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CONFORMAL INVARIANCE ON A GENERIC RIEMANN SURFACE
IN THE PRESENCE OF A NON-FLAT BACKGROUND METRIC

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ABSTRACT

By using bases of meromorphic differentials, recently proposed by Krichever and Novikov, we calculate the brackets of the moments of the energy momentum tensor of a bosonic string theory in the presence of a generic target metric. To do this we expand the target metric in normal co-ordinates. To the lowest perturbative order we find that the BRST operator is nilpotent if we impose the condition that the Ricci tensor of the target metric vanish.

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1. Introduction The recent introduction by Krichever and Novikov [1] of a basis for any kind of tensor fields over a Riemann surface has considerably enhanced the potentialities of the operator formalism in describing conformal field theories over genus g Riemann surfaces. The operator formalism constructed over the Krichever-Novikov (KN) bases is both conceptually simpler and practically more powerful than analogous existing formalisms. It has been shown in a series of papers [2,3,4,5] that many problems of the operator formalism on Riemann surfaces can be conveniently formulated and simply solved by means of the KN bases.

In this paper we attack a more ambitious problem. In [2] it was shown how to define a BRST operator over any Riemann surface in relation to a bosonic string theory (in the presence of a flat background metric) and that this charge is nilpotent for $D = 26$ (in [3] this was extended to the superstring case). Here we do the same thing in the presence of a non-flat background metric $G_{\mu\nu}(x)$. The method consists of defining a classical stress-energy tensor T by mutuating it from the corresponding sigma-model Lagrangian, then expanding it in normal coordinates ($x^\mu = x_0^\mu + \xi^\mu$) and quantizing it by using canonical commutation relations and normal ordering.

The dependence on a given Riemann surface is taken into account by expanding every tensor involved into the appropriate KN basis characteristic of the given surface. In this way we are able to define an appropriate BRST charge Q_{BRST} for any Riemann surface in the presence of an arbitrary background. We then compute the anticommutator of this charge with itself. We find that Q_{BRST}^2 does not contain only the expected central charge (as in the flat case). At the first perturbative order it contains also q -terms proportional to $R_{\mu\nu}$ saturated with genus dependent terms containing normal products of $\xi, d\xi$ and P , the conjugate momentum of ξ . One of these q -terms turns out to be "trivial", i.e. it can be absorbed by means of a redefinition of Q_{BRST} . But the other terms can vanish only if $R_{\mu\nu} = 0$. So we recover the well-known equation of motion of the background metric at one loop [6]. We recall that the above result (i.e., $R_{\mu\nu} = 0$) has already been obtained with various methods from the request of nilpotency for the BRST charge [7]. Recent papers on this field are to be found in ref. [8]. The novelty of our result is that it holds for any Riemann surface.

The paper is organized as follows. First we introduce the notation, in particular the KN bases needed in this work, and define the $BRST$ charge in the presence of a background metric. Then we calculate the one-loop approximation to Q_{BRST}^2 . The final part is devoted to the technical details of the calculations.

2. Notation. The KN bases. Given a compact genus g Riemann surface Σ we

consider a theory defined by the action:

$$S = -\frac{1}{2} \int_{\Sigma} d\sigma d\tau (-\dot{X}^{\mu} \dot{X}^{\nu} + X'^{\mu} X'^{\nu}) G_{\mu\nu}(X) \quad (2.1)$$

where $\mu = 1, \dots, D$.

After performing a Wick rotation $\sigma \longrightarrow i\sigma$ we obtain the stress energy tensor

$$T = \frac{1}{4} (X'^{\mu} + P_{\lambda} G^{\mu\lambda}) G_{\mu\nu} (X'^{\nu} + P_{\rho} G^{\nu\rho}) \quad (2.2)$$

where the canonical momentum P_{μ} is given by

$$P_{\mu} = i \dot{X}^{\nu} G_{\mu\nu} \quad (2.3)$$

Now we interpret X'_{μ} as the differential dX^{μ} , P_{μ} as a 1-differential and T as a quadratic differential. Then we expand these quantities in the appropriate KN bases. The latter are defined as follows.

On Σ let us consider two distinguished points P_+ and P_- and local coordinates z_+ and z_- around them, such that $z_{\pm}(P_{\pm}) = 0$. On Σ we will consider bases of meromorphic vector fields e_i , functions A_i , 1-differentials ω^i and quadratic differentials Ω^i , which are holomorphic outside $P_{\pm}$. Here i is integer or half integer according to whether g is even or odd. The behaviour near $P_{\pm}$ is defined by

$$A_i(z_{\pm}) = a_i^{\pm} z_{\pm}^{\pm i - \frac{g}{2}} (1 + \mathcal{O}(z_{\pm})) \quad (2.4a)$$

$$\omega^i(z_{\pm}) = b_i^{\pm} z_{\pm}^{\mp i + \frac{g}{2}} (1 + \mathcal{O}(z_{\pm})) (dz_{\pm}) \quad (2.4b)$$

$$e_i(z_{\pm}) = c_i^{\pm} z_{\pm}^{\pm i - g_0 + 1} (1 + \mathcal{O}(z_{\pm})) \frac{\partial}{\partial z_{\pm}} \quad g_0 \equiv \frac{3}{2}g \quad (2.4c)$$

$$\Omega^i(z_{\pm}) = d_i^{\pm} z_{\pm}^{\mp i + g_0 - 2} (1 + \mathcal{O}(z_{\pm})) (dz_{\pm})^2 \quad (2.4d)$$

For $|i| \leq \frac{g}{2}$ the definitions (2.4a), (2.4b) must be modified, due to the Weierstrass gap theorem (see [1] or [5]). The above basis elements are determined up to numerical constants by the Riemann-Roch theorem. The constants $a_j^{\pm}$ in eqs. (2.4a) can be determined for example by setting $a_j^+ = 1$ and then a_j^- are completely determined. We can do the same for the other constants. We will need the following duality relations

$$\frac{1}{2\pi i} \oint_{C_{\tau}} A_i(Q) \omega^j(Q) = \delta_i^j \quad (2.5a)$$

$$\frac{1}{2\pi i} \oint_{C_{\tau}} e_i(Q) \Omega^j(Q) = \delta_i^j \quad (2.5b)$$

Here C_{τ} denotes a level curve of the univalent function $\tau(Q) = \operatorname{Re} \int_{Q_0}^Q \varphi$, where Q_0 is a generic fixed point in Σ and φ is the third kind differential with poles $\pm(z_{\pm})^{-1}$ in

$P_{\pm}$. The Lie brackets of the meromorphic vector fields e_i are

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s e_{i+j-s}, \quad g_0 \equiv \frac{3}{2}g \quad (2.6)$$

The structure constants C_{ij}^s can be calculated from the constants appearing in the Laurent expansion, eq (2.4c). Eq. (2.6) is the defining equation for the Krichever-Novikov algebra. Its central extension is defined by means of the cocycle

$$\chi(e_i, e_j) = \frac{1}{24\pi i} \oint \tilde{\chi}(e_i, e_j) \quad (2.7)$$

where the integral is along a contour surrounding P_+ and $\tilde{\chi}(f, g)$ for any two meromorphic vector fields which are holomorphic outside $P_{\pm}$, $f = f(z_+) \frac{\partial}{\partial z_+}$ and $g = g(z_+) \frac{\partial}{\partial z_+}$ near P_+ , is given by

$$\tilde{\chi}(f, g) = \left(\frac{1}{2} (f'''g - g'''f) - R(f'g - fg') \right) dz_+$$

R is a Schwarzian connection. The properties of this cocycle can be found in [1]. Finally the extended KN algebra is defined by

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s e_{i+j-s} + t\chi(e_i, e_j), \quad [e_i, t] = 0 \quad (2.8)$$

We will expand all the fields in the relevant KN bases. Let us notice that in this way we allow for singularities in all the fields at $P_{\pm}$.

Let us return to the energy momentum tensor and, for the time being, let us assume that $G_{\mu\nu} = \eta_{\mu\nu}$, the flat metric in the target manifold. In [2] $T_{\eta}(Q)$ was decomposed according to the basis of meromorphic quadratic differentials

$$T_{\eta}(Q) = \sum_i L_i^{\eta} \Omega_i(Q) \quad (2.9)$$

where

$$L_j^{\eta} = \frac{1}{2\pi i} \oint_{C_{\tau}} T_{\eta}(Q) e_j(Q) \quad (2.10)$$

The canonical commutation relations

$$[X^{\mu}(Q), P^{\nu}(Q')] = \eta^{\mu\nu} \Delta(Q, Q'), \quad Q, Q' \in C_{\tau} \quad (2.11)$$

were assumed, where

$$\Delta(Q, Q') = \sum_i A_i(Q) \omega^i(Q') \quad Q, Q' \in C_{\tau} \quad (2.12)$$

plays the rôle of a delta function over C_{τ} , i.e. for any smooth function over C_{τ} we have

$$f(Q) = \frac{1}{2\pi i} \oint_{C_{\tau}} \Delta(Q, Q') f(Q')$$

As a consequence it was found that

$$[: L_i^\eta :, : L_j^\eta :] = \sum_{s=-g_0}^{g_0} C_{ji}^s : L_{i+j-s}^\eta : + D \hat{\chi}_{ij} \quad (2.13)$$

where $\hat{\chi}_{ij}$ is proportional to $\chi(e_i, e_j)$, up to trivial cocycles and D is the target space dimension. Then it was proved that, by considering also the ghost energy-momentum tensor,

$$T^{gh} \equiv c\partial b + 2\partial cb \quad (2.14)$$

the corresponding components L_j^{gh} satisfy analogous commutation rules with different central charges. Moreover, when $D = 26$, $L_i^\eta + L_i^{gh}$ satisfy an algebra (2.8) with $t = 0$, i.e. the central term generated by the $b - c$ ghost system exactly cancels the central term due to the X part of the energy momentum tensor in the presence of a flat background.

The problem we set out to solve in this paper is to find the generalization of eq. (2.13) in the presence of a non-flat background. In other words, we define

$$L_j^G \equiv L_j = \frac{1}{2\pi i} \oint_{C_\tau} T(Q) e_j(Q) \quad (2.15)$$

where T is given by eq. (2.2) and ask ourselves what are the conditions on G such that, after quantization,

$$[L_i, L_j] = \sum_{s=-g_0}^{g_0} C_{ji}^s L_{i+j-s} + D \hat{\chi}_{ij} \quad (2.16)$$

is satisfied, where normal order in L_j is understood.

3. Lowest order approximation. In order to solve the just stated problem, we consider a normal coordinate expansion of all the relevant quantities [9]

$$x^\mu(Q) = x_0^\mu + \xi^\mu(Q) \quad (3.1)$$

$$G_{\mu\nu}(X) = G_{\mu\nu}^0 - \frac{1}{3} R_{\mu\lambda\nu\rho} \xi^\lambda(Q) \xi^\rho(Q) \dots \quad (3.2)$$

where X_μ^0 is constant (w.r.t. Q). Now, as usual, we consider ξ^μ as the quantum field and

$$P_\mu = i\dot{\xi}^\nu (G_{\mu\nu}^0 - \frac{1}{3} R_{\mu\lambda\nu\rho} \xi^\lambda \xi^\rho + \dots) \quad (3.3)$$

its conjugate momentum. We assume again the c.c.r.

$$[\xi^\mu(Q), P_\nu(Q')] = \delta_\mu^\nu \Delta(Q, Q') \quad (3.4)$$

where $\Delta(Q, Q')$ is defined by (2.12). In order to make sense of the quantities involving monomials of ξ and P we have to define normal ordering.

To this end we expand $\xi^\mu(Q)$ and $P_\nu(Q)$ in the KN basis of meromorphic functions and 1-differentials, respectively,

$$\xi^\mu(Q) = \sum_j \xi_j^\mu A_j(Q) \quad (3.5a)$$

$$P_\nu(Q) = \sum_j P_\nu^j \omega^j(Q) \quad (3.5b)$$

As a consequence of (3.4) we have:

$$[\xi_i^\mu, P_\nu^j] = \delta_i^j \delta_\nu^\mu \quad (3.6)$$

We now define normal ordering by considering $\xi_j^\mu, j > \frac{g}{2}$ and P_μ^j for $j \leq \frac{g}{2}$ as annihilation operators and the complementary set as creation operators. We will comment later on the possible choice of different normal orderings.

Then the relevant L_j , see eq. (2.15), are

$$\begin{aligned} L_j = & \frac{1}{2\pi i} \oint_{C_r} \{ e_j(Q) : \left(\frac{1}{4} d\xi^\mu(Q) G_{\mu\nu}^0 d\xi^\nu(Q) + \frac{1}{4} P_\mu(Q) G^{0\mu\nu} P_\nu(Q) \right. \\ & - \frac{1}{2} d\xi^\mu(Q) P_\mu(Q) - \frac{1}{12} R_{\mu\nu\sigma\tau} d\xi^\mu(Q) \xi^\sigma(Q) d\xi^\nu(Q) \xi^\tau(Q) \\ & \left. + \frac{1}{12} R^\mu{}_{\sigma\nu\tau} P_\mu(Q) \xi^\sigma(Q) \xi^\tau(Q) P_\nu(Q) + \dots \right) : \} \end{aligned} \quad (3.7)$$

where we have explicitly written only the lowest perturbative order. The commutation relations for the L_i 's are calculated with standard methods (Wick theorem). The result is

$$[L_i, L_j] = \sum_{s=-g_0}^{g_0} C_{ji}^s L_{i+j-s} + M_{ij}^{(0)} + M_{ij}^{(2)}(\xi, P) + o_{ij}(R) \quad (3.8)$$

where

$$M_{ij}^{(0)} = D\hat{\chi}_{ij},$$

$$\begin{aligned} M_{ij}^{(2)}(\xi, P) = & \frac{1}{24} R_{\mu\nu} \frac{1}{(2\pi i)^2} \oint \oint \{ (K_1(Q, Q') : \xi^\mu(Q) \xi^\nu(Q) : \\ & + K_2(Q, Q') : d\xi^\mu(Q) d\xi^\nu(Q) : + K_3(Q, Q') : P_\mu(Q) P_{\nu'}(Q) : G^{0\mu\mu'} G^{0\nu\nu'} \\ & + K_4(Q, Q') : \xi^\mu(Q) d\xi^\nu(Q) : + K_5(Q, Q') : \xi^\mu(Q) P_{\nu'}(Q) : G^{0\mu\nu'}) \\ & e_i(Q) e_j(Q') - \{Q \leftrightarrow Q'\} \} \end{aligned} \quad (3.9)$$

while $o_{ij}(R)$ contains any term proportional at least to DR or R^2 . The integrals are both along the same C_r , but one with respect to Q and the other with respect to Q' ,

and

$$\begin{aligned}
K_1(Q, Q') &= d' \Delta^-(Q', Q) d' \Delta^-(Q', Q) - d \Delta^+(Q, Q') d \Delta^+(Q, Q') \\
&\quad - d' \Delta^+(Q', Q) d' \Delta^+(Q', Q) + d \Delta^-(Q, Q') d \Delta^-(Q, Q') \\
K_2(Q, Q') &= \Delta^-(Q, Q') \Delta^-(Q, Q') - \Delta^+(Q, Q') \Delta^+(Q, Q') \\
K_3(Q, Q') &= -K_2(Q, Q') \\
K_4(Q, Q') &= 2d \Delta^-(Q, Q') \Delta^-(Q, Q') - 2d \Delta^+(Q, Q') \Delta^+(Q, Q') \\
K_5(Q, Q') &= K_4(Q', Q)
\end{aligned}$$

Here

$$\Delta^+(Q, Q') = \frac{1}{2\pi i} \sum_{j>0} A_j(Q) \omega_j(Q'), \quad \Delta^- = \Delta - \Delta^+ \quad (3.10)$$

Let us examine one by one the terms appearing in eq.(3.8). The first proportional to $:\xi\xi:$ vanishes (see below). The second and third terms can be written (see below) in the following way:

$$\begin{aligned}
&\frac{1}{24\pi i} \frac{1}{(2\pi i)^2} \oint \oint \{ (R_{\mu\nu} K_2(Q, Q') : d\xi^\mu(Q) d\xi^\nu(Q') : \\
&\quad - R^{\mu\nu} K_2(Q, Q') : P_\mu(Q) P_\nu(Q') :) e_i(Q) e_j(Q') - (i \leftrightarrow j) \} = \\
&= \sum_{s=-g_0}^{g_0} C_{ij}^s \{ \frac{1}{24} R^{\mu\nu} \frac{1}{2\pi i} \oint : P_\mu(Q) P_\nu(Q') : e_{i+j-s}(Q) \\
&\quad - \frac{1}{24} R_{\mu\nu} \frac{1}{2\pi i} \oint : d\xi^\mu(Q) d\xi^\nu(Q') : e_{i+j-s}(Q) \} \quad (3.11)
\end{aligned}$$

This equation is a consequence of a remarkable cocycle identity which was studied in ref. [4]. The last two terms on the r.h.s. of eq. (3.8) cannot be written in such a form, i.e. in the form $\sum_s C_{ij}^s \chi_{i+j-s}$, where χ_i contains only one contour integral as on the r.h.s. of eq. (3.11). Whenever in a commutator $[L_i, L_j]$ we come across a term $\sum_s C_{ij}^s \chi_{i+j-s}$, where χ_i is a c -number, it is possible to eliminate it by a redefinition of the L_i 's: $L'_i = L_i + \chi_i$. However when χ_i is a q -number term as in eq. (3.11) this operation is not always possible. Let us see this in detail for the terms in eq. (3.11). We consider the redefinition

$$L'_i = L_i + S_i \quad (3.12)$$

where

$$S_i = a \frac{1}{2\pi i} \oint R_{\mu\nu} : d\xi^\mu d\xi^\nu : e_i + \frac{b}{2\pi i} \oint R^{\mu\nu} : P_\mu P_\nu : e_i + \frac{c}{2\pi i} \oint R^\mu_\nu : d\xi^\mu P_\nu : e_i \quad (3.13)$$

S_i in eq.(3.12) is the most general coordinate independent "counterterm" involving a single contour integral we can write in connection with eq. (3.11), with arbitrary constants a, b and c . After carrying out the calculation of $[L'_i, L'_j]$ we find that it is impossible to eliminate simultaneously both terms in eq. (3.11). The best we can

do is to absorb either one, e.g. the one proportional to $:d\xi d\xi:$. In such a case the solution is

$$a = -b \quad \text{and} \quad c = \frac{1}{12} \quad (3.14)$$

and the commutator is

$$\begin{aligned} [L'_i, L'_j] = & \sum_{s=-g_0}^{g_0} C_{ij}^s L'_{i+j-s} + (D + \frac{1}{12}R) \hat{\chi}_{ij} \\ & + \sum_{s=-g_0}^{g_0} C_{ij}^s (\frac{1}{12} R^{\mu\nu} \frac{1}{2\pi i} \oint : P_\mu P_\nu : e_{i+j-s}) \\ & + \frac{1}{24} \frac{1}{(2\pi i)^2} \oint \oint \{ (K_4(Q, Q') : \xi^\mu(Q) d\xi^\nu(Q) : \\ & + K_4(Q', Q) : \xi^\mu(Q) P_\lambda(Q) : G_{(0)}^{\mu\lambda} e_i(Q) e_j(Q') - (Q \leftrightarrow Q') \} + \dots \end{aligned} \quad (3.15)$$

where R is the Ricci scalar and the dots denote higher order terms. Here we see that as a consequence of the redefinition of the L_i 's a modification of the central charge appears. This modification does not depend on the values of a and b satisfying eq. (3.14).

Now, taking into account the ghost contribution, see eq. (2.14), we conclude that in order to have conformal invariance, i.e. invariance under the KN algebra, we must impose that

$$D = 26, \quad R_{\mu\nu} = 0 \quad (3.16)$$

We see that eq. (3.16) is not only sufficient for the closure of the conformal algebra, but also necessary due to the fact that the anomalous term proportional to $R_{\mu\nu}$ is a q -number term. Notice that the dependence on the Riemann surface is contained in the integrands in eq.(3.15) and so it can manifest itself only outside the "critical point" (3.16).

Another comment is in order concerning the modification of the central charge by the Ricci scalar. This modification turns out to be of no importance in the present context due to eq.(3.16). However, first of all it is important that our operator formalism contains a mechanism able to produce such a modification of the central charge (which is induced in other approaches by tadpole graphs, for instance). Secondly this modification will become very important in the presence of a more general background involving the dilaton and the antisymmetric tensor. Indeed, an analogous central term is expected when the target manifold is a group manifold.

Finally, a comment concerning the normal ordering chosen in this paper. It is apparently the most convenient. We could of course have chosen another value instead of $\frac{g}{2}$ as initial point to label the creation and annihilation operators. The difference

in the results amounts to terms which can be absorbed by means of the redefinitions considered above.

4. Calculations. In order to give an idea of the techniques used in this paper we present two calculations in some detail. First we show that the term proportional to $:\xi^\mu\xi^\nu:$ in (3.9) vanishes.

First we rewrite this term by means of the explicit bases of tensor fields

$$\begin{aligned}
& R_{\mu\nu} \frac{1}{(2\pi i)^2} \oint_Q \oint_{Q'} [d'\Delta^-(Q', Q)d'\Delta^-(Q', Q) - d'\Delta^+(Q', Q)d'\Delta^+(Q', Q) \\
& + d\Delta^-(Q, Q')d\Delta^-(Q, Q') - d\Delta^+(Q, Q')d\Delta^+(Q, Q')] : \xi^\mu(Q)\xi^\nu(Q) : e_i(Q)e_j(Q') \\
& - (Q \leftrightarrow Q') = \\
& = R_{\mu\nu} : \xi^\mu_\nu \xi^\nu_\rho : \frac{1}{(2\pi i)^2} \{ [(\sum_{m,n < \frac{g}{2}} - \sum_{m,n \geq \frac{g}{2}}) \oint \omega^m \omega^n e_i A_r A_s \oint dA_m dA_n e_j - (i \leftrightarrow j)] \\
& + [(\sum_{m,n < \frac{g}{2}} - \sum_{m,n \geq \frac{g}{2}}) \oint dA_m dA_n e_i A_r A_s \oint \omega_m \omega_n e_j - (i \leftrightarrow j)] \}
\end{aligned} \tag{4.1}$$

Then we show that the first term inside the curly bracket cancels the second.

$$\begin{aligned}
& [(\sum_{m,n < \frac{g}{2}} - \sum_{m,n \geq \frac{g}{2}}) \oint \omega^m \omega^n e_i A_r A_s \oint dA_m dA_n e_j - (i \leftrightarrow j)] = \\
& = [(\sum_{m,n < \frac{g}{2}} - \sum_{m,n \geq \frac{g}{2}}) \oint \omega^m \omega^n e_i A_r A_s \oint \omega^p \omega^q \gamma_{pm} \gamma_{qn} e_j - (i \leftrightarrow j)] = \\
& = [\sum_{\substack{m \geq \frac{g}{2} \\ p \geq \frac{g}{2}}} + \sum_{\substack{m+p \geq -g-1 \\ m,p < \frac{g}{2}}} (\sum_{\substack{q \geq \frac{g}{2} \\ n \geq \frac{g}{2}}} + \sum_{\substack{n+q \geq -g-1 \\ n,q < \frac{g}{2}}}) - (\sum_{\substack{m < \frac{g}{2} \\ p < \frac{g}{2}}} - \sum_{\substack{m+p \geq -g-1 \\ m,p < \frac{g}{2}}}) (\sum_{\substack{n < \frac{g}{2} \\ q < \frac{g}{2}}} - \sum_{\substack{n+q \geq -g-1 \\ n,q < \frac{g}{2}}})] \\
& \quad \oint \omega^m \omega^n \gamma_{pm} \gamma_{qn} e_i A_r A_s \oint \omega^p \omega^q e_j - (i \leftrightarrow j) = \\
& = [\sum_{m,n} \sum_{\substack{p \geq \frac{g}{2} \\ q \geq \frac{g}{2}}} - \sum_{mp} \sum_{\substack{n+q \geq -g-1 \\ n,q < \frac{g}{2}}} + \sum_{qn} \sum_{\substack{m+p \geq -g-1 \\ m,p < \frac{g}{2}}}] \\
& \quad \oint \omega^m \omega^n \gamma_{pm} \gamma_{qn} e_i A_r A_s \oint \omega^p \omega^q e_j - (i \leftrightarrow j) =
\end{aligned}$$

$$\begin{aligned}
&= [(\sum_{\substack{p \geq \frac{g}{2} \\ q \geq \frac{g}{2}}} - \sum_{\substack{p < \frac{g}{2} \\ q < \frac{g}{2}}}) \oint dA_p dA_q e_i A_r A_s \oint \omega^p \omega^q e_j - (i \leftrightarrow j)] \\
&\quad + \sum_{\substack{n+q \geq -g-1 \\ n, q < \frac{g}{2}}} \oint d(i_{e_i} \omega^q) \gamma_{qn} \omega^n A_r A_s e_j - (i \leftrightarrow j)] = \\
&= -[(\sum_{\substack{m < \frac{g}{2} \\ n < \frac{g}{2}}} - \sum_{\substack{m \geq \frac{g}{2} \\ n \geq \frac{g}{2}}}) \oint dA_m dA_n e_i A_r A_s \oint \omega^m \omega^n e_j - (i \leftrightarrow j)]
\end{aligned} \tag{4.2}$$

where we used the following properties of γ_{qn}

$$\gamma_{pn} = -\gamma_{np}$$

$$\gamma_{pn} = 0 \quad \text{for} \quad |p+n| > g+1$$

$$\sum_{\substack{n \geq \frac{g}{2} \\ p}} \gamma_{pn} f(p, n) = \sum_{\substack{p < \frac{g}{2} \\ n}} \gamma_{pn} f(p, n) - \sum_{\substack{n+p \geq -g-1 \\ n, p < \frac{g}{2}}} \gamma_{pn} f(p, n)$$

valid for any $f(p, n)$.

Now we prove eq. (3.11). Consider for example

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} R^{\mu\nu} \oint \oint \{K_3(Q, Q') : P_\mu(Q) P_\nu(Q) : e_i(Q) e_j(Q') - \{i \leftrightarrow j\}\} = \\
&= R^{\mu\nu} \sum_{\substack{r, s \\ n \geq \frac{g}{2}}} (\sum_{\substack{m \geq \frac{g}{2} \\ n \geq \frac{g}{2}}} - \sum_{\substack{m < \frac{g}{2} \\ n < \frac{g}{2}}}) \cdot
\end{aligned} \tag{4.3}$$

$$\cdot \left\{ \frac{1}{2\pi i} \oint A_m A_n e_i \omega^r \omega^s \frac{1}{2\pi i} \oint \omega^m \omega^n e_j - (i \leftrightarrow j) \right\} : P_\mu^r P_\nu^s :$$

Now using $\alpha_{ml}^r = \frac{1}{2\pi i} \oint A_m A_l \omega^r$, we can write

$$\begin{aligned}
&(\sum_{\substack{m \geq \frac{g}{2} \\ n \geq \frac{g}{2}}} - \sum_{\substack{m < \frac{g}{2} \\ n < \frac{g}{2}}}) \left\{ \frac{1}{2\pi i} \oint A_m A_n e_i \omega^r \omega^s \frac{1}{2\pi i} \oint \omega^m \omega^n e_j - (i \leftrightarrow j) \right\} = \\
&= (\sum_{\substack{m \geq \frac{g}{2} \\ n \geq \frac{g}{2}}} - \sum_{\substack{m < \frac{g}{2} \\ n < \frac{g}{2}}}) \frac{1}{2\pi i} \oint e_i \omega^k \omega^l \alpha_{ml}^r \alpha_{nk}^s \frac{1}{2\pi i} \oint e_j \omega^m \omega^n - \\
&- (\sum_{\substack{l \geq \frac{g}{2} \\ n \geq \frac{g}{2}}} - \sum_{\substack{l < \frac{g}{2} \\ k < \frac{g}{2}}}) \frac{1}{2\pi i} \oint e_i \omega^k \omega^l \alpha_{ml}^r \alpha_{nk}^s \frac{1}{2\pi i} \oint e_j \omega^m \omega^n
\end{aligned} \tag{4.4}$$

On the other hand we have ^[1]

$$\frac{1}{2\pi i} \oint \omega^r \omega^s [e_i, e_j] = l_i^r l_j^s \gamma_{mn} - (i \leftrightarrow j) \tag{4.5}$$

where $l_i^{nm} = \frac{1}{2\pi i} \oint e_i \omega^n \omega^m$, $\gamma_{nm} = \frac{1}{2\pi i} \oint A_n dA_m$. Now we use a remarkable cocycle identity proved in [4]

$$\gamma_{mn} = \left(\sum_{\substack{l \geq \frac{g}{2} \\ k < \frac{g}{2}}} - \sum_{\substack{l < \frac{g}{2} \\ k \geq \frac{g}{2}}} \right) \alpha_{ml}^k \alpha_{nk}^l \quad (4.6)$$

Inserting this in (4.5) and expanding on the basis elements we find

$$\begin{aligned} & \frac{1}{2\pi i} \oint \omega^r \omega^s [e_i, e_j] = \\ & = \left(\sum_{\substack{l \geq \frac{g}{2} \\ k < \frac{g}{2}}} - \sum_{\substack{l < \frac{g}{2} \\ k \geq \frac{g}{2}}} \right) \left\{ \frac{1}{2\pi i} \oint e_i \omega^k \omega^{l'} \alpha_{kl}^r \alpha_{ll'}^s \frac{1}{2\pi i} \oint e_j \omega^{k'} \omega^l - (i \leftrightarrow j) \right\} = \\ & = \left[\left(\sum_{\substack{l < \frac{g}{2} \\ n \geq \frac{g}{2}}} - \sum_{\substack{l \geq \frac{g}{2} \\ n < \frac{g}{2}}} \right) - \left(\sum_{\substack{m < \frac{g}{2} \\ k \geq \frac{g}{2}}} - \sum_{\substack{m \geq \frac{g}{2} \\ k < \frac{g}{2}}} \right) \right] \frac{1}{2\pi i} \oint e_i \omega^l \omega^k \alpha_{lm}^r \alpha_{nk}^s \oint e_j \omega^n \omega^m \end{aligned} \quad (4.7)$$

The last result is obtained by first observing that in the intermediate expression exchanging i with j corresponds to the Aexchange of k and k' and l and l' , and then by relabelling the indices in the following way: $k' \rightarrow m, k \rightarrow m, l \rightarrow n, l' \rightarrow k$. Comparing now the last expression in (4.7) with the RHS of eq. (4.4) one can realize that the two expressions are in fact identical.

Finally collecting the results we have

$$\begin{aligned} & \frac{1}{(2\pi i)^2} R^{\mu\nu} \oint \oint \{ K_3(Q, Q') : P_\mu(Q) P_\nu(Q) : e_i(Q) e_j(Q') - (i \leftrightarrow j) \} = \\ & = \frac{1}{2\pi i} R^{\mu\nu} \oint [e_i, e_j](Q) : P_\mu(Q) P_\nu(Q) : \end{aligned} \quad (4.8)$$

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