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PARTITION FUNCTION OF STRING SIGMA MODEL ON A COMPACT TWO -SPACE

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ABSTRACT

We develop a manifestly covariant procedure of computation of the partition function Z of a general renormalizable 2-d sigma model defined on a compact 2-space. Special attention is paid to a proper separation of the path integral into the integrals over the constant and nonconstant parts of the coordinate. The important role of the path integral measure in establishing the covariance and renormalizability properties of Z is stressed. We explicitly check the validity of representation of closed string effective action as the partial derivative of Z over the logarithm of the cutoff. We also comment on the status of the "c-theorem" and suggest a resolution of a problem in renormalization of the closed string loop corrections corresponding to the disc topology.

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I. The central role played by the partition function Z of the general 2-d sigma model in the path integral approach to string theory is well known at present [1-3]. Z is closely related to the generating functional for string S-matrix and the string theory effective action (see [4] for a review and extensive references). It provides a starting point for off-shell and off-conformal point extensions of string theory.

While we are free to choose any of the conformally equivalent representations for the string world sheet while computing "tree" string amplitudes in a conformal vacuum, it is essential to use a compact world sheet in extending string theory away from tree level conformal point. Working on a compact 2-space we automatically cut off possible infrared infinities and deal with mathematically well defined problems (the spectra of the Laplace operators are discrete, etc.).

Computing Z on a compact 2-space one should carefully account for the contribution of the path integral measure [5]. It was pointed out in [6] that a proper choice of a regularized expression for the measure is important for obtaining a covariant expression for Z in terms of the integral over the constant ("zero-mode") part of the coordinates x^μ [3]. We shall see that the contribution of the measure is also essential for the renormalizability of Z . Another important contribution comes from the nontrivial "ghost" determinant which appears in splitting the path integral $\int dx$ into $\int dy$ and $\int d\zeta$ where ζ is the "normal coordinate" variable.

Below we shall explicitly compute the $O(\alpha')$ (two-loop) term in the expansion of Z in derivatives of the fields. As a result we shall check the validity of the following representation of the string effective action: $S(\varphi) = \frac{\partial}{\partial t} Z(\varphi, t)$ ($t = \log \mu$ and φ are the renormalized couplings) [7]. We shall also discuss the question of validity of the "c-theorem" [8] in the general 2-d renormalisable sigma model (cf. [9, 10]).

Since the higher loop contributions to the string theory partition function are given by the modular integrals of the sigma model partition function Z defined on compact two-spaces of corresponding genera the knowledge of the structure of Z is essential for understanding whether the total (summed over genera) string partition function can be made finite by renormalising both local and modular infinities [11-13, 4, 6]. In particular, we shall suggest how to resolve the inconsistency found previously in the attempt to apply the renormalisation program to the partition function containing the contributions of the sphere and the disc topologies [13].

2. We shall consider the following bosonic sigma model

$$Z = \int \mathcal{D}x \exp(-I[x]) \quad , \quad (1)$$

$$\mathcal{D}x = \prod_z dx(z) \sqrt{G(x(z))} \quad , \quad G = \det G_{\mu\nu} \quad , \quad (2)$$

$$I = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} \left(\partial_\alpha x^\mu \partial^\alpha x^\nu G_{\mu\nu}(x) + \alpha' R^{(2)} \phi(x) \right) \quad (3)$$

defined on a compact 2-space M^2 with some fixed non-singular metric (for example, we may assume that $g_{\alpha\beta}$ has a constant curvature, $R^{(2)} = \text{const}$). We shall first consider the case when M^2 has no boundary. For simplicity we do not include in (3) the antisymmetric tensor coupling and the tachyonic term $\Lambda^2 \int d^2z \sqrt{g} \Phi(x)$ (we shall assume that the renormalised value of Φ is equal to zero). The fields G and ϕ in (I) are the bare fields depending on an UV cutoff Λ . The theory (I) is defined by the action and the measure. The usual choice of the measure (2) guarantees the formal invariance of the theory under the simultaneous transformations of x and G , $x \rightarrow x'(x)$, $G \rightarrow G'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} G_{\alpha\beta}$. However, one is still to specify what the formal product in (2) really means. To preserve the general coordinate invariance in a regularised theory it is important to regularise both the action and the measure in a consistent (universal) way. We shall use the heat kernel (proper time) regularisation, inserting $e^{-\epsilon\Delta}$, $\epsilon \rightarrow 0$ ($\Delta = -\nabla^2$ is the scalar Laplacian which appears in (3)) into functional traces. We shall thus replace Δ in the free part of the action (3) by $e^{\epsilon\Delta} \Delta$ and define the formal measure in (2) as

$$\mathcal{D}x = \prod_z dx(z) \mu[x], \quad \mu = e^{\mathcal{M}} \quad (4)$$

$$\mathcal{M} = \frac{1}{2} \text{tr}(\ln G e^{-\epsilon\Delta}) = \frac{1}{2} \int d^2z \sqrt{g} \ln G(x(z)) K_\epsilon(z, z) \quad (5)$$

Here we have regularised the δ -function " $\delta^{(2)}(0)$ " replacing it by the heat kernel

$$K_\epsilon(z, z') = \langle z | e^{-\epsilon\Delta} | z' \rangle, \quad \epsilon = 1/\Lambda^2 \quad (6)$$

$$\lim_{\epsilon \rightarrow 0} K_\epsilon(z, z') = \delta^{(2)}(z, z') = \frac{1}{\sqrt{g}} \delta^{(2)}(z - z'), \quad z \neq z' \quad (7)$$

As is well known [17]

$$K_\epsilon(z, z) = \frac{1}{4\pi\epsilon} + \frac{1}{24\pi} R^{(2)} + O(\epsilon) \quad (8)$$

Hence the measure factor (4), (5) can in fact be rewritten as a combination of the tachyon and the dilaton terms in the bare action [6]. As we shall see below, the above choice of the regularised expression for the measure implies that that the resulting partition function Z (i) is a covariant functional of G and ϕ ; (ii) is renormalisable, and (iii) is free of non-trivial power ("quadratic") infinities. We would like to emphasize again that in order to ensure these properties it is crucial to regularise the measure in a way consistent with the regularisation of the action.

To compute Z in (I) we are to split x^μ on a constant and non-constant part ,
 $x^\mu = y^\mu + \eta^\mu$. This can be done by inserting the following "unite" into (I) [5] (see also [3])

$$1 = \int d^D y \int \prod_z d\eta(z) \delta^{(D)}(x(z) - y - \eta(z)) \delta^{(D)}(P^\mu[\eta]) Q[\eta], \quad (9)$$

$$Q = \det (\partial P^\mu[\eta - a, \eta + a] / \partial a^\nu)_{a=0},$$

where $P=0$ is a "gauge condition" which fixes the symmetry of the δ -function and Q is the corresponding "ghost" determinant. For example, we may choose

$$P^\mu = \int d^2 z \sqrt{g} \eta^\mu(z), \quad Q = V^D, \quad V = \int d^2 z \sqrt{g}. \quad (10)$$

$P=0$ implies that η can be represented as a sum of the non-zero modes of the Laplacian. Substituting $x=y+\eta$ into the action and the measure (3), (5) and expanding in powers of η we get

$$Z = \int d^D y \int [d\eta] \exp \left\{ - \frac{1}{4\pi\alpha'} \int d^2 z \sqrt{g} [\partial^\alpha \eta^\mu \partial_\alpha \eta^\nu G_{\mu\nu}(y) \right. \\ + \partial^\alpha \eta^\mu \partial_\alpha \eta^\nu \eta^\lambda \partial_\lambda G_{\mu\nu}(y) + \frac{1}{2} \partial^\alpha \eta^\mu \partial_\alpha \eta^\nu \eta^\lambda \eta^\rho \partial_\lambda \partial_\rho G_{\mu\nu}(y) \\ + O(\eta^5)] - \chi \phi(y) - \frac{1}{4\pi} \int d^2 z \sqrt{g} R^{(2)} [\frac{1}{2} \eta^\mu \eta^\nu \partial_\mu \partial_\nu \phi(y) + O(\eta^3)] \\ \left. + \frac{1}{2} \int d^2 z \sqrt{g} [\ln G(y) + \frac{1}{2} \eta^\lambda \eta^\rho G^{\alpha\beta} \partial_\alpha \partial_\beta G_{\alpha\beta} - \frac{1}{2} \eta^\mu \eta^\nu (G^{-1} \partial_\mu G)(G^{-1} \partial_\nu G) + \dots] \right\} \\ [d\eta] \equiv \prod_z d\eta^\mu(z) \delta(P^\mu) Q, \quad \chi \equiv \frac{1}{4\pi} \int d^2 z \sqrt{g} R^{(2)} \quad (12)$$

We have dropped $O(\eta)$ terms in I and ~~II~~ assuming that $R^{(2)} = \text{const}$. The regularised Green function for η is thus

$$\langle \eta^\mu(z) \eta^\nu(z') \rangle = 2\pi\alpha' G^{\mu\nu}(y) \mathcal{D}(z, z') \quad (13)$$

$$\mathcal{D}(z, z') = \langle z | \Delta^{-1} e^{-\epsilon \Delta} | z' \rangle_\perp = \sum_{n \neq 0} \frac{e^{-\epsilon \lambda_n}}{\lambda_n} \varphi_n(z) \varphi_n(z') \quad (14)$$

where φ_n and λ_n are the eigenfunctions and eigenvalues of the Laplacian on M^2

$$\Delta \varphi_n = \lambda_n \varphi_n, \quad \int d^2 z \sqrt{g} \varphi_n \varphi_m = \delta_{nm}, \quad \int d^2 z \sqrt{g} \varphi_n(z) = 0, \quad n \neq 0, \\ \varphi_0 = \frac{1}{\sqrt{V}}, \quad \lambda_0 = 0, \quad \Delta \mathcal{D}(z, z') = \delta^{(2)}(z, z') - \frac{1}{V} \quad (15)$$

The leading order (one-loop) contribution of the integral over η is thus

$$[\det' G_{\mu\nu}(y) \Delta]^{-1/2} = \exp \left[- \frac{1}{2} N' \ln G(y) - \frac{D}{2} \ln \det' \Delta \right], \quad (16)$$

where N' is the regularised number of the non-zero eigenvalues of Δ

$$N' = \sum_{n \neq 0} e^{-\epsilon \lambda_n} = \sum_n e^{-\epsilon \lambda_n} - 1 \equiv N - 1 \quad (17)$$

Using (6), (8) we can represent N' and the total number of the eigenvalues N as

$$N' = \int d^2z \sqrt{g} K'_\epsilon(z, z) \quad , \quad K'_\epsilon(z, z) = K_\epsilon(z, z) - \frac{1}{V} \quad , \quad (I8)$$

$$K_\epsilon(z, z') = \sum_n e^{-\epsilon \lambda_n} \varphi_n(z) \varphi_n(z') \quad , \quad (I9)$$

$$N = \int d^2z \sqrt{g} K_\epsilon(z, z) = \frac{V}{4\pi\epsilon} + \frac{1}{6} \chi + O(\epsilon) \quad . \quad (20)$$

Combining (I6) with the contribution of the measure in (II) we conclude that the $\ln G$ -terms corresponding to the non-zero modes cancel out and hence what remains is the $\sqrt{G}$ - factor coming from the measure. Thus Z can be represented as [3,4]

$$Z = \lambda \int d^D y \sqrt{G} e^{-\chi \phi} \mathcal{Z}(G, \partial G, \partial \phi) \quad , \quad \mathcal{Z} = e^{-W} \quad (21)$$

Here λ is the free partition function $\sim (\det' \Delta)^{-D/2}$,

$$\lambda = (2\pi\alpha')^{-D/2} \exp \left[-\frac{1}{2} D \left(\frac{V}{4\pi\epsilon} + \frac{1}{6} \chi \ln \epsilon + f[g_{ab}] \right) \right] \quad (22)$$

and W depends on the derivatives of the fields. Since $\int d^2z \sqrt{g} \chi' = 0$, $\int d^2z \sqrt{g} \mathcal{D}(z, z) = 0$, W is given by the sum of one-particle irreducible vacuum graphs in the theory (II). We expect that $\mathcal{Z}$ can be represented in a manifestly covariant way as a series of terms constructed of the curvature R of $G_{\mu\nu}$ and covariant derivatives of R and ϕ . However, this is not obvious in the direct non-manifestly covariant expansion made in (9)-(II). It is possible to check [I8] (assuming for simplicity that g_{ab} is such that $\mathcal{D}_0 \mathcal{D}(z, z) = \text{const}$, what is true, e.g., for homogeneous spaces) that the covariantization does indeed take place in the 2-loop expression for Z if the following non-trivial 2-loop correlator

$$J = \int d^2z \sqrt{g} \int d^2z' \sqrt{g'} \nabla_a \nabla'_b \mathcal{D}(z, z') \nabla^a \mathcal{D}(z, z') \nabla'^b \mathcal{D}(z, z') \quad (23)$$

is equal to $-\frac{1}{2} \mathcal{D}_0$. One finds then that various ∂G and $\partial \phi$ -dependent terms group together to form

$$\mathcal{Z} = 1 - \pi \alpha' \mathcal{D}_0 (R + \chi \mathcal{D}^2 \phi) + O(\alpha'^2) \quad . \quad (24)$$

The quadratic infinities cancel in the final result: the contributions of the action ($\langle \partial \eta \partial \eta \eta \eta \rangle \sim \int K'_\epsilon \mathcal{D}_0$) and of the measure ($\sim \int K_\epsilon \mathcal{D}_0$) combine so that $\mathcal{D}_0$ multiplies $\int K_\epsilon - \int K'_\epsilon = N - N' = 1$.

3. The straightforward non-covariant expansion used in (II) is certainly inconvenient. It is desirable to have a manifestly covariant procedure of computation of W based on a sort of normal coordinate expansion (usually applied in studying sigma-models, see e.g. [5] and references there). To this end we are to make a change of the qua-

ntum variable $\eta \rightarrow \xi(y, \eta)$ where ξ^μ is the tangent vector at y to the geodesic which connects y and $y + \eta$,

$$\eta^\mu = \xi^\mu - \frac{1}{2} \Gamma_{\alpha\beta}^\mu(y) \xi^\alpha \xi^\beta - \frac{1}{6} (\partial_\gamma \Gamma_{\alpha\beta}^\mu - 2 \Gamma_{\gamma\alpha}^\lambda \Gamma_{\lambda\beta}^\mu) \xi^\alpha \xi^\beta \xi^\gamma + \dots \quad (25)$$

Then the expansion of the action (3) and of the measure factor (5) in powers of ξ takes the manifestly covariant form

$$I = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} \left[\partial_a \xi^\mu \partial^a \xi^\nu G_{\mu\nu}(y) + \frac{1}{3} \partial_a \xi^\mu \partial^a \xi^\nu \xi^\lambda \xi^\rho R_{\mu\lambda\rho\nu}(y) + O(\xi^5) \right] + \frac{1}{4\pi} \int d^2z \sqrt{g} \left[\phi(y) + \xi^\mu \partial_\mu \phi(y) + \frac{1}{2} \xi^\mu \xi^\nu \partial_\mu \partial_\nu \phi(y) + O(\xi^3) \right] \quad (26)$$

$$\mathcal{M} = \frac{1}{2} \int d^2z \sqrt{g} K_\epsilon(z, z) \left[\ell_h G(y) - \frac{1}{3} R_{\mu\nu}(y) \xi^\mu \xi^\nu + O(\xi^3) \right] \quad (27)$$

Since the kinetic term in (26) is invariant under $\xi \rightarrow \xi + a$, $a = \text{const}$ and since (25), (10) do not imply that ξ is non-constant, it is necessary to fix the "splitting symmetry" $y \rightarrow y - a$, $\eta \rightarrow \eta + a$ by imposing a gauge condition different from (10). The natural choice is [5]

$$P^\mu = \int d^2z \sqrt{g} \xi^\mu(z) \quad (28)$$

implying that $\xi^\mu = \sum_{n \neq 0} \xi_n^\mu \varphi_n(z)$. Then the ghost determinant Q in (9) is already non-trivial

$$Q = \det \left[\int d^2z \sqrt{g} \lambda_{\nu}^\mu \right], \quad \lambda_{\nu}^\mu = \frac{\partial \xi^\mu(y, \eta)}{\partial \eta^\nu} - \frac{\partial \xi^\mu(y, \eta)}{\partial y^\nu} \quad (29)$$

As discussed in [5], λ_{ν}^μ can be represented as a covariant derivative $-\partial_\nu \xi^\mu$. It is easy to compute the normal coordinate expansion of Q directly, inverting (25), $\xi^\mu = \eta^\mu + \frac{1}{2} \Gamma_{\alpha\beta}^\mu \eta^\alpha \eta^\beta + \frac{1}{6} (\partial_\gamma \Gamma_{\alpha\beta}^\mu + \Gamma_{\gamma\alpha}^\lambda \Gamma_{\lambda\beta}^\mu) \eta^\alpha \eta^\beta \eta^\gamma + \dots$, differentiating and using that in the normal coordinates $\Gamma_{\alpha\beta}^\mu(y) = 0$, $\partial_\gamma \Gamma_{\alpha\beta}^\mu = 2 \Gamma_{(\gamma\alpha}^\lambda \Gamma_{\lambda\beta)}^\mu$, $\partial_\alpha \Gamma_{\lambda\rho}^\mu = \frac{1}{3} (R_{\lambda\alpha\rho}^\mu + R_{\rho\alpha\lambda}^\mu)$, etc. (see also [5]). We find

$$\lambda_{\nu}^\mu = \delta_{\nu}^\mu - \frac{1}{3} R_{\alpha\nu\beta}^\mu \xi^\alpha \xi^\beta + O(\xi^3) \quad (30)$$

$$Q = V^D \exp \left[-\frac{1}{3V} \int d^2z \sqrt{g} R_{\mu\nu}(y) \xi^\mu \xi^\nu + \dots \right]$$

The account of the contribution of Q is crucial for obtaining the correct result for Z ,

$$Z = \lambda \int d^D y \sqrt{G} e^{-x\phi} \left\{ 1 + \alpha' (b_1 R + b_2 \partial^2 \phi) + O(\alpha'^2) \right\}, \quad (31)$$

$$b_1 = a_1 + a_2 + a_3$$

Here a_1 and b_2 are the contributions of the action (26), a_2 - of the measure (27), and a_3 - of the ghost determinant (30),

$$a_1 = \frac{1}{6} 2\pi \int d^2z \sqrt{g} \left[\left(\nabla_a \nabla'^a \mathcal{D}(z, z') \right)_{z=z'} \mathcal{D}(z, z) - \left(\nabla_a \mathcal{D}(z, z') \right)_{z=z'} \left(\nabla^a \mathcal{D}(z, z') \right)_{z=z'} \right] \quad (32)$$

$$a_2 = -\frac{1}{6} 2\pi \int d^2z \sqrt{g} K_\epsilon(z, z) \mathcal{D}(z, z) \quad (33)$$

$$a_3 = -\frac{1}{3V} 2\pi \int d^2z \sqrt{g} \mathcal{D}(z, z) \quad (34)$$

$$b_2 = -\frac{1}{4} \int d^2z \sqrt{g} R^{(2)} \mathcal{D}(z, z) \quad (35)$$

$$\left(\nabla_a \nabla'^a \mathcal{D} \right)_{z=z'} = \sum_{h \neq 0} \frac{e^{-\epsilon \lambda_h}}{\lambda_h} \partial_a \varphi_h(z) \partial^a \varphi_h(z) = \frac{1}{4\pi\epsilon} + \dots \quad (36)$$

$$\left(\nabla_a \mathcal{D} \right)_{z=z'} = \sum_{h \neq 0} \frac{e^{-\epsilon \lambda_h}}{\lambda_h} \varphi_h(z) \partial_a \varphi_h(z) = \frac{1}{2} \partial_a \mathcal{D}(z, z)$$

These results are easily found by substituting $\xi = \sum_{h \neq 0} \xi_h \varphi_h$ and averaging over ξ_h . Note that to determine the $O(\alpha')$ -term in Z in this manifestly covariant approach it is not necessary to compute the complicated 2-loop correlator (23). In general, the Green function $\mathcal{D}(z, z)$ can be represented as

$$\mathcal{D}(z, z) = \mathcal{D}_0 + h(z) \quad , \quad \mathcal{D}_0 = -\frac{1}{4\pi} \ln \epsilon^2 \quad , \quad \epsilon = \frac{1}{\Lambda \tau} \quad (37)$$

where ϵ is the dimensionless combination of the UV cutoff and of a scale τ of a compact 2-space (e.g., $\tau = V^{1/2}$) and h is a finite function. Then

$$a_1 = \frac{\pi}{3} N' \mathcal{D}_0 + \bar{a}_1 \quad , \quad a_2 = -\frac{\pi}{3} N \mathcal{D}_0 + \bar{a}_2 \quad , \quad a_3 = -\frac{2\pi}{3} \mathcal{D}_0 + \bar{a}_3 \quad (38)$$

$$b_1 = \left(-\frac{\pi}{3} - \frac{2\pi}{3} \right) \mathcal{D}_0 + \bar{b}_1 = -\pi \mathcal{D}_0 + \bar{b}_1 \quad , \quad b_2 = -\pi \chi \mathcal{D}_0 + \bar{b}_2$$

where $\bar{a}_i$ and $\bar{b}_i$ are the finite functionals of g_{ab} . Hence

$$Z = \int d^2y \sqrt{G} e^{-\chi \phi} \left[1 + \frac{1}{2} \alpha' \ln \epsilon (R + \chi \mathcal{D}^2 \phi) + \alpha' (\bar{b}_1 R + \bar{b}_2 \mathcal{D}^2 \phi) + O(\alpha'^2) \right] \quad (39)$$

in agreement with the result (24) found using the non-covariant expansion. We see that the cancellation of the quadratic infinities takes place for arbitrary g_{ab} and that the divergent part of the $O(\alpha')$ -term in Z (39) is "universal (independent of g_{ab})". The latter property need not be true for higher order terms in Z : the coefficients of the logarithmic divergences may, in general, be non-trivial functionals of g_{ab} . This does not contradict the usual claim that the coefficients of the UV counterterms are universal since Z is a global functional (the coefficients in the local β -functions are, of course, universal).

Recalling the standard expressions for the one-loop β -functions (which are found by computing the counterterms on a non-constant background $\bar{x}$) [5,9,19]

$$\begin{aligned}\beta_{\mu\nu}^G &= \alpha' R_{\mu\nu} + \frac{\alpha'^2}{2} R_{\mu\alpha\beta\gamma} R_{\nu}{}^{\alpha\beta\gamma} + O(\alpha'^3) \\ \beta^\phi &= \frac{1}{6} D - \frac{\alpha'}{2} \partial^2 \phi + \frac{\alpha'^2}{16} R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} + O(\alpha'^3)\end{aligned}\quad , \quad (40)$$

we can check the renormalisability property of Z by expressing the bare fields in (39) in terms the renormalised ones

$$\begin{aligned}G_{\mu\nu} &= G_{\mu\nu}^{(R)} - \alpha' R_{\mu\nu}^{(R)} \ln \epsilon + \dots \\ \phi &= \phi^{(R)} - \frac{1}{6} D \ln \epsilon + \frac{\alpha'}{2} \partial^2 \phi^{(R)} \ln \epsilon + \dots\end{aligned}\quad , \quad (41)$$

We find that $Z(\varphi(\epsilon), \epsilon) = Z^{(R)}(\varphi^{(R)})$ -finite. Note that this result depends crucially on the values of the coefficients in (39) and hence is true only if all the three contributions (32)-(34) to b_1 are included. It is interesting to note that the universality of the $\ln \epsilon R$ -term in (39) is related to the absence of the $O(R)$ -term in β^ϕ (40). For example, the coefficient of $\ln \epsilon R_{\lambda\mu\nu\rho}^2$ -term in Z consistent with the renormalisability of Z is no longer universal

$$Z = \lambda \int d^D y \sqrt{G} e^{-\chi \phi} \left[1 + \ln \epsilon \left(\frac{1}{2} \alpha' R + \frac{1}{16} \alpha'^2 (4 - \chi) R_{\lambda\mu\nu\rho}^2 \right) + \dots \right]^{(42)}$$

It is straightforward to generalize the above computation of Z to the case of M^2 with boundary. In view of the application to the open string theory we assume the Neumann boundary conditions on x . Then the Laplacian again has the zero mode. We find just the same expressions (31)-(39) where now $N = \int K_\epsilon = \frac{1}{4\pi\epsilon} V + \frac{1}{8\sqrt{\pi\epsilon}} \int_{\partial M} ds + \frac{1}{6} \chi + O(\epsilon)$, $\chi = \frac{1}{4\pi} \left(\int_M R^{(2)} + 2 \int_{\partial M} K \right)$, etc. (note that in general one should also couple the dilaton to the geodesic curvature of the boundary to ensure that ϕ couples to the Euler number).

4. Let us now discuss some applications. As was argued in [7], the tree-level effective action (EA) in the closed string theory can be represented as

$$S = \left(\frac{\partial}{\partial \ln \epsilon} Z_{\chi=2} \right)^{(R)} \quad (43)$$

where Z corresponds to the sigma model defined on M of the spherical topology.

Taking $\frac{\partial}{\partial \ln \epsilon}$ corresponds to a subtraction of the Mobius infinity in the on-shell amplitudes. In fact, the computation of Z is equivalent to the computation of the regularised correlators of the vertex operators **without** fixing the Mobius gauge. For example, the $O(R)$ -term in Z is directly related to the k^2 -term in the regularised correlator of the three graviton vertex operators and hence to the Mobius group volume. Thus the $\ln \epsilon R$ -term in (39) can be interpreted as the Mobius infinity. Renormalisation group invariance implies that we should not simply divide but differentiate over $\ln \epsilon$ (then the result is RG invariant and hence finite being expressed in terms of the renormalised fields). Substituting (39) into (43) we find

$$S = \lambda' \int d^2y \sqrt{G} e^{-2\phi} \left[D-26 - \frac{3}{2} \alpha' (R + 2 \partial^2 \phi) + O(\alpha'^2) \right] \quad (44)$$

$$= \lambda' \int d^2y \sqrt{G} e^{-2\phi} \left[D-26 - \frac{3}{2} \alpha' (R + 4 \partial_a \phi \partial^a \phi) + O(\alpha'^2) \right]$$

(we have accounted for the $\ln \epsilon$ -dependence of λ (22) and replaced D by $D-26$ assuming that the contribution of the reparametrisation ghosts is included). S in (44) is the EA that reproduces the leading order terms in the graviton and dilaton amplitudes in the closed string theory [20]. Eq. (43) thus makes precise the old suggestion [3] that the sigma model partition function on a 2-sphere is related to the string theory EA. To determine the EA it is thus not necessary to go through the complicated procedure of matching of the field-theory and string-theory on-shell scattering amplitudes but it is sufficient to compute one global functional defined in (43).

It is possible to generalize the above results to the case of the supersymmetric sigma model. As was already found in [3, 21] the leading terms in Z and the β -functions remain the same. Assuming that ψ^m satisfies "antiperiodic" boundary conditions (so that the $\psi^m = \text{const}$ zero mode is excluded) we find that the measure factor reduces to the $\sqrt{G(\psi)}$ corresponding to the zero mode of x^m (as for the quadratic infinities, they cancel automatically due to the "asymptotic" 2-d supersymmetry present in the UV limit, see [22] for a discussion of a similar cancellation in the open superstring case). It is easy to check that the fermi and mixed bose-fermi contributions to the coefficients of the $\ln \epsilon R$ -term in (39) cancel each other. As a result, we find again (39), (44) (with $D-26$ replaced by $\frac{3}{2}(D-10)$ [2]).

5. Next, let us comment on the status of the "c-theorem" [8] when applied to the sigma models. In [8] it was suggested to define the "c-function" in terms of the correlators of the energy-momentum tensor $\langle T_{ab}(z) T_{cd}(z') \rangle$ on a flat background.

It is natural to define these correlators in terms of the $\delta/\delta g_{ab}$ -derivatives of the 2-d effective action $W = -\ln Z$ on a weakly curved 2-background. It was noted recently [10] that the presence of the IR divergent $\partial^2 R$ -term in W and hence in $\langle T T \rangle$ invalidates the proof of the "c-theorem" given in [8]. Equivalently, defining c in terms of the IR finite coefficients which appear in the operator product $T(z) T(z')$ one finds [10, 23] that the original relation of ref. [8] is modified by the additional term γc ($\gamma \sim \partial^2 + \dots$ is the scalar anomalous dimension operator). Trying to reformulate the "c-theorem" in the context of the general sigma models [9] we have pointed out that (i) computing Z on a curved background (or $\langle T T \rangle$) it is necessary to introduce the "running" dilaton coupling, and (ii) it is natural (in view of correspondence with string theory and in order to cut off IR infinities) to

define Z or TT on a compact 2-space of a spherical topology. Then Z (see (2I)) and hence $\langle T T \rangle$ and c are given by the integrals over the zero mode of x . While the c -function defined in [8] and discussed in [10,23] is a local function of G and ϕ and their derivatives ($c \sim D + a_1 R + a_2 R^2 + \dots$), c introduced in [9] is a global functional of the couplings, $c \sim \int d^2 y \sqrt{G} e^{-2\phi} (D + a_1 R + \dots)$. If Z is given by (2I), the $D^2 R$ -term in Z disappears after we integrate by parts [4,23] (this is true even for $\phi \neq \text{const}$). This could be considered as an indication that the formal difficulty found in [10] could be avoided.

Still, it seems to us that the "c-theorem" can not be true in the general sigma models (and hence in string theory) in its original formulation given in [8]: there exists such $c(\varphi)$ that: (1) $\frac{\partial c}{\partial \varphi^i} = 0$ is equivalent to $\beta^i = 0$; (2) c decreases along the trajectories of RG and is equal to the central charge at the fixed points. While the first statement (and even $\frac{\partial c}{\partial \varphi^i} = \alpha_{ij} \beta^j$) is certainly true in the perturbation theory (see e.g. ref. [24] and refs. there) with c being a functional, essentially equivalent to the string theory effective action S [7], the second statement can not be true. The reason is that since the dilaton coupling must be introduced into the theory, it participates in the RG flow ($\varphi^i = (G, \phi)$) the fixed points of which are thus $\beta^G = 0$ and $\beta^\phi = 0$. It is easy to see that if $\beta^G = 0$ and $\beta^\phi = 0$ the central charge vanishes. c is thus the same ($=0$) in all fixed points and hence does not decrease. This, in fact, is what should be for the string tree vacua (which are solutions of $\frac{\delta S}{\delta G} = 0$ and $\frac{\delta S}{\delta \phi} = 0$). Thus the condition of decreasing along the RG trajectories is inconsistent with the requirement of being equal to the central charge at the critical points.

We suggest that in the general string sigma models the "c-theorem" is replaced by the following statement: there exists a functional $\mathcal{C}$ such that (1) $\frac{\partial \mathcal{C}}{\partial \varphi^i} = \alpha_{ij} \bar{\beta}^j$ where $\varphi^i = (G, \phi)$, $\bar{\beta}^i$ are the Weyl anomaly coefficients α is non-degenerate; (2) there exists a choice of φ^i such that $\mathcal{C} = S_0 = \int d^2 y \sqrt{G} e^{-2\phi} \bar{\beta}^\phi$, where $\bar{\beta}^\phi = \beta^\phi - \frac{1}{4} \beta^\mu{}_\mu G^\mu{}_\mu$ is the basic Weyl anomaly coefficient ("central charge"). Then $S_0 = 0$ at the critical points. The representation (43) implies that S_0 is equivalent to the string theory effective action. As was noted in [22], $\beta^i \frac{\partial}{\partial \varphi^i} S_0 = A^2 - B^2 \geq 0$, where A and B are some functionals of $\mathcal{G}$, and hence S_0 does not decrease or increase along the RG flow.

6. Finally, we would like to discuss the issue of the renormalisability of the summed over genera string partition function $\hat{Z}(\varphi)$,

$$\hat{Z} = \sum_{n=0}^{\infty} \hat{Z}_n, \quad \hat{Z}_0 = \frac{\partial}{\partial \ell_{h_E}} Z_0, \quad \hat{Z}_n = \int [d\tau]_n Z_n. \quad (45)$$

Z_n are the sigma model partition functions on compact 2-spaces of the corresponding genera. $\hat{Z}$ is the generating functional for string amplitudes and its renormalised value should be equal to the loop-corrected string EA (see [4]). Regularizing the modular integrals by the same cutoff as used in the 2-d propagator we should find that $\hat{Z}_n$ contain both local infinities (present in Z_n) and modular infinities.

The crucial property that should make possible to extend the RG approach to loops (and that should be related to the factorization property of string amplitudes [25]) should be the renormalisability of $\hat{Z}$ with respect to both local and modular infinities [4] (each Z_n is, of course, renormalisable with respect to local infinities). It should imply that $\frac{\partial \hat{Z}}{\partial \ln \epsilon} + \hat{\beta}^i \frac{\partial \hat{Z}}{\partial \varphi^i} = 0$ and hence make possible to introduce the loop-corrected β -functions (see, e.g., [II-13]), $\hat{\beta} = \sum_{n=0}^{\infty} \beta_n$, $\beta_0 = \beta_{\text{local}}$.

In what follows we shall consider the leading sphere and disc corrections to $\hat{Z}$ in the open-closed Bose string theory (see [13] and [4,26]). The corresponding EA is (we set $D=26$ and $\phi = \text{const}$)

$$\hat{S} = \lambda_0 \int d^D y \sqrt{G} e^{-2\phi} (\alpha' R + \dots) + \lambda_1 \int d^D y \sqrt{G} e^{-\phi} (1 + \dots) \quad (46)$$

Assuming that $\frac{\partial \hat{S}}{\partial \varphi^i} = 0$ is equivalent to $\hat{\beta}^i = 0$ we can guess the disc corrections to the $\hat{\beta}$ -functions [13]

$$\begin{aligned} \hat{\beta} &= \beta_0 + \beta_1 + \dots, \quad \beta_1^G = -\frac{1}{4} \rho e^{\phi} G_{\mu\nu} + \dots, \quad \rho \equiv \frac{\lambda_1}{\lambda_0} \\ \beta_1^{\phi} &= -\frac{1}{16} \rho e^{\phi} (D+2) \end{aligned} \quad (47)$$

so that the full bare fields are (cf. (41))

$$\begin{aligned} G_{\mu\nu} &= G_{\mu\nu}^{(R)} - \alpha' R_{\mu\nu}^{(R)} \ln \epsilon + \frac{1}{4} \rho e^{\phi^{(R)}} G_{\mu\nu}^{(R)} \ln \epsilon + \dots, \\ \phi &= \phi^{(R)} + \frac{1}{2} \alpha' D^2 \phi^{(R)} \ln \epsilon + \frac{1}{16} \rho (D+2) e^{\phi^{(R)}} \ln \epsilon + \dots \end{aligned} \quad (48)$$

Hence in order for the renormalisability property to be true we should find that [26]

$$\begin{aligned} \hat{Z} &= \hat{Z}_0 + \hat{Z}_1 + \dots = \lambda_0 \int d^D y \sqrt{G} e^{-2\phi} (\alpha' R + \dots) \\ &+ \lambda_1 \int d^D y \sqrt{G} e^{-\phi} (1 + \alpha' \ln \epsilon R + \dots) \end{aligned} \quad (49)$$

Note that the coefficient of the $\ln \epsilon R$ -term in the string partition function on the disc should be two times larger than corresponding coefficient in the sigma model one (39),(42): $\frac{1}{2}$ of the infinity in Z is the local one present in (39) and $\frac{1}{2}$ is the new "modular" infinity special for the disc [26]. The local infinity is cancelled by substituting the local part of the counterterms (47) into $\sqrt{G}$ in the disc piece of (49) while the modular is cancelled by substituting the modular counterterms in (48) into the $\sqrt{G} e^{-2\phi} R$ in the sphere part of (49) [26].

To confirm this picture of the renormalisation and hence to resolve the paradox of ref. [13] we are thus to prove by explicit computation that the coefficient of the $\ln \epsilon R$ -infinity in the disc correction $\hat{Z}_1$ is precisely the one given in (49). It may seem strange that we expect additional "modular" infinities on the disc since its metric has no moduli. However, one should not assume that $\hat{Z}_1$ is simply equal to the sigma model partition function on the disc with a fixed 2-metric. $\hat{Z}$ extends the

string theory off shell, so the only guiding principle should be the correspondence with the usual expressions for the on-shell scattering amplitudes. Since there exists a number of representations for the graviton string amplitudes on the disc which are formally equivalent on-shell we should decide which one to choose for an off-shell definition of $\hat{Z}_1$. The representation which seems to be the only suitable one was used in [13] and is based on considering the disc as plane with a hole (of radius a and the center at w), integrating over a and w and dividing by the volume of the $SL(2, C)$ Mobius group of the plane. We thus suggest to define $\hat{Z}_1$ in terms of the partition function Z on the round 2-sphere with a hole of radius a , with the center at w . $Z(a, w)$ should be integrated over a and w . To be able to interpret $\hat{Z}$ as a generating functional for the string amplitudes we are also to "divide" by the Mobius volume. We expect that this can be done just as in the case of the sphere [7]: by dropping the power infinities and taking the derivative $\partial/\partial \ln \epsilon$. Thus we propose the following representation for the string partition function on the disc (the minus sign is simply for convenience)

$$\hat{Z}_1 = - \frac{\partial}{\partial \ln \epsilon} \left[\int_{\epsilon}^1 da \mu(a) \int d^2 w Z_1(a, w) \right] \quad (50)$$

Here a is the ratio of the radius of the hole and the radius of the sphere and ϵ is the same cutoff we used for the local infinities (in fact, the scale of ϵ does not matter since we are interested only in the $\ln \epsilon$ -terms). According to (39) we should have

$$Z_1 = \lambda_1 \int d^2 y \sqrt{G} e^{-\phi} \left(1 + \frac{1}{2} \alpha' \ln \epsilon R + \alpha' f(a) R + \dots \right) \quad (51)$$

where $\lambda_1 = \text{const}$ (we assume that the possible a -dependence in λ is absorbed into $\mu(a)$ in (50)). The function $f(a) = f_1 + f_2 a^2 + \dots$ should be regular at $a \rightarrow 0$ ($a=0$ corresponds to the case of the sphere). A natural guess for the measure $\mu(a)$ is $\mu \sim \frac{1}{a}$. Then (50) implies that

$$\begin{aligned} \hat{Z}_1 &= - \frac{\partial}{\partial \ln \epsilon} \int_{\epsilon}^1 \frac{da}{a} \int d^2 y \sqrt{G} e^{-\phi} \left(1 + \frac{1}{2} \alpha' \ln \epsilon R + \dots \right) = \\ &= \int d^2 y \sqrt{G} e^{-\phi} \left(1 + \alpha' \ln \epsilon R + \dots \right) \end{aligned} \quad (52)$$

This result is in agreement with (49). The mechanism of the doubling of the coefficient of the $O(\ln \epsilon)$ term in Z is thus very simple and universal: we "multiply" by $\int \frac{da}{a} = -\ln \epsilon + \dots$ and take $\partial/\partial \ln \epsilon$. This universal doubling is consistent with the observation [26] that the coefficients of the $\ln \epsilon R^2$ and $\ln \epsilon R^3$ terms in Z (k^4 and k^6 terms in the 3-graviton amplitude) should be 2 times the corresponding coefficients in Z ("local" and "modular" contributions turn out to be equal in the leading order on the disc).

We expect that a generalization of the representation (50) is true also for higher genus surfaces. Representing a surface as a sphere with a number of holes (identified in pairs if necessary) and integrating over the radii and positions of the

holes (cf. [I2]) we are also to "divide" by the Mobius volume (i.e. to take $1/2\ell_n \epsilon$) in order not to overcount. Then instead of (45) we should get

$$\hat{Z} = \frac{\partial}{\partial \ell_n \epsilon} \bar{Z}, \quad \bar{Z} = \sum_{n=0} \bar{Z}_n, \quad \bar{Z}_n = \int [da d\bar{w}]_n Z_n \quad (53)$$

$\bar{Z}_n$ should have natural factorization properties and both $\bar{Z}$ and $\hat{Z}$ should be renormalisable in the generalized sense.

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FOOTNOTES

1) A choice of a local measure is in principle equivalent to a choice of the bare values of the tachyon and dilaton couplings [6] (see also [14]).

2) We shall not go into details of the rigorous definition of the path integral measure (see [5] and also [15]).

3) Similar definition of the measure was used in [16]. While the zero mode term in K_ϵ was omitted in [16] it is of crucial importance in computing the partition function.

4) The result for J is different [19] if we do not separate the zero mode and use dimensional regularisation (thus effectively setting N in (18) equal to zero).

5) To obtain this result it is necessary to integrate by parts in (21) and to use that $\int d^2y \sqrt{G} R = \frac{1}{4} \int d^2y \sqrt{G} G^{\alpha\beta} G^{\mu\nu} G^{\sigma\tau} [\partial_\sigma G_{\mu\tau} \partial_\tau G_{\alpha\beta} - \partial_\sigma G_{\alpha\tau} \partial_\tau G_{\mu\beta} - 2 \partial_\sigma G_{\mu\tau} \partial_\alpha G_{\beta\tau} + 2 \partial_\sigma G_{\alpha\tau} \partial_\mu G_{\beta\tau}]$

6) We, of course, account for the (properly regularised) jacobian of the transformation $z \rightarrow \zeta(y, z)$.

7) Similar expression have already appeared in [3].

8) Instead of analyzing particular graviton scattering amplitudes we directly consider the renormalisation of their generating functional $\hat{Z}$. The computation of the $\alpha' R$ -term in (49) is of course related to the computation of the k^2 -term in the 3- graviton amplitude on the disc.

9) Being related by conformal transformations, they, in fact, would be equivalent if not for the infinities present in them (see the remarks in [25,27]).

10) This choice is apparently different from the one ($\mu \sim \frac{1}{a^3}$) in [13]. $\mu \sim \frac{1}{a}$ should probably be the effective measure which appears after we integrate over w .

This apparent discrepancy may be related to the "non-commutativity" of taking $\partial/\partial \ln \epsilon$ and computing $\int_\epsilon da \mu(a)$.

II) Taking $\partial/\partial \ln \epsilon$ for all genera is consistent with our interpretation of the $\ln \epsilon$ in (39) as "Möbius" infinity.

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