & k \\ j & l \end{bmatrix} \\
&= \varepsilon_{kl}^p \varepsilon_{il}^{p'} e^{i\pi(\Delta_p + \Delta_{p'} - \Delta_i - \Delta_k)} C_{pp'} \begin{bmatrix} j & l \\ i & k \end{bmatrix}
\end{aligned} \quad (2.8)$$

ε_{ij}^p is a + or - sign depending on whether p appears symmetrically or antisymmetrically in the operator product of i and j. This formula makes explicit the relation between C and N pointed out in [7].

One can interpret the non-vanishing three-point functions in an operator language in terms of the chiral vertices introduced in [9]. If the fusion coefficients $N_{ij}^k \neq 0$, we can construct an operator $\Phi \begin{pmatrix} j \\ i \ k \end{pmatrix}(z)$ mapping the representation space $\mathcal{H}_k$ into $\mathcal{H}_i$. This is just a "dual" way of thinking about the vertex in fig.3. The analytic properties of $\Phi \begin{pmatrix} j \\ i \ k \end{pmatrix}$ are such that it behaves as a primary field of the algebra $\mathcal{U}$ with respect to the label j. One can certainly construct similar operators transforming as descendant fields, and their properties would be completely determined by those of $\Phi \begin{pmatrix} j \\ i \ k \end{pmatrix}(z)$. Equivalently, we can think about these operators in terms of "colored" pants (fig.5), where the colors indicate the representation of the chiral algebra we consider on each boundary circle. If we saturate or close the j-boundary with the highest weight vector of the representation $\mathcal{H}_j$, then it is clear that fig.5 can be interpreted as a map from $\mathcal{H}_k$ to $\mathcal{H}_i$. In terms of chiral vertices, we can write the conformal blocks (fig.6):

$$\begin{aligned}
&\mathcal{G}_{p_1 \dots p_{n-3}}^{i_1 i_2 \dots i_n} (z_1, \dots, z_{n-2}) = \\
&= \langle i_1 | \Phi \begin{pmatrix} i_2 \\ i_1 \ p_1 \end{pmatrix}(z_1) \Phi \begin{pmatrix} i_3 \\ p_1 \ p_2 \end{pmatrix}(z_2) \dots \Phi \begin{pmatrix} i_{n-1} \\ p_{n-3} \ i_n \end{pmatrix}(z_{n-2}) | i_n \rangle
\end{aligned} \quad (2.9)$$

Hence given a CFT with a set of fusion rules N_{ij}^k , we automatically will have a set of vertices $\Phi \begin{pmatrix} j \\ i \ k \end{pmatrix}$ and the number of vertices with the same labels equals N_{ij}^k . Out of these operators one can in principle construct the blocks $\mathcal{G}_{p_1 \dots p_{n-3}}^{i_1 \dots i_n}(z_n)$ and the physical amplitudes. The braiding property of the blocks is equivalent to an exchange property of the chiral vertices [9,10]:

$$\Phi \begin{pmatrix} j \\ i \ p \end{pmatrix}(z) \Phi \begin{pmatrix} k \\ p \ l \end{pmatrix}(w) = \sum_{p'} C_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \Phi \begin{pmatrix} k \\ i \ p' \end{pmatrix}(w) \Phi \begin{pmatrix} j \\ p' \ l \end{pmatrix}(z) \quad (2.10)$$

and one should understand this equation in the sense of analytic continuation. Notice furthermore that in the sewing of colored pants (fig.5) one has to make sure that the colors match. We have a sewing operation in the space of chiral vertices $\star$, which can be

represented as

$$i \begin{array}{c} j \\ | \\ \hline p \end{array} \star_{p,m} \begin{array}{c} k \\ | \\ \hline m \\ \hline \ell \end{array} = \delta_{pm} i \begin{array}{c} j \\ | \\ \hline p \\ \hline \ell \end{array} \quad (2.11)$$

One can also define a trace over the representations $\mathcal{X}_i$, leading to characters or multiple point functions on surfaces of genus higher than zero. We can write this operation as

$$\tilde{\star}_{ii} \left(\begin{array}{c} j \\ | \\ \hline k \\ \hline m \end{array} \right) = \delta_{ik} \delta_{im} \text{Tr}_{\mathcal{X}_i} \begin{array}{c} j \\ | \\ \hline k \\ \hline m \end{array} = \begin{array}{c} j \\ | \\ \hline \bigcirc_i \end{array} \quad (2.12)$$

and similarly for higher point chiral vertices. When j is the identity operator, this operation produces the character of π_L in the representation $\mathcal{X}_i$. If V denoted the space of chiral vertices, using the operators $\star_{ij}$ and $\tilde{\star}_{ii}$ acting on $V \otimes V \otimes \dots \otimes V$ we can obtain the blocks for arbitrary amplitudes in any genus surface. More precisely, we obtain a basis of sections of a flat vector bundle over the moduli space of the genus g surfaces with N distinguished points and a particular φ^3 -marking of the surface. The φ^3 -skeleton specifies the way we sew together the colored three-point vertices. Different φ^3 -marking give different basis for the same vector bundle. For example, the dimension does not change as a consequence of the associativity of the fusion algebra, which is expressed by :

$$N_i N_j = N_j N_i \quad (2.13)$$

For example, the dimension of the bundle whose basis is represented by the blocks of fig.6 as the p 's vary is given by:

$$\sum_{p_1, \dots, p_{n-3}} N_{i_1 i_2}^{p_1} N_{p_1 i_3}^{p_2} \dots N_{p_{n-3} i_{n-1}}^{i_n} = d_{i_1 \dots i_n} \quad (2.14)$$

(see [6] for further details and properties of the fusion algebra).

Summarizing, when we are given a CFT with some fusion rules $\{N_{ij}^k\}$ we are given a space of chiral vertices $V = \{ \phi(i; j) \mid i, j \in \mathbb{Z} \}$ and two operations : $\{ \star_{ij}, \tilde{\star}_{ii} \}$. The first one is sewing, and the second is to take traces over the i -th representation. Furthermore, the algebra $\{V, \star_{ij}, \tilde{\star}_{ii}, N_{ij}^k\}$ should also satisfy the duality properties as well as modular invariance. Duality in particular implies that the blocks constructed from $\phi(i; j)$ with the operation $\star_{ij}$ generate representations of the braid and the modular (mapping class) group. The duality matrices $C_{pp'} \left[\begin{smallmatrix} j & k \\ i & \ell \end{smallmatrix} \right]$ and $N_{pp'} \left[\begin{smallmatrix} j & k \\ i & \ell \end{smallmatrix} \right]$ satisfy the polynomial equations in [7] which will be obtained later as a consequence of the quantum symmetry of any CFT. There are several useful

properties of any RCFT which can be obtained from the general properties required of V . We recall that the braid group on n -strands has $n-1$ generators σ_i [12] shown in fig.7, and satisfying the relations :

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| \geq 2\end{aligned}\quad (2.15)$$

the fact that the blocks provide representations of the braid group implies that the C -matrices must satisfy relations similar to (2.15). This condition is best expressed in terms of a matrix $R_{s_1 s_2, s_3 s_4}$ where each s_a is one of the labels with three indices $\begin{pmatrix} j_a \\ i_a \end{pmatrix}$ appearing in the chiral vertices. Then the exchange algebra can be written as

$$\Phi_{s_1}(z) \Phi_{s_2}(w) = \sum_{s_3, s_4} R_{s_1 s_2, s_3 s_4} \Phi_{s_3}(w) \Phi_{s_4}(z) \quad (2.16a)$$

with

$$\begin{aligned}R_{s_1 s_2, s_3 s_4} &= \delta_{i_1 i_3} \delta_{k_2 k_4} \delta_{j_1 j_4} \delta_{j_2 j_3} \delta_{k_1 i_2} \delta_{k_3 i_4} C_{k_1 k_3} \begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix} \\ s_a &= \begin{pmatrix} j_a \\ i_a \end{pmatrix} \quad a=1,2,3,4\end{aligned}\quad (2.16b)$$

the δ -function keeps track of the sewing constraints and of the fields being braided. Before writing down explicitly the constraints on R due to (2.15) we discuss some notation. If V is a vector space of dimension N , R is a matrix in $V \otimes V$. The equation following from (2.15) can be most easily written down as an equation in $V \otimes V \otimes V$. We write R_{12} as a matrix in $V \otimes V \otimes V$ acting as R in the first and second vector spaces, and as the identity matrix in the third. Similarly we can define the operators R_{13} and R_{23} . The constraint on C implied by the first equation in (2.15) is the quantum Yang-Baxter equation :

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23} \quad (2.16c)$$

The second consequence of the general structure is due to the fact that the n -point blocks at three level define a representation of the sphere braid group $B_n(S^2)$, which is generated by n elements as in (2.15) plus two extra relations [12]. One of them is

$$\sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2 \sigma_1 = 1 \quad (2.17)$$

(the best way to understand why this is true , is to look at fig.8. On the sphere , the strand that wraps around the others can be taken around infinity to give the trivial braid). As it stands however, (2.17) will not hold for the $R(\sigma_i)$ matrices. The reason for this is simple. When we consider the braid group B_n , the points $P_1, \dots, P_n$ are given without any further specification. In conformal field theory (or more generally in field theory) we have the points P_i together with local parameters z_i in a neighborhood of P_i satisfying $z_i(P_i)=0$. Having a local parameter at each P_i implies in particular that there is a vector assigned to each P_i . Thus each braid comes with a frame. We may end up with Dehn twists around the points P_i after performing some braiding operations, and in principle we should not expect (2.17) to hold for the $R(\sigma_i)$ matrices. The right-hand side may become a phase. A Dehn twist around P_i is represented by $\exp 2\pi i L_0^{(i)}$, where $L_0^{(i)}$ is the generator of dilatations in the i -th Hilbert space $\mathcal{H}_i$. Hence to find the possible extra phase appearing in (2.17) we want to separate the phases due to the framing of the braid from those intrinsically due to analytic continuation. Using the $SL(2, \mathbb{C})$ invariance of the n -point amplitudes on the sphere , we can write the general n -point block in the form :

$$\mathcal{H}_{\mathbf{I}}^{i_1 \dots i_n}(z_1, \dots, z_n) = \left(\prod_{i < j} z_{ij}^{-\gamma_{ij}} \right) f_{\mathbf{I}}^{i_1 \dots i_n}(\eta_1, \dots, \eta_{n-3})$$

where

$$\sum_{j \neq i} \gamma_{ij} = 2 \Delta_i$$

$\mathbf{I}$ represents collectively the internal states in the block , and η_a $a=1,2,\dots,n-3$ are harmonic ratios between four points for some z_a, z_b, z_c, z_d . When we braid the external legs, the matrix R will have two contributions : One coming from the permutations of the $f_{\mathbf{I}}^l$ s due to analytic continuation , and the other will be a phase coming from the pre-factor which keeps track of how the frame moves. Since braiding z_i and z_j , corresponds to the process of analytic continuation $z_{ij} \rightarrow e^{i\pi} z_{ji}$ it is now easy to see the overall phase generated by the prefactor after we perform the braiding in (2.17) to be :

$$e^{-2\pi i \sum_{j \neq i} \gamma_{ij}}$$

since from $SL(2, \mathbb{C})$ invariance $\sum_{j \neq i} \gamma_{ij} = 2 \Delta_i$ [2] , we finally obtain :

$$R(\sigma_1) \dots R(\sigma_{n-2}) R(\sigma_{n-1}^2) R(\sigma_{n-2}) \dots R(\sigma_1) = e^{-4\pi i \Delta_i} \quad (2.18)$$

This relation implies constraints on the possible anomalous dimensions of a conformal field theory which are equivalent to those found in [13] using the defining relations for the mapping class group on the sphere with four distinguished points with local parameters. Some examples will be worked out in detail in sec.IV . As a further example consider the block $\mathcal{F}_i^{\uparrow\uparrow\uparrow\uparrow}$ in a theory with fusion rule $\phi \times \phi = \psi_1 + \dots + \psi_k$. Taking the determinant of (2.18) we obtain :

$$e^{2\pi i (4k\Delta\phi - 3 \sum_{i=1}^k \Delta_i)} = 1$$

It is an interesting exercise to work out the implications of (2.18) in known conformal theories.

IIb. The main purpose of this section is to find the symmetries of the algebra of chiral operators V compatible with the sewing and duality operations. As we will see one representation of the automorphisms of this structure is generated by a set of operators which can be interpreted as the "open path" version of the Verlinde operators [6] ; and the compatibility conditions become the polynomial equations in [7]. Before presenting the details of the construction, we would like to stress some of the properties of this new symmetry algebra Q

- a). There is an operator in the algebra for each primary field and every path in the ϕ^3 -like diagram defining the basis of conformal blocks.
- b). The defining relations of the algebra associate a quantum group to every conformal theory.
- c). The structure of this algebra seems more basic than the polynomial equations in [7]. In fact , from an abstract point of view , knowledge of the fusion algebra $\{N_{ij}^k\}$ together with the requirement of having a representation of the braid group imply that the symmetry algebra generates the polynomial equations.

We want to find a set of operators $X : V \rightarrow V$ which are compatible with three properties : 1). Sewing 2). Braiding and 3). s-t duality (2.7) which is closely related to the associativity of the operator algebra. First we write down these conditions in abstract form , and then we find representations which will make clear that the X operators generate a quantum group. Next we will show that 1-3) imply the polynomial equations , and finally we will compute the action of these operators on characters. This together with the polynomial equations imply the validity of Verlinde's conjecture ([6],[7], see also [14]). To simplify the notation we write the chiral vertices as $\mathcal{F}_s$, where s stands for the three label index $s = \begin{pmatrix} j \\ i \ k \end{pmatrix}$. Conditions 1)-3) become in equations :

$$X(\Phi_{s_1} * \Phi_{s_2}) = X(\Phi_{s_1}) * X(\Phi_{s_2}) \quad (2.19a)$$

$$X(R(\Phi_{s_1} * \Phi_{s_2})) = R(X(\Phi_{s_1}) * X(\Phi_{s_2})) \quad (2.19b)$$

$$X(N(\Phi_{s_1} * \Phi_{s_2})) = N(X(\Phi_{s_1}) * X(\Phi_{s_2})) \quad (2.19c)$$

(2.19a) is the statement that $X \in Q$ is compatible with sewing. (2.19b) indicates that the operations of braiding and acting with elements of Q commute, and (2.19c) implies that X preserves s-u duality (2.7), and it also makes manifest the relation between Q and the operator algebra. The operation N defined in (2.7) can be thought of as performing the OPE between the primary fields labelled by $j(s_1)$ and $j(s_2)$. If one stares long enough at (2.19a,b,c) it becomes clear that the operators $X \in Q$ will depend on paths in the block generated by $\Phi_{s_1} * \Phi_{s_2}$, and on the operators whose OPE is represented by the operation $N_{pp'}[j^k]$. We would like to emphasize that for the moment we take R and N as defined abstractly, representing the operations of s-u and s-t duality in the space of φ^3 -colored graphs compatible with the fusion coefficients N_{ij}^k .

There are several ways of finding operators X satisfying the defining relations (2.19a,b,c) for Q . The combination of a) and b) suggests that the X 's are closely related to a quantum group. Defining :

$$X(\Phi_{s_1}) = \sum_{s_2} C_{s_1 s_2}^{(1)}(X) \Phi_{s_2} \quad (2.20)$$

then

$$\begin{aligned} X(\Phi_{s_1} * \Phi_{s_2}) &= \sum_{s'_1, s'_2} C_{s_1 s_2, s'_1 s'_2}^{(2)}(X) \Phi_{s'_1} * \Phi_{s'_2} \\ &= \sum_{s'_1, s'_2} C_{s_1 s'_1}^{(1)}(X) C_{s_2 s'_2}^{(1)}(X) \Phi_{s'_1} * \Phi_{s'_2} \end{aligned} \quad (2.21)$$

and similarly for higher products, $C_{s_1 \dots s_n, s'_1 \dots s'_n}^{(n)}$ is constructed in terms of $C_{ss'}^{(1)}$. In components, equation (2.19b) becomes :

$$\sum_{s'_1, s'_2} R_{s_1 s_2, s'_1 s'_2} C_{s'_1 s'_2, s_3 s_4}^{(2)}(X) = \sum_{s'_1, s'_2} C_{s_1 s_2, s'_1 s'_2}^{(2)}(X) R_{s'_1 s'_2, s_3 s_4} \quad (2.22)$$

(in order not to clutter the equations with more indices than necessary, we are not including the δ -functions $\delta_{h(s_1), i(s_2)}$ that follow from the products $\Phi_{s_1} * \Phi_{s_2}$, although the reader should be aware that such constraints are implicit in the definitions of R and $C^{(n)}(X)$).

When we combine (2.21) and (2.22) we obtain an equation of the form $R C_1 C_2 = C_2 C_1 R$; strongly reminiscent of the defining relation for a quantum group [15]. The constraint implied by (2.19c) relates $C^{(2)}(x)$ with $C^{(1)}(x)$ and N . This equation should be read as follows : On the right hand side ,we construct first the block $x(\Phi_{s_1}) * x(\Phi_{s_2})$ and then perform the OPE of two of the legs. This will generate a term of the form $N C^{(2)}(X)$, or $N C^{(1)}(X) C^{(1)}(X)$ if we use (2.21). In the left hand side we are first performing the OPE and then acting on with X . Thus X only acts on the chiral vertices resulting from the OPE , and therefore we obtain symbolically something of the form $C^{(1)}(x) N = N C^{(1)}(x) C^{(1)}(x)$. We will show later that the pentagon identity in [7] follows from this relation. Before pursuing in detail the consequences of (2.19c) , we want to explore in more detail the analogy of (2.19a,b) with quantum groups. One of the possible ways of introducing a quantum group is as follows : Let V be a vector space of dimension N , and R a matrix in $V \otimes V$; i.e. R has four indices $R_{ij,kl}$; $i,j,k,l=1,\dots,N$. We introduce a matrix of operators $T=(T_{ij})$ with each component acting on the same Hilbert space. If R satisfies the Yang-Baxter equation (2.16) , then , the quantum group (of Hopf algebra) $A(R)$ associated to the Yang-Baxter matrix R is generated by the operators 1 and T_{ij} satisfying the relation

$$T_1 T_2 R P = R P T_2 T_1 \quad (2.23a)$$

$$T_1 = T \otimes 1, \quad T_2 = 1 \otimes T, \quad P(v_1 \otimes v_2) = v_2 \otimes v_1$$

or in components :

$$\sum_{k,l} R_{ij,kl} T_{lk} T_{ls} = \sum_{k,l} T_{ik} T_{jl} R_{kl,rs} \quad (2.23b)$$

(notice that in (2.16c) and (2.23) there is a slight difference with the standard quantum group literature [13], this is due to the form we have written the exchange algebra between chiral vertices) the algebra $A(R)$ is automatically equipped with a co-multiplication $\Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}$, and the axioms for a quantum group are satisfied (since we will not need the formal definition of a quantum group , we refer the interested reader to the literature for further details [15]). One can think of $A(R)$ as the algebra of functions of the quantum group. For example, if $R=1$, $A(1)$ is the algebra of functions of $GL(N,C)$. Given $A(R)$, we can study its representation theory as with ordinary groups. For our discussion however we only need the adjoint representation , which is in many respects similar to the adjoint representation of a Lie algebra. In this representation the operators T_{ij} act on the space V itself and in a basis of V , they can be represented as

$$(T_{ij})_{ab} = R_{ibaj} \quad (2.24)$$

it is a simple exercise to verify that (2.23b) is indeed satisfied for this representation using the Yang-Baxter equation. If we take V to be the space of chiral vertices, we have a natural action of the adjoint representation (2.24) on it :

$$T_{s_1 s_2}(\Phi_{t_2}) = \sum_{t_1} R_{s_1 t_2 t_1 s_2} \Phi_{t_1} \quad (2.25)$$

we will find a representation of Q in terms of the algebra $A(R)$ associated to the Yang-Baxter matrix R (2.16). The reason why (2.25) is not enough to preserve the braiding relations is that nowhere in (2.25) is included the constraint that $\Phi_{s_1} * \Phi_{s_2}$ is zero unless $k(s_1) = i(s_2)$. However a linear combination of these operators does the job. In the algebra defined by the fusion rules, given a primary operator corresponding to the label a : ϕ_a , we can always find another operator $\phi_{\bar{a}}$ conjugate to it in the sense that $\phi_a \times \phi_{\bar{a}} = id + \dots$; this is certainly true for RCFT. In many cases $\phi_{\bar{a}}$ is not the same as ϕ_a . However, in order to make the arguments simpler, we will consider self-conjugate fields $\phi_a \times \phi_a = id + \dots$. The arguments below have a straightforward extension to the more general case at the expense of including explicitly some charge conjugation matrices. Hence, to each primary field ϕ_a we can associate the following operator acting on V :

$$\chi^a \left(\begin{array}{c} j \\ i \quad k \end{array} \right) = \sum_{m,n} R \left(\begin{array}{c} a \\ m \end{array} \right) \left(\begin{array}{c} j \\ i \quad k \end{array} \right), \left(\begin{array}{c} j \\ m \quad n \end{array} \right) \left(\begin{array}{c} a \\ n \end{array} \right) \begin{array}{c} j \\ m \quad n \end{array} \quad (2.26)$$

or more formally

$$\chi^a = \sum_{m,n} T_{\left(\begin{array}{c} a \\ m \end{array} \right) \left(\begin{array}{c} a \\ n \end{array} \right)} \quad (2.27)$$

where the dots represent the indices to be contracted with the chiral vertices. The proof that (2.26) satisfies (2.19b) uses the fact that the R matrix gives a representation of the braid group, in other words, we use the Yang-Baxter property of R . Although (2.26) looks a bit complicated at first sight, it has a simple geometrical interpretation. Consider the vertex in fig.9 and choose the path γ from i to j . Since $\phi_a \times \phi_{\bar{a}} = id + \dots$, imagine inserting the identity in the middle of γ factorized into $\phi_a, \phi_{\bar{a}}$ and then move ϕ_a to i and $\phi_{\bar{a}}$ to k . What (2.26) says is that for non-zero fusion coefficients N_{ai}^p and $N_{\bar{a}k}^{p'}$ we obtain new chiral vertices. In this case, the coefficients of this operator are not the fusion constants N_{ij}^k , but rather the braiding matrix R . Since a representation of the braid group gives a representation of the monodromy

group , it is clear that the operators χ^a will be compatible with the representation of the mapping class group carried by the given RCFT. To see that the operator (2.27) is truly the open path version of the operators in [6] , we compute how it acts on characters. Since the algebra of operators V contains a trace , we can obtain the character for the representation i as :

$$\chi_{ii} \left(\begin{array}{c|c} & \text{id} \\ \hline i & i \end{array} \right) = \text{Tr}_{\chi_i} \left(\begin{array}{c|c} & \text{id} \\ \hline i & i \end{array} \right) = \chi_i \quad (2.28)$$

the action of χ^a on χ_i is obtained by first acting on the chiral vertex , and then taking the trace. Using that braiding with the identity is equal to 1 , we obtain :

$$\chi^a(\chi_i) = \sum_m \text{Tr} \left(R \left(\begin{array}{c} a \\ m \end{array} \right) \left(\begin{array}{c} \text{id} \\ i \end{array} \right), \left(\begin{array}{c} \text{id} \\ m \end{array} \right) \left(\begin{array}{c} a \\ m \end{array} \right) \begin{array}{c|c} & \text{id} \\ \hline m & m \end{array} \right) \quad (2.29)$$

since the (non-vanishing) entries of R appearing in (2.29) are all equal to 1 , all the sum counts is the independent number of ways of coupling the fields a and i . Thus :

$$\chi^a(\chi_i) = \sum_m N_{ai}^m \chi_m \quad (2.30)$$

(the traces in (2.28,29) are taken over the representations of the chiral algebra of the theory). This shows how the general operators appearing in (2.19) should depend on paths drawn on a colored ψ^3 -graph. A more precise notation would be $\chi^a(\gamma; k)$, indicating the path where the identity is split , and the colors of the end points. (There is really no better substitute to understand these operations than working out a few examples. May be this is a good time to go on to section III before continuing to the end of this section).

Using the representation of X^a in (2.26) ,we now want to write in detail the constraints (2.19b,c). First , from the fusion rules and the braiding interpretation of (2.26) we can define a matrix C in analogy with (2.16b). Then X^a becomes :

$$X^a \Phi \left(\begin{array}{c} j \\ i \quad k \end{array} \right) = \sum_{b,c} C_{ic} \left[\begin{array}{c} a \quad j \\ b \quad k \end{array} \right] \Phi \left(\begin{array}{c} j \\ b \quad c \end{array} \right) \quad (2.31)$$

since the braiding of the chiral vertices is

$$\Phi \left(\begin{array}{c} j \\ i \quad r \end{array} \right) (1) \Phi \left(\begin{array}{c} k \\ r \quad \ell \end{array} \right) (2) = \sum_{r'} C_{rr'} \left[\begin{array}{c} j \quad k \\ i \quad \ell \end{array} \right] \Phi \left(\begin{array}{c} k \\ i \quad r' \end{array} \right) (2) \Phi \left(\begin{array}{c} j \\ r' \quad \ell \end{array} \right) (1) \quad (2.32)$$

(2.19b) implies that the two operations (2.31,32) commute. We obtain :

$$\begin{aligned}
& \sum_c C_{ic} \begin{bmatrix} a & j \\ b & r \end{bmatrix} C_{rd} \begin{bmatrix} a & k \\ c & e \end{bmatrix} C_{cc'} \begin{bmatrix} j & k \\ b & d \end{bmatrix} \\
& = \sum_{r'} C_{rr'} \begin{bmatrix} j & k \\ i & e \end{bmatrix} C_{ic'} \begin{bmatrix} a & k \\ b & r' \end{bmatrix} C_{r'd} \begin{bmatrix} a & j \\ c & e \end{bmatrix}
\end{aligned} \tag{2.33}$$

This is the hexagon identity [7]. Next consider (2.19c). We start with the relation :

$$\begin{array}{c} j & k \\ | & | \\ i & \text{---} r & \text{---} e \end{array} = \sum_{r'} N_{rr'} \begin{bmatrix} j & k \\ i & e \end{bmatrix} \begin{array}{c} j & k \\ \diagdown & \diagup \\ & r' \\ \text{---} i & \text{---} e \end{array} \tag{2.34}$$

applying first the quantum group to the left-hand side , and then implementing the OPE yields :

$$\sum_{b, c, d, r'} C_{ic} \begin{bmatrix} a & j \\ b & r \end{bmatrix} C_{rd} \begin{bmatrix} a & k \\ c & e \end{bmatrix} N_{cr'} \begin{bmatrix} j & k \\ b & d \end{bmatrix} \begin{array}{c} j & k \\ \diagdown & \diagup \\ & r' \\ \text{---} b & \text{---} d \end{array} \tag{2.35}$$

doing things in the reverse order leads to :

$$\sum_{r', b, d} N_{rr'} \begin{bmatrix} j & k \\ i & e \end{bmatrix} C_{id} \begin{bmatrix} a & r' \\ b & e \end{bmatrix} \begin{array}{c} j & k \\ \diagdown & \diagup \\ & r' \\ \text{---} b & \text{---} d \end{array} \tag{2.36}$$

combining both we obtain :

$$\sum_c C_{ic} \begin{bmatrix} a & j \\ b & r \end{bmatrix} C_{rd} \begin{bmatrix} a & k \\ c & e \end{bmatrix} N_{cr'} \begin{bmatrix} j & k \\ b & d \end{bmatrix} = N_{rr'} \begin{bmatrix} j & k \\ i & e \end{bmatrix} C_{id} \begin{bmatrix} a & r' \\ b & e \end{bmatrix} \tag{2.37}$$

as the explicit form of (2.19c). When we consider a RCFT , we know that there is a relation between C and N (2.8). Taking $\epsilon_{ij}^P = 1$ for simplicity :

$$N_{rr'} \begin{bmatrix} j & k \\ i & e \end{bmatrix} = e^{i\pi(\Delta_r + \Delta_{r'} - \Delta_j - \Delta_e)} C_{rr'} \begin{bmatrix} i & k \\ j & e \end{bmatrix}$$

substituting in (2.37) :

$$\begin{aligned}
& \sum_c e^{i\pi(\Delta_c - \Delta_d)} C_{ic} \begin{bmatrix} a & j \\ b & r \end{bmatrix} C_{rd} \begin{bmatrix} a & k \\ c & e \end{bmatrix} C_{cr'} \begin{bmatrix} j & k \\ b & d \end{bmatrix} \\
& = e^{i\pi(\Delta_r - \Delta_e)} C_{rr'} \begin{bmatrix} i & k \\ j & e \end{bmatrix} C_{id} \begin{bmatrix} a & r' \\ b & e \end{bmatrix}
\end{aligned} \tag{2.38}$$

the pentagon identity in [7] follows from the set of operations illustrated in fig.10 , and it gives

$$\sum_s N_{as} \begin{bmatrix} j & k \\ i & b \end{bmatrix} N_{bp} \begin{bmatrix} s & \ell \\ i & m \end{bmatrix} N_{sq} \begin{bmatrix} k & \ell \\ j & p \end{bmatrix} = N_{ap} \begin{bmatrix} j & q \\ i & m \end{bmatrix} N_{bq} \begin{bmatrix} k & \ell \\ a & n \end{bmatrix} \quad (2.39)$$

using (2.8) in (2.39) gives exactly (2.38). Hence the operation (2.19) is closely related to the associativity of the OPE.

This concludes our proof that the representation $\hat{X}^a$ in (2.31) of the algebra Q (2.19a-c) automatically generates the tree level polynomial equations. Since X acting on character is the same as the Verlinde operator, and since we have the pentagon equation, the requirement of modular invariance gives the proof of the Verlinde conjecture along the lines of [7] (see also [14]) and the rest of the polynomial equations.

In the previous discussion we have only considered representations of Q based on rather simple $\mathcal{Q}^3$ -graphs. If one considers more general representations, then one might be able to drop (2.19b,c) and replace them by more general conditions in the algebra Q . Some progress has been made in this regard, and it will be reported elsewhere [16]. One of the problems which is still puzzling in CFT is deciding a priori when a given operator algebra gives automatically a modular invariant spectrum. We believe that there should be some simple conditions on the algebra Q which guarantee modular invariance. These conditions seem to be closely related to the construction of knot invariants out of a CFT. In the next section we illustrate the ideas of this section in a simple example : the Ising model.

III. AN EXPLICIT EXAMPLE.

Since the discussion in the previous section was rather general, in this section we will compute explicitly the braiding matrices and the adjoint representation of the quantum group in some examples. We will use the Coulomb gas representation presented in [17] to compute the conformal blocks in the minimal models. Since some of the formulae for the general theory in the minimal series [2] are quite involved, we concentrate on the Ising model. We will also compute the form of the braiding matrices in the minimal series when there are only two internal states. This will be useful in the relation between RCFT and Jones polynomials to be explored in sec.IV.

The starting point of the Coulomb gas representation in [17] is to consider a scalar field defining an anomalous $U(1)$ -current algebra. This is done by adding an amount of charge

$-2\alpha_0$ at infinity. The energy-momentum tensor of ϕ is :

$$T(z) = -\frac{1}{4} : \partial_z \phi(z) \partial_z \phi(z) : + i\alpha_0 \partial_z^2 \phi(z) \quad (3.1)$$

with the standard logarithmic behavior of the two point function

$$\langle \phi(z) \phi(w) \rangle = -2 \ln(z-w)$$

(we will consistently ignore whenever possible the antiholomorphic part of the algebra). The central charge of the Virasoro algebra is then $c=1-24\alpha_0^2$. The charge added at infinity means that the only non-vanishing two point functions of normal ordered exponentials are :

$$\langle : e^{i(2\alpha_0-\alpha)\phi(z)} : : e^{i\alpha\phi(w)} : \rangle = (z-w)^{-2\alpha(\alpha-2\alpha_0)} \quad (3.2)$$

so that the field conjugate to $: e^{i\alpha\phi} :$ according to the inner product defined by (3.3) is $: e^{i(2\alpha_0-\alpha)\phi} :$. The two point function (3.2) implies that the conformal dimension of the field $: e^{i\alpha\phi} :$ of charge α is $\Delta(\alpha) = \alpha(\alpha-2\alpha_0) = \Delta(2\alpha_0-\alpha)$, and charge conservation states that in any correlation function $\langle : e^{i\alpha_1\phi(z_1)} : \dots : e^{i\alpha_n\phi(z_n)} : \rangle$, the condition $\sum \alpha_i = 2\alpha_0$ must hold. It is very important to realize that we can always construct two fields of dimension 1 :

$$\begin{aligned} \mathcal{J}_{\pm}(z) &= : e^{i\alpha_{\pm}\phi(z)} : \\ \alpha_+ + \alpha_- &= 2\alpha_0, \quad \alpha_+ \alpha_- = -1 \end{aligned} \quad (3.3)$$

since $\mathcal{J}_{\pm}$ have dimension 1, the charges Q_+, Q_- associated to $\mathcal{J}_{\pm}$ via contour integration will not change the conformal properties of the correlation functions, but they will alter the balance of charge. Using charge conservation and other simple arguments, it is shown in [17] that to obtain the Coulomb gas representation of the 4-point blocks in the minimal models, the charge α must be quantized according to the Kac formula :

$$\begin{aligned} \alpha = \alpha_{n,m} &= \frac{1-n}{2} \alpha_+ + \frac{1-m}{2} \alpha_- ; \quad n, m \geq 1 \\ \Delta_{n,m} = \Delta(\alpha_{n,m}) &= -\alpha_0^2 + \frac{1}{4} (n\alpha_+ + m\alpha_-)^2 \end{aligned} \quad (3.4)$$

and the charge balance is achieved by the insertion of the Q_+ and Q_- operators. For example, in the four point function of $: e^{i\alpha_{n,m}\phi} :$ one has to insert $(n-1) Q_+^1$ s and $(m-1) Q_-^1$ s operators. In this way one obtains an integral representation for the correlation functions :

$$\mathcal{H}_{C_N}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(z_i) = C \int_{C_N} dF_1 \dots dF_p d\gamma_1 \dots d\gamma_q \times$$

$$V_\alpha =: e^{i\alpha\phi} : \quad \langle V_{\alpha_1}(z_1) \dots V_{\alpha_4}(z_4) \prod_{i=1}^p \mathcal{O}_+(F_i) \prod_{j=1}^q \mathcal{O}_-(\gamma_j) \rangle \quad (3.5)$$

where C_N runs over a set of independent contours for the screening operators. The different contours are related to the internal representations appearing as intermediate states in the block, and C is a constant that has to be adjusted to give the correct normalization of the blocks [2]. With this explicit representation we can compute the monodromy and braiding properties of the conformal blocks, and the full amplitudes are determined by combining the holomorphic and antiholomorphic blocks into monodromy invariants :

$$\langle \phi_{\alpha_1}(z_1, \bar{z}_1) \dots \phi_{\alpha_4}(z_4, \bar{z}_4) \rangle$$

$$= \sum_{p, \bar{p}} h_{p\bar{p}} \mathcal{H}_p^{\alpha_i}(z_i) \overline{\mathcal{H}_{\bar{p}}^{\alpha_i}}(\bar{z}_i) \quad (3.6)$$

In the minimal series of Virasoro representations [2], we have

$$C = 1 - 6 \frac{(p-q)^2}{pq}, \quad p > q$$

$$\Delta_{r,s} = \frac{1}{4pq} \left[(rp - sq)^2 - (p-q)^2 \right], \quad \begin{matrix} 1 \leq r \leq p-1 \\ 1 \leq s \leq q \end{matrix} \quad (3.7)$$

a particularly simple example of (3.5) occurs when we compute 4-point functions involving two $\phi_{1,2}$ or two $\phi_{2,1}$ fields (in the Ising model for example, $c=1/2$, $\phi_{1,2}$ is the spin operator and $\phi_{2,1}$ is the energy operator, $\phi_{1,1}$ is the identity operator). For two of these fields, it is necessary to include a single contour in (3.5), and the blocks can be expressed as hypergeometric functions. Using $SL(2, \mathbb{C})$ invariance, and some simple manipulations, it is not difficult to obtain

$$\langle \phi_1(z_1) \dots \phi_4(z_4) \rangle = \left(\frac{z_{12} z_{14}}{z_{24}} \right)^{-\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4} \frac{(1-\eta)^{2\alpha_2\alpha_3} \eta^{2\alpha_3\alpha_4}}{z_{12}^{2\Delta_2} z_{13}^{2\Delta_3} z_{14}^{2\Delta_4}}$$

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} \quad \times \oint_C dt (1-t)^{2\alpha_2\alpha_4} (\eta-t)^{2\alpha_3\alpha_4} t^{2\alpha_4\alpha_2} \quad (3.8)$$

$$z_{ij} = z_i - z_j$$

(we have used an $SL(2, \mathbb{C})$ transformation to send $z_1 \rightarrow \infty, z_2 \rightarrow 1, z_3 \rightarrow \eta, z_4 \rightarrow 0$). Generically there are two contours for C . From the fusion rules of the minimal theories we know that

$$\begin{aligned}\phi_{112} \times \phi_{112} &= \phi_{111} + \phi_{113} \\ \phi_{211} \times \phi_{211} &= \phi_{111} + \phi_{311}\end{aligned}\quad (3.9)$$

If we specialize (3.8) to blocks with four ϕ_{112} 's or four ϕ_{211} 's, using $\alpha_{112} = -\frac{\alpha_-}{2}$, $\alpha_{211} = -\frac{\alpha_+}{2}$, $\Delta_{112} = \alpha_0 \alpha_- + \frac{1}{4} \alpha_-^2$, $\Delta_{211} = \alpha_0 \alpha_+ + \frac{1}{4} \alpha_+^2$, $\alpha_+ = \left(\frac{p}{q}\right)^{1/2}$, $\alpha_- = -\left(\frac{q}{p}\right)^{1/2}$ we obtain

$$\left\langle \prod_{i=1}^4 \phi_{112}(z_i) \right\rangle = (z_{13} z_{24})^{-2\Delta_{112}} [\gamma(1-\gamma)]^{q/2p} \oint_c dt [t(1-t)(\gamma-t)]^{-q/p} \quad (3.10a)$$

$$\left\langle \prod_{i=1}^4 \phi_{211}(z_i) \right\rangle = (z_{13} z_{24})^{-2\Delta_{211}} [\gamma(1-\gamma)]^{p/2q} \oint_c dt [t(1-t)(\gamma-t)]^{-p/q} \quad (3.10b)$$

the two possible contours appear in fig.11. This yields :

$$\begin{aligned}\mathcal{H}_{c_1}^{(112)}(z_i) &= N_1^{(112)} (z_{13} z_{24})^{-2\Delta_{112}} (1-\gamma)^{q/2p} \gamma^{1-q/2p} F\left(\frac{q}{p}, \frac{p-q}{p}, \frac{2(p-q)}{p}, \gamma\right) \\ \mathcal{H}_{c_2}^{(112)}(z_i) &= N_2^{(112)} (z_{13} z_{24})^{-2\Delta_{112}} (1-\gamma)^{q/2p} \gamma^{q/2p} F\left(\frac{q}{p}, \frac{3q-p}{p}, \frac{2q}{p}, \gamma\right)\end{aligned}\quad (3.11)$$

and similar results hold for $\mathcal{H}_{c_i}^{(211)}(z_i)$. The particular result for the unitary series [18] can be obtained if we set $p=q+1$. Choosing $q=3$, $p=4$ we obtain the Ising model with only three primary fields : $1, \epsilon, \sigma$. With the notation :

$$\mathcal{H}_p^{ijkl}(z_2, z_3) = \begin{array}{ccc} & j & k \\ & | & | \\ & (z_2) & (z_3) \\ i & \text{---} p & l \\ \text{---} & & \text{---} \\ (z_1=\infty) & & (z_4=0) \end{array} \quad (3.12)$$

(recall that before taking the limit $z_1 \rightarrow \infty$ we have to multiply the amplitude by $z_1^{2\Delta_1}$), and after some algebra we obtain the complete set of blocks for the Ising model :

$$\begin{aligned}\mathcal{H}_0^{0000} &= 1, & \mathcal{H}_\sigma^{0000} &= \frac{-1/8}{z_{23}}, & \mathcal{H}_\epsilon^{0000} &= \frac{-1}{z_{23}} \\ \mathcal{H}_\epsilon^{0000} &= \frac{-1/2}{z_{23}}, & \mathcal{H}_\sigma^{0000} &= \frac{-1/2}{z_{23}}, & \mathcal{H}_\sigma^{0000} &= \frac{3/8}{z_{23}} \\ \mathcal{H}_\sigma^{0000} &= \frac{3/8}{z_2^{1/2} z_{23}^{1/2}}, & \mathcal{H}_\epsilon^{0000} &= \frac{3/8}{z_2^{1/2} z_{23}^{1/2}}, & \mathcal{H}_\sigma^{0000} &= \frac{3/8}{z_2^{1/2} z_{23}^{1/2}}\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_0^{EEEE} &= \frac{z_2^2 + z_3^2 - z_2 z_3}{z_2 z_3 z_{23}} \\
\mathcal{T}_0^{EEEE} &= \frac{1}{\sqrt{2}} \left(\frac{1}{z_2 z_3 z_{23}} \right)^{1/8} (z_2^{1/2} + z_{23}^{1/2})^{1/2} \\
\mathcal{T}_\epsilon^{EEEE} &= \sqrt{2} \left(\frac{1}{z_2 z_3 z_{23}} \right)^{1/8} (z_2^{1/2} - z_{23}^{1/2})^{1/2} \\
\mathcal{T}_0^{EE\epsilon\epsilon} &= \frac{z_2 - \frac{1}{2} z_3}{z_2^{1/2} z_3^{1/2} z_{23}^{1/2}}, \quad \mathcal{T}_\epsilon^{EE\epsilon\epsilon} = \frac{z_2 - 2 z_3}{z_2^{1/2} z_3^{1/2} z_{23}^{1/2}} \\
\mathcal{T}_\epsilon^{EE\epsilon\epsilon} &= \frac{z_2 + z_3}{z_2^{1/2} z_3^{1/2} z_{23}^{1/2}}, \quad \mathcal{T}_\epsilon^{\epsilon\epsilon\epsilon\epsilon} = \frac{z_2 - 2 z_3}{z_2 z_3^{1/2} z_{23}^{1/2}} \\
\mathcal{T}_0^{\epsilon\epsilon\epsilon\epsilon} &= \frac{z_2 - \frac{1}{2} z_3}{z_2^{1/2} z_3 z_{23}^{1/2}}, \quad \mathcal{T}_\sigma^{\epsilon\epsilon\epsilon\epsilon} = \frac{z_2 + z_3}{z_2^{1/2} z_3^{1/2} z_{23}}
\end{aligned} \tag{3.13}$$

and all the blocks are normalized according to [2] so that for $z_2=1$, and $z_3=\gamma \rightarrow 0$, the blocks behave as

$$\mathcal{T}_p^{ij\ k\ell}(\gamma) \rightarrow \gamma^{\Delta_p - \Delta_k - \Delta_\ell} \quad (1 + \dots)$$

With the explicit formulae (3.13), we can evaluate the braiding matrices $C_{pp'} \left[\begin{smallmatrix} j & k \\ i & \ell \end{smallmatrix} \right]$. The only complication comes with the braiding of the first two blocks in (3.13), where the identity

$$\left[y^{1/2} + i(x-y)^{1/2} \right]^{1/2} = \frac{e^{i\pi/4}}{\sqrt{2}} \left[x^{1/2} + (x-y)^{1/2} \right]^{1/2} + \frac{e^{-i\pi/4}}{\sqrt{2}} \left[x^{1/2} - (x-y)^{1/2} \right]^{1/2}$$

has to be used. We will write instead of the matrix $C_{pp'}$ the matrix $R_{s_1 s_2, s_3 s_4}$. To do this, we label the chiral vertices as follows $1 = \begin{pmatrix} \epsilon \\ \epsilon o \end{pmatrix}$, $2 = \begin{pmatrix} \epsilon \\ o \epsilon \end{pmatrix}$, $3 = \begin{pmatrix} \epsilon \\ \sigma \sigma \end{pmatrix}$, $4 = \begin{pmatrix} \sigma \\ \sigma o \end{pmatrix}$, $5 = \begin{pmatrix} \sigma \\ o \sigma \end{pmatrix}$, $6 = \begin{pmatrix} \sigma \\ \sigma \epsilon \end{pmatrix}$, $7 = \begin{pmatrix} \sigma \\ \epsilon \sigma \end{pmatrix}$. We can also include the chiral vertices $\begin{pmatrix} o \\ o o \end{pmatrix}$, $\begin{pmatrix} o \\ \epsilon \epsilon \end{pmatrix}$, $\begin{pmatrix} o \\ \sigma \sigma \end{pmatrix}$, but their braiding properties are straightforward. All one has to do is to braid the identity operator. Hence, when one of the last three operators appears in R we have:

$$\begin{aligned}
R \begin{pmatrix} o \\ aa \end{pmatrix} \begin{pmatrix} i \\ ak \end{pmatrix}, \begin{pmatrix} i \\ ak \end{pmatrix} \begin{pmatrix} o \\ kk \end{pmatrix} &= 1 \\
R \begin{pmatrix} i \\ ak \end{pmatrix} \begin{pmatrix} o \\ kk \end{pmatrix}, \begin{pmatrix} o \\ aa \end{pmatrix} \begin{pmatrix} i \\ ak \end{pmatrix} &= 1
\end{aligned} \tag{3.14}$$

The non-trivial entries in the R-matrix are

$R_{s_1 s_2 s_3 s_4}$	12	21	33	45	67	54	56	74	76	15	27	34	36	42	61	53	73
12	-1																
21		-1															
33			-1														
45				$\frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}$	$\frac{e^{3\pi i/4}}{2\sqrt{2}}$												
67				$\frac{1}{\sqrt{2}} e^{-\frac{\pi i}{4}}$	$\frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}$												
54						$e^{-i\frac{\pi}{4}}$											
56							$e^{i\frac{\pi}{4}}$										
74								$e^{i\frac{3\pi}{4}}$									
76									$e^{-i\frac{\pi}{4}}$								
15																	$-\frac{e^{-i\frac{\pi}{2}}}{2}$
27																	$e^{-i\frac{\pi}{2}}$
34																	$e^{-i\frac{\pi}{2}}$
36														$-2e^{-i\frac{\pi}{2}}$			
42														$-\frac{e^{-i\frac{\pi}{2}}}{2}$			
61															$e^{-i\frac{\pi}{2}}$		
53																$e^{-i\frac{\pi}{2}}$	
73																	$-2e^{-i\frac{\pi}{2}}$

(3.15)

and we can also write down the non-vanishing entries of the adjoint representation on the quantum group associated to R :

$$\begin{aligned}
 (t_{12})_{12} &= R_{1212} = -1, & (t_{21})_{21} &= R_{2121} = -1 \\
 (t_{33})_{33} &= R_{3333} = -1 \\
 (t_{46})_{32} &= R_{4236} = -\frac{1}{2} e^{-i\pi/2}, & (t_{64})_{31} &= R_{6134} = e^{-i\pi/2} \\
 (t_{57})_{23} &= R_{5327} = e^{-i\pi/2}, & (t_{76})_{13} &= R_{7315} = -2 e^{-i\pi/2}
 \end{aligned}$$

$$\begin{aligned}
(t_{13})_{75} &= R_{1573} = -\frac{1}{2} e^{-i\pi/4} & (t_{31})_{64} &= R_{3461} = e^{-i\pi/2} \\
(t_{23})_{57} &= R_{2753} = e^{-i\pi/2} & (t_{32})_{46} &= R_{3642} = -2 e^{-i\pi/2} \\
(t_{45})_{45} &= R_{4545} = \frac{1}{\sqrt{2}} e^{i\pi/8} & (t_{54})_{54} &= R_{5454} = e^{-i\pi/8} \\
(t_{47})_{65} &= R_{4567} = \frac{1}{2\sqrt{2}} e^{-3\pi i/8} & (t_{74})_{74} &= R_{7474} = e^{3\pi i/8} \\
(t_{56})_{56} &= R_{5656} = e^{3\pi i/8} & (t_{65})_{47} &= R_{6745} = \sqrt{2} e^{-3\pi i/8} \\
(t_{67})_{67} &= R_{6767} = \frac{1}{\sqrt{2}} e^{i\pi/8} & (t_{76})_{76} &= R_{7676} = e^{-i\pi/8}
\end{aligned} \tag{3.16a}$$

together with the elements coming from (3.14)

$$\begin{aligned}
\left(t \begin{pmatrix} 0 \\ a \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \right) \begin{pmatrix} i \\ a \end{pmatrix} \begin{pmatrix} i \\ h \end{pmatrix} &= 1 \\
\left(t \begin{pmatrix} i \\ a \end{pmatrix} \begin{pmatrix} i \\ h \end{pmatrix} \right) \begin{pmatrix} 0 \\ a \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} &= 1
\end{aligned} \tag{3.16b}$$

with this explicit representation it is now easy to check the general structure presented in the previous section. In particular we can write down the action of the operators X^σ, X^ϵ on several blocks :

$$\begin{aligned}
X^\sigma \left(\begin{array}{c} \sigma \\ \hline 0 \quad \sigma \end{array} \right) &= \frac{e^{i\pi/8}}{\sqrt{2}} \begin{array}{c} \sigma \\ \hline \sigma \quad 0 \end{array} + \frac{e^{-3\pi i/8}}{2\sqrt{2}} \begin{array}{c} \sigma \\ \hline \sigma \quad \epsilon \end{array} \\
X^\sigma \left(\begin{array}{c} \sigma \\ \hline \epsilon \quad \sigma \end{array} \right) &= \sqrt{2} e^{-3\pi i/8} \begin{array}{c} \sigma \\ \hline \sigma \quad 0 \end{array} + \frac{1}{\sqrt{2}} e^{i\pi/8} \begin{array}{c} \sigma \\ \hline \sigma \quad \epsilon \end{array} \\
X^\epsilon \left(\begin{array}{c} \sigma \\ \hline 0 \quad \sigma \end{array} \right) &= -\frac{1}{2} e^{-i\pi/2} \begin{array}{c} \sigma \\ \hline \epsilon \quad \sigma \end{array}, \text{ etc}
\end{aligned}$$

It is a useful exercise to verify explicitly in this example the compatibility of the operators with braiding, sewing and duality.

IV. HECKE ALGEBRAS, JONES POLYNOMIALS AND RCFT.

In this section we want to show that to any RCFT we can uniquely associate a set of knot invariants which is a specialization of the two variable Jones polynomial [19]. Before we embark into knots and links, we want to show that as a consequence of duality, any RCFT containing a subset of operators ϕ, ψ_1, ψ_2 with fusion rules

$$\phi \times \phi = \psi_1 + \psi_2 \quad (4.1)$$

gives a two dimensional representation of the Hecke algebra of type A_n ; $H_n(q)$. The defining relations of $H_n(q)$ are exactly those of the braid group B_n on n strands, plus a quadratic relation. $H_n(q)$ is generated by $n-1$ elements $g_1, g_2, \dots, g_{n-1}$ satisfying :

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i \quad |i-j| \geq 2 \\ g_i^2 &= (q-1) g_i + q \end{aligned} \quad (4.2)$$

when $q=1$ we obtain the permutation group S_n . The defining relations (4.2) imply that a representation of the Hecke algebra provides a special representation of the braid group B_n . Any matrix representation of (4.2) implies that all matrices $R(g_i)$ have the same two distinct eigenvalues λ_1, λ_2 .

For a theory with three operators satisfying (4.1) (in many instances ψ_1 or ψ_2 is the identity operator), we have two four-point blocks $\mathcal{F}_1^{\phi\phi\phi\phi} = \mathcal{F}_1$ and $\mathcal{F}_2^{\phi\phi\phi\phi} = \mathcal{F}_2$ contributing to the four point amplitude $\langle \phi(1) \phi(2) \phi(3) \phi(4) \rangle$. If g is any element of the braid group B_4 , it will act on the blocks via the R matrix :

$$g \in B_4 : \quad \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \rightarrow R(g) \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \quad (4.3)$$

and $R(g)$ is a 2×2 matrix. The representation is completely specified by the matrices $R(\sigma_i)$ $i=1,2,3$ where $\sigma_i \in B_4$ are the generators of B_4 . The action of σ_1 and σ_3 on $\mathcal{F}_i$ does not change the label i , and it is diagonal. It only changes $\mathcal{F}_i$ by a phase. If $\Delta_\phi, \Delta_1, \Delta_2$ are respectively the conformal dimensions of ϕ, ψ_1, ψ_2 , we have :

$$R(\sigma_1) = R(\sigma_3) = \begin{pmatrix} \epsilon_1 e^{i\pi(\Delta_1 - 2\Delta\phi)} & 0 \\ 0 & \epsilon_2 e^{i\pi(\Delta_2 - 2\Delta\phi)} \end{pmatrix} \quad (4.4)$$

and $\epsilon_i = +1$ (-1) depending on whether ψ_i appears symmetrically or antisymmetrically in the operator product $\phi \times \phi$. The action of σ_2 on $\mathcal{H}_i$ will in general change the channel, and $R(\sigma_2)$ will not be diagonal. However the eigenvalues of $R(\sigma_2)$ are the same as those of $R(\sigma_1)$ and $R(\sigma_3)$ due to the s-t duality (2.7). The matrix N is a similarity transformation in this case, and it does not change the eigenvalues. Since all the $R(\sigma_i)$ have the same eigenvalues, they provide a representation of the Hecke algebra $\mathcal{H}_4(q)$ with a value of q to be computed presently. To obtain the standard form (4.2) we have to normalize $R(\sigma_i)$. If we define

$$g_i \equiv -\epsilon_2 e^{i\pi(2\Delta\phi - \Delta_2)} R(\sigma_i) \quad (4.5)$$

then g_i satisfies (4.2) with

$$q = -\epsilon_1 \epsilon_2 e^{i\pi(\Delta_1 - \Delta_2)} \quad (4.6)$$

hence, q only depends on the spins of the internal states in the block. Since we are working on the sphere, we can use the relation :

$$R(\sigma_1) R(\sigma_2) R(\sigma_3^2) R(\sigma_2) R(\sigma_1) = e^{-4\pi i \Delta\phi} \mathbb{1} \quad (4.7)$$

to obtain constraints on the possible values of $\Delta\phi$, Δ_1 and Δ_2 . Using the explicit form of the R matrices in this case, we obtain :

$$q^3 = e^{2\pi i (4\Delta\phi - 3\Delta_2)}$$

or

$$e^{i\pi(8\Delta\phi - 3\Delta_1 - 3\Delta_2)} = -\epsilon_1 \epsilon_2 \quad (4.8)$$

There are many cases where one finds the rules (4.1). For example, we can take as chiral algebra $\mathcal{SV}(2)_k$, $k > 1$, and $\phi = \phi_{1/2}$ the spin 1/2 primary field. Then ψ_2 is the identity or spin zero operator, and ψ_1 is the spin 1 representation. In this case $\epsilon_1 = 1$, $\epsilon_2 = -1$. Thus :

$$q_{su(2)_k} = e^{2\pi i/(k+2)} \quad (4.9)$$

a result which first appeared in [9]. Similarly, if we consider $su(N)_k$, $k>1$, we can take two fundamental representations N and two $\bar{N}$. The fusion rule is $N \times N = \mathbb{1} + \square$, $N \times \bar{N} = 1 + (\text{adjoint})$, and a slight modification of the previous arguments yields [11]

$$q_{su(N)_k} = e^{2\pi i/(k+N)} \quad (4.10)$$

In the minimal models we have

$$\begin{aligned} \phi_{1,2} \times \phi_{1,2} &= \phi_{1,1} + \phi_{1,3} \\ \phi_{2,1} \times \phi_{2,1} &= \phi_{1,1} + \phi_{3,1} \end{aligned} \quad (4.11)$$

for any

$$c = 1 - 6 \frac{(r-s)^2}{rs} \quad (4.12)$$

then $\epsilon_1 = \epsilon_2 = 1$ and

$$q = -e^{i\pi \Delta_{1,3}} = e^{2\pi i s/r} \quad (4.12a)$$

in the first case, and

$$q = -e^{i\pi \Delta_{3,1}} = e^{2\pi i r/s} \quad (4.12b)$$

in the second.

Representations of higher Hecke algebras $\mathcal{H}_n(q)$ are obtained by considering blocks with n external legs of type ϕ . This guarantees that the matrix obtained has only two distinct eigenvalues, and the value of q is the same as the one obtained for the four point blocks.

Another interesting case occurs when $\psi_1 = \phi$, $\psi_2 = c\phi$. Then the condition (4.8) reads

$$e^{5\pi i \Delta \phi} = -\epsilon_1 \epsilon_2 \quad (4.13)$$

corresponding to the $k=1$ WZW model for the non-simply laced groups G_2 and F_4 with $(c, \Delta \phi, \epsilon_1 \epsilon_2) = (\frac{14}{5}, \frac{2}{5}, -1)$ and $(\frac{26}{5}, \frac{3}{5}, 1)$ respectively [6]. The q values are

$$q_{G_2}^{k=1} = e^{2\pi i/5}, \quad q_{F_4}^{k=1} = e^{-2\pi i/5} \quad (4.14)$$

the explicit form of the representation of $H_4(q)$ for all models satisfying (4.1) is :

$$\begin{aligned} g_1 = g_3 &= \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix} \\ g_2 &= \begin{pmatrix} -\frac{1}{1+q} & \alpha \\ \beta & \frac{q^2}{1+q} \end{pmatrix} \end{aligned} \quad (4.15)$$

where

$$\alpha\beta = \frac{q}{(1+q)^2} (1+q+q^2) \quad (4.16)$$

The freedom in choosing α and β is explained by the fact that a similarity transformation $g_i \rightarrow h g_i h^{-1}$ with h diagonal does not change the defining properties of g_i , and the fact that we have not imposed any normalization condition on the fields. It can be shown for example that $|\alpha|$ is related to the structure constants of the operator algebra. Once the normalizations are chosen for each conformal theory, the values of α and β are fixed up to a phase.

The irreducible representations of $H_n(q)$ for generic values of q are in one-to-one correspondence with those of the permutation group, and can be labelled by Young tableaux. For instance, $H_4(q)$ has generically five irreducible representations of dimensions (1,3,2,3,1) according to the Tableaux :

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} \quad (4.17)$$

the CFT provides us with a representation of dimension 2, given by the middle tableau in (4.17). For certain values of q an irreducible representation may become reducible. This is indeed the case when q is a root of unity [20], and the case of most interest in CFT, where this is the generic situation.

To relate CFT to knot and link invariants, we use the connection between knots and braids [21]. It can be shown that any knot or link can be obtained by closing a braid. Given a braid $\alpha \in \mathcal{B}_n$, the closure $\widehat{\alpha}$ is the operation where the initial and end points of the braid are sewn. Three examples are shown in fig.12. Knots and links are certainly objects living in three dimensional space, while a CFT is defined in two dimensions. In a recent paper [11] Witten has shown that knot theory is deeply related to a three-dimensional topological field theory defined by the Chern-Simons action for a gauge theory with arbitrary group G where knot and link invariants are naturally associated to expectation values of Wilson lines. He also

shows the relation between this three dimensional theory and the two dimensional WZW-conformal field theory.

What we will now show, is how the Jones polynomials arise in the context of RCFT. A link polynomial $P_L(t, x)$ is a polynomial in two complex variables t, x satisfying the skein rule:

$$t^{-1} P_{L_+} - t P_{L_-} = x P_{L_0} \quad (4.18)$$

where the links L_+ , L_- , and L_0 are obtained from the original link L by looking at a single crossing and performing the operations according to the rules of fig.13. It is clear that any polygonal knot or link can be reduced to the trivial one by a finite sequence of skein moves. The polynomial $P_L(t, x)$ is normalized so that it equals 1 when L consists of a single unknotted component (i.e. the oriented circle) $P(\bigcirc) = 1$. This normalization is chosen so that P_L behaves multiplicatively under the operation of connected sum (fig.14) :

$$P_{L_1 \# L_2} = P_{L_1} \cdot P_{L_2} \quad (4.19)$$

An important theorem in knot theory [22] asserts that (4.18) together with the normalization $P(\bigcirc) = 1$ determines $P_L(t, x)$ uniquely. The hard part of the proof is in showing that the polynomial is independent of the way the link is disentangled using the Skein rules.

Jones' work [19] provides a systematic way to evaluate $P_L(t, x)$ for any knot. In the computation of $P_L(t, x)$ one needs to use all the irreducible representations of $U_n(q)$ given by the Young tableau of S_n . The value of $P_L(t, x)$ is essentially a weighted sum over the traces on these representations of the matrix $R(\beta)$ where $\beta \in B_n$, and the closure $\hat{\beta}$ is a representation of the link. From our previous discussion of the relation between CFT and Hecke algebras, we know that the CFT will contribute to the link polynomial in the form :

$$P_L = \text{Tr}_{L=\hat{\beta}} R_{\text{CFT}}(\beta) + \dots \quad (4.20)$$

the dots in (4.20) correspond to Young tableaux which appear at first sight not to be related to the CFT, although they may appear naturally when the theory is related to a 3D field theory. It is easy to understand why one needs representations other than those furnished by the CFT. The braid $\sigma_1 \sigma_2 \sigma_3$ corresponds to the trivial knot, however, in the examples worked at the beginning of this section one finds $\text{Tr} R(\sigma_1 \sigma_2 \sigma_3) = 0$ which does not agree with the normalization condition. The uniqueness of the invariant with the chosen normalization will allow us to determine the other representations uniquely. All we need to know are the skein

rules for the CFT.

The skein rules are defined because a CFT satisfying (4.1) provides a representation of the Hecke algebra. In fact, for an arbitrary link $L_0 = \widehat{\alpha}$, $\alpha \in \mathcal{B}_n$ we easily find that L_+ , L_- are respectively represented by $\widehat{\alpha \sigma_i}$ and $\widehat{\alpha \sigma_i^{-1}}$, for some i , and therefore we can read off the skein rule from the quadratic relation $g_i^2 = (q-1)g_i + q$. Here we have to be a little careful. We stressed in sec.II that the conformal blocks do not simply provide a representation of the braid group. Since one also needs to introduce local parameters at the points where the operators are inserted, the relations of the braid group on the sphere hold up to Dehn twists around the points z_i . We showed that $\beta = \sigma_1 \sigma_2 \sigma_3^2 \sigma_2 \sigma_1$, which would be the identity without local parameters, becomes $\exp(-4\pi i \Delta \phi)$. In the process of successively implementing the braid operations indicated by β we end up with twists around the points $z_i(t) = 0$, and these have to be taken into account. Similarly, the operations implied by the skein rule also require that we perform some twists around the distinguished points to maintain the original framing of knot. Hence, in order to construct the skein rules for a CFT we have to include phases of the form $\exp(-2\pi i \Delta \phi)$ to compensate the effects of twists. After this is done, the skein parameters t and x are determined by q and $\exp(-2\pi i \Delta \phi)$. The polynomial P is then

$$P_L(g) = e^{-2\pi i \Delta \phi \cdot \omega(g)} \frac{1}{K} R_{\text{CFT}}(g) + \dots \quad (4.21)$$

where $\omega(g)$ is the length of the word g in the braid group (for example $\omega(\sigma_i) = 1$, $\omega(\sigma_i^{-1}) = -1$, $\omega(gg') = \omega(g) + \omega(g')$). The extra dots in (4.21) are necessary, because the first term by itself does not satisfy the normalization condition $P_L(\emptyset) = 1$ although it completely determines the skein rule. Now (4.18) follows essentially from the Hecke algebra relation which we write for convenience as

$$q^{-1/2} g_i - q^{1/2} g_i^{-1} = (q^{1/2} - q^{-1/2}) \quad (4.22)$$

using the relation between g_i and R_i (4.5), (4.22) becomes :

$$\begin{aligned} & -\epsilon_2 e^{i\pi(2\Delta\phi - \frac{\Delta_1 + \Delta_2}{2} - \frac{1 + \epsilon_1 \epsilon_2}{4})} R(\sigma_i) \\ & + \epsilon_2 e^{i\pi(-2\Delta\phi + \frac{\Delta_1 + \Delta_2}{2} + \frac{1 + \epsilon_1 \epsilon_2}{4})} R(\sigma_i^{-1}) \\ & = q^{1/2} - q^{-1/2} \end{aligned} \quad (4.23)$$

since $\omega(\alpha \sigma_i) = \omega(\alpha) + \omega(\sigma_i)$ and $R(\alpha \sigma_i) = R(\alpha) R(\sigma_i)$ then (4.18) is equivalent to

$$t^{-1} e^{-2\pi i \Delta \phi} R(\epsilon_i) - t e^{2\pi i \Delta \phi} R(\epsilon_i^{-1}) = x \mathbb{1} \quad (4.24)$$

comparing (4.23) and (4.24) we obtain

$$\begin{aligned} x &= q^{1/2} - q^{-1/2} \\ t &= -\epsilon_2 e^{i\pi(-4\Delta\phi + \frac{\Delta_1 + \Delta_2}{2} + \frac{1 + \epsilon_1 \epsilon_2}{4})} \end{aligned} \quad (4.25)$$

For example, for the $SU(N)$ level k WZW-model, (4.25) gives $t = q^{-N/2}$ with $q = e^{2\pi i/(N+k)}$, and the skein rule agrees with the one in reference [11].

For the minimal models, we can take ϕ to be either ϕ_{112} or ϕ_{211} . Since now $\epsilon_1 = \epsilon_2 = 1$, for ϕ_{112} we have

$$t = e^{i\pi(-4\Delta_{112} + \frac{1}{2}\Delta_{113} - \frac{1}{2})} = -e^{-2\pi i s/r} \quad (4.26a)$$

and for ϕ_{211}

$$t = e^{i\pi(-4\Delta_{211} + \frac{1}{2}\Delta_{311} - \frac{1}{2})} = -e^{-2\pi i r/s} \quad (4.26b)$$

since for these models q is given by (4.12a,b) we learn that for the minimal series :

$$qt = -1 \quad (4.27)$$

and in this case we obtain the Jones polynomial $V_L(t)$ [19] ; a specialization of $P_L(t, x)$.

To summarize, to every CFT which contains some fields satisfying (4.1) we can associate a Skein rule (4.18) with coefficients x and t determined by the dimension of the field ϕ and the value of q . Even though the CFT is only defined in two dimensions and it only provides rectangular Young tableau representations of the Hecke algebra, the Skein rule and the normalization condition uniquely determine the other representations needed to obtain the link polynomial. There is a geometrical reason why CFT only provide rectangular Young tableau representations of the Hecke algebra : Only the latter furnish representations of the mapping class group of the sphere with distinguished points [19]. Since the conformal blocks are also representations of the sphere mapping class group, this property determines the type of representations of $H_n(q)$ they can generate.

It is reasonable to speculate that the extra representations of $H_n(q)$ determined by the Skein rule of the CFT and the normalization condition give information about a possible three-dimensional topological field theory whose "boundary" is the CFT, in the same way that one can think of the WZW-theories as boundaries of the 3D Chern-Simons theory [11].

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FIGURE CAPTIONS.

FIG1. The four point block. $\mathcal{F}_p^{ijkl}$.

FIG2. Associativity of the OPE.

FIG3. Three point vertex.

FIG4. Constructing N from C in two different ways.

FIG5. Colored pant description of the three point vertex.

FIG6. n-point block.

FIG7. Definition of the braid group generator σ_i .

FIG8. One of the relations in the braid group of the sphere $\mathcal{B}_n(S^2)$

FIG9. Graphical interpretation on the operator (2.31).

FIG10. Operations involved in the pentagon identity.

FIG11. The two independent contours of integration with a single screening operator in the 4-point block.

FIG12. Examples of knots and links obtained by closing braids : The trivial knot, two linked circles with linking number one , and the trefoil knot.

FIG13. Skein moves.

FIG14. Connected sum of two knots.

Fig. 1

Fig. 2

Fig. 3

Fig. 4a

Fig. 4b

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14