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A UNIQUENESS THEOREM FOR WORMHOLES IN QUANTUM GRAVITY

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ABSTRACT

We prove a uniqueness theorem for wormhole spaces with vanishing Ricci scalar. We show that amongst such metrics and manifolds the unique extremum of the Euclidean action is the Tolman wormhole. We also collect arguments to support the conjecture that this metric is a global minimum in this class.

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Recently there has been much interest in wormholes in Quantum Gravity [1,2,3]. A wormhole space is one possessing two or more Asymptotically Euclidean regions that are linked by one or more “throats”. Such spaces are believed to be of relevance to tunnelling processes between the asymptotic regions in the path integral approach to Quantum Gravity. By deforming such a manifold one may obtain an Asymptotically Euclidean region linked to a small baby universe. The Riemannian metric or “infanton” on this manifold governs tunnelling processes which give rise to the creation of a baby universe.

Conventionally one describes this process by finding a stationary point of the Euclidean action with the required topology. However, some years ago it was found [4] that there were no Asymptotically Euclidean vacuum solutions of the wormhole type. Thus if we wish to adopt this approach we must couple matter fields to gravity [2].

An alternative direction is suggested by the regulated path integral for gravity proposed by Gibbons, Hawking and Perry [5]. Here one performs the path integral in two stages; first doing a Wick rotated integral over the conformal factor relating conformally equivalent metrics and then doing an integral over conformal equivalence classes. It was in this setting that Hawking suggested [6] that relevant configurations for wormhole physics would be those of least action in the conformal “gauge” $R = 0$ where R is the Ricci scalar, for in [5] it was proposed that the representatives of equivalence classes should be Ricci scalar flat metrics. Hawking then claimed that the unique such minimum for given wormhole size was that of the Tolman wormhole [7] given by

$$ds^2 = \left(1 + \frac{b^2}{r^2}\right)^2 (dr^2 + r^2 d\Omega_3^2), \quad (1)$$

where $d\Omega_3^2$ is the round metric on the unit three sphere.

The object of this letter is to prove that indeed this metric is the unique extremum of the Euclidean action amongst metrics on manifolds of wormhole type with vanishing Ricci scalar and for given wormhole size. We have not, as yet, been able to establish the minimality of the action for this critical point of our constrained problem.

First we formulate our problem more carefully. Instead of considering manifolds with two or more asymptotic regions we slice each throat at a minimal three surface whose existence is assumed and thus treat manifolds whose boundary is the union of minimal surfaces and just one Asymptotically Euclidean region. The wormhole size is taken to be the total volume of minimal surfaces. We also make the following restriction on the topology of the manifold and the fall-off conditions at infinity.

Restriction. There is a compact $K \subset \Sigma$ such that $\Sigma - K$ is diffeomorphic to the complement of the closed unit ball in R^4 such that in the coordinates induced by this diffeomorphism (and with the unit ball centred on the origin) the metric behaves like

$$g_{\alpha\beta} = (1 + \frac{b^2}{r^2})^2 \delta_{\alpha\beta} + h_{\alpha\beta} \quad (2)$$

where $h_{\alpha\beta}$ and its derivatives are $O(\frac{1}{r^3})$ as $r \rightarrow \infty$. This is the standard definition of Asymptotically Euclidean (see [4]). We now state our theorem.

Theorem. Subject to the above restrictions the unique extremum of the Euclidean action amongst metrics with $R = 0$ and with boundary comprising an Asymptotically Euclidean region and the union of minimal surfaces with prescribed total volume is the metric given by equation (1).

The expression to be varied is

$$\begin{aligned} S(g, V, \phi, A, \Lambda) = & \int_{\Sigma} R \sqrt{g} d^4 x + 2 \int_{\infty} (K - K^{\circ}) \sqrt{h} d^3 x \\ & + \int_{\Sigma} V R \sqrt{g} d^4 x + \int_B \phi K \sqrt{h} d^3 x + \Lambda (\int_B \sqrt{h} d^3 x - A) \end{aligned} \quad (3)$$

where V, ϕ, Λ are Lagrange multipliers and A is the wormhole 3-volume. We take the boundary of Σ to be the disjoint union $\partial\Sigma = \infty \cup B$. K is the trace of the extrinsic curvature of the boundary pieces over which we integrate and K° is that of the boundary at infinity embedded into flat space. The first two terms of equation (3) are -16π times the Euclidean

action [5]. The last three terms are the three constraints with their Lagrange multipliers; for recall that the condition that a hypersurface be minimal is that the trace of the second fundamental form should vanish. The metric variations that we consider here are those appropriate to the boundary terms, namely that they vanish at infinity.

The Euler-Lagrange equations for this variational problem are then obtained by considering successively more general variations within this class. Considering variations satisfying $\nabla\delta g = 0$ on $\partial\Sigma$ leads to

$$(1 + V)(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) = \nabla_\alpha\nabla_\beta V - g_{\alpha\beta}\nabla^2 V \quad (4)$$

$$R = \nabla^2 V = 0 \quad (5)$$

Relaxing these to $\nabla\delta g$ not necessarily vanishing at infinity but $\delta g = \nabla\delta g = 0$ on B implies that $V = 0$ at infinity. We now relax conditions at B . Variations satisfying the above with δg vanishing at B but $\nabla\delta g \neq 0$ on B yield

$$2(1 + V) = \phi \quad (6)$$

on B . Finally the most general variations in our class lead to the equations

$$\phi = K = 0 \quad (7)$$

$$2n^\alpha\nabla_\alpha V + \Lambda = 0 \quad (8)$$

where n_β is the unit normal to B . Equations (4) and (5) may be interpreted as the static field equations in five dimensions where $g_{00} = -(1 + V)^2$. This leads to the fall-off behaviour of V near infinity:

$$V = -\frac{2b^2}{r^2} + O(\frac{1}{r^3}) \quad (9)$$

Since V is harmonic and vanishes at infinity we see that Λ cannot vanish for V is not constant and the mixed Neumann-Dirichlet problem has a unique solution. The extrinsic curvature of B is given $K_{ij} = \Lambda^{-1}(n_i n^k \nabla_k \nabla_j - (1 + V)R_{ij})$ which vanishes on B . Thus B is totally geodesic in Σ .

We now perform conformal transformations on two copies of Σ

$$g_1 = \frac{1}{4}(2 + V)^2 g \quad g_2 = \frac{1}{4}V^2 g \quad (10)$$

Both of these factors are nowhere vanishing from the maximum principle [8] except at infinity for the second metric. g_1 has zero action since the fall-off of g and V implies that the boundary term vanishes and $R_1 = 0$. g_2 may be defined on $\Sigma \cup \infty$ similarly. Moreover we can join these two copies of Σ along the two copies of B to induce a smooth metric on the “doubled-up” manifold since $g_1 = g_2$ on B and $K_{1ij} = -K_{2ij}$ where $K_{1,2ij}$ are the extrinsic curvatures of B in the two conformally related metrics. The behaviour of the metric g_2 in the asymptotic region of the second copy tends to that of the flat metric in the neighbourhood of the coordinate singularity in spherical coordinates at the origin. Adding a point at infinity on the second copy thus yields a zero action metric with $R = 0$ with one asymptotically flat region. We may now appeal to the positive action theorem of Schoen and Yau [9]. This tells us that the metric on the joined manifold is, in fact, the flat metric on R^4 . This enables us to find g . We know that $g = 4(2 + V)^{-2}\delta$ where δ is the flat metric on Σ regarded as a piece of R^4 , which is identified via the positive action theorem. We now deduce the spherical symmetry of g . Using the well-known conformal properties of the Ricci tensor we may show that

$$(1 - (1 + V)^2)\nabla^\beta \nabla_\alpha V + (1 + V)(4\nabla^\beta V \nabla_\alpha V - \delta^\beta_\alpha (\nabla V)^2) = 0 \quad (11)$$

This expresses the second derivatives of V as bilinears in the gradients of V . We now show that the extrinsic curvature of a hypersurface of constant V is pure trace. Such surfaces are known as totally umbilic surfaces and have been much studied by differential geometers.

Let S be such a surface with unit normal $n_\alpha = \rho \nabla_\alpha V$. Then the extrinsic curvature of S with respect to g satisfies

$$K_{\alpha\beta} = (g_{\alpha\gamma} - n_\alpha n_\beta) \nabla_\gamma n_\beta = (1 + V)(1 - (1 + V)^2)^{-1} \rho^{-1} (g_{\alpha\beta} - n_\alpha n_\beta) \quad (12)$$

Thus the extrinsic curvature is pure trace, but this is a conformally invariant notion, so S is totally umbilic in the flat metric on R^4 . Then the Codazzi equations [10]

$$K^i_{i;j} = K^i_{j;i} \quad (13)$$

show that constant V hypersurfaces in R^4 have constant mean curvature and Alexandrov's theorem [11] says that such surfaces carry the metrics of round spheres. Thus we see that each constant V surface has the metric of a sphere of radius $R(V)$ in g . Expressing the Ricci scalar R' of the induced metric in terms of that of g and $K_{\alpha\beta}$ gives

$$R' = 6\rho^{-2}(1 - (1 + V)^2)^{-2}, \quad (14)$$

but R' can only depend on V , hence g has the form

$$g_{\alpha\beta} dx^\alpha dx^\beta = \rho^2(V) dV^2 + R^2(V) d\Omega_3^2 \quad (15)$$

The explicit form in equation (1) is found by demanding that g have vanishing Ricci scalar. More accurately, the above discussion holds in a neighbourhood of infinity, where, for example, we have connected hypersurfaces of constant V . However, solutions of equation (4) are analytic and hence we can deduce the form (1) for the whole of Σ by appealing to the uniqueness of the analytic extension [12].

The above proof generalises to any dimension for which one can appeal to the relevant positive action theorem. Note, however, that in the case of dimension four we must use the Schoen and Yau version, rather than the Witten version [13], proved with the use of spinors, as it is not clear that spinors exist on the joined manifold. What remains to be shown is that the action is minimal in the class considered, rather than being merely extremal. A

perturbative discussion is extremely messy due to the constraints. However, there is strong evidence by analogy with black holes in four dimensions. In this case one may express the cosmic censorship principle as an inequality. We consider an asymptotically flat initial data set obeying the dominant energy condition. Let M be the A.D.M mass of the data set and A the area of the outermost apparent horizon. An apparent horizon is a closed two surface whose outgoing orthogonal null geodesics have zero convergence. In the time-symmetric case any apparent horizon is a minimal surface. Then cosmic censorship implies:

$$M \geq \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} \quad (16)$$

with equality if and only if the data evolves to Schwarzschild. A survey of the evidence for (16) is given in reference [14].

Now recall that any vacuum time-symmetric data set has $R = 0$. A generalisation to 5 dimensions is the conjecture that any Riemannian 4-manifold with $R = 0$ has action I obeying

$$I \geq \left(\frac{27}{2^{11}\pi}\right)^{\frac{1}{3}} V^{\frac{2}{3}} \quad (17)$$

where V is the volume of the outermost minimal surface with equality if and only if the metric is the time symmetric slice of 5 dimensional Schwarzschild i.e. metric (1). Thus there appear to be strong reasons for believing that our extremum is indeed minimal.

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