VERTEX OPERATOR CONSTRUCTION OF UNITARY CONFORMAL THEORIES

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ABSTRACT

We give a new, manifestly unitary construction of large classes of two-dimensional conformal theories, including the minimal series. This construction is based on the inclusion of vertex operators of naive dimension two in the conformal generators. Closure of the conformal algebra imposes restrictions on the corresponding momenta of the vertex operators which in particular are satisfied by the scaled root diagram of simply-laced groups. This provides a new direct connection between Lie groups and certain conformal theories.

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1. - INTRODUCTION

The discovery [1] that the structure of the Virasoro algebra could by itself determine the anomalous dimensions occurring in known critical statistical mechanics models has generated a tremendous burst of interest and activity in two-dimensional conformal field theories. In particular, the unitary theories [2,3] have been found to occur in discrete series, depending on the central charge c of the Virasoro algebra, together with some internal discrete symmetry group. Explicit unitary representations of these theories have been constructed in Ref. [4] using the Sugawara construction.

In this paper, we propose a systematic approach based on vertex operators of dimension two, the naive dimension of the Virasoro generators. This can be seen as a natural extension to conformal algebras of the Frenkel-Kac construction [5,6] of Kac-Moody algebras, which uses vertex operators of dimension one. Our strategy is to look for a suitable set of vertex operators of dimension two which, when supplemented with the usual bilinear forms in the bosonic creation-annihilation operators, closes under commutation to give the Virasoro algebras. One finds that the central charge naturally takes discrete values yielding many such conformal algebras. In this paper we concentrate on the simplest cases, in which the vertex operators are built on the root systems (rescaled by a factor $\sqrt{2}$, to lead to dimension two) of the simply-laced A,D,E algebras; even in this case, there are many possibilities. Among them we find the minimal series, and the $Z_N \times Z_N$ models. We find very naturally highest-weight states, built on the weight lattice, and the weight lattice shifted by half roots of the corresponding A,D,E algebras. We discuss in more detail the cases $c = \frac{1}{2}$ and $c = 7/10$, which are associated with the A_1 and A_2 root systems respectively.

For a given highest weight, there does not exist a unique corresponding state in our construction. Indeed, there is an infinite set of them. This arises from the fact that there are explicit vertex operators, which we call laddering operators, which completely commute with all the Virasoro generators; there are also, for a given ADE root system, several mutually commuting Virasoro algebras, each with its own set of laddering operators (among which, of course, are the other Virasoro algebras). The existence of laddering operators in a given representation of the conformal theory implies a non-irreducibility of the representation, and in particular an infinite degeneracy of the highest-weight states.

The results presented in this paper overlap with previous works. Particular cases of vertex operator representations of conformal algebras were first found by Eguchi and Higashijima [7], and developed by Bernard and Thierry-Mieg [8]. Independently, the same cases were encountered by Corrigan [9] in the twisted sector. Eguchi and Higashijima [10] also realized that they could be obtained by

bosonizing the coset construction of Goddard, Kent and Olive [4], originally based on the Sugawara construction. Hence, some of the results presented here have already been found in this framework. However, from past experience with the Frenkel-Kac construction, we can hope to gain new insights into the structure and classification of conformal theories by using more systematically their representation with vertex operators.

In Section 2, we present the simplest case, corresponding to the theory with $c = \frac{1}{2}$, built out of a single real bosonic field. Many general features can already be seen there. In Section 3, we give the general scheme for the construction of conformal theories with bosonic vertex operators, simultaneously finding the Virasoro operators and the central charges. This construction gives yet another connection between simply-laced algebras (via their root systems) and conformal theories. Although we find a very large set of theories, it is not at all clear that we exhaust all possibilities that vertex operators might provide. There are other constructions which we have not explored because they involve more complicated combinations of vertex operators; however, they might lead to interesting structures and deserve further study. This would go beyond the scope of this paper.

In Section 4, we discuss highest weights, highest-weight states and primary fields in the general case; in this connection, the weight lattice, as well as the weight lattice shifted by half roots, come into play.

Section 5 is devoted to a more thorough discussion of the theory with $c = 7/10$.

2. - THE $c = \frac{1}{2}$ THEORY IN TERMS OF BOSONIC OPERATORS

Several of the results of this section can be found in Ref. [7]. Introducing the standard notation

$$P(z) = \sum_{n=-\infty}^{+\infty} \alpha_n z^{-n} = i z \partial_z Q(z) \quad (2.1)$$

$$[\alpha_n, \alpha_m] = n \delta_{n,-m}, \quad n, m \in \mathbb{Z}, \quad (2.2)$$

one defines

$$L_n^\pm = \frac{1}{2} \frac{1}{2i\pi} \oint \frac{dz}{z} z^n : \left[\frac{1}{2} P^2(z) \pm z^2 \cos 2Q(z) \right] : . \quad (2.3)$$

It is straightforward to verify that

$$[L_n^\pm, L_m^\pm] = (n-m) L_{n+m}^\pm + \frac{1}{24} (n^3 - n) \delta_{n,-m} , \quad (2.4)$$

$$[L_n^\pm, L_m^\mp] = 0 . \quad (2.5)$$

Hence L_n^+ (or L_n^-) provide a unitary representation of the $c = \frac{1}{2}$ conformal algebra in terms of one bosonic field. They are both constructed with P^2 and $\cos 2Q$, which are (up to c numbers) conformal operators of (naive) dimension two under the (naive) algebra

$$L_n^{\text{naive}} = L_n^+ + L_n^- = \frac{1}{4i\pi} \oint \frac{dz}{z} z^n P^2(z) . \quad (2.6)$$

These conformal operators act on states such that $\cos 2Q(z)$ is single-valued. Hence, the eigenvalue of the zero mode α_0 should be integer or half-integer.

The highest weights of the $c = \frac{1}{2}$ theory are known to be 0, $\frac{1}{2}$ and $1/16$. The highest weight zero can be obtained by simply taking the vacuum of the oscillators $\alpha_n > 0$ at $\alpha_0 = 0$, a state which we note by $|0,0\rangle$.

It is straightforward to verify that the highest-weight state corresponding to the weight $\frac{1}{2}$ can be obtained by taking

$$|\frac{1}{2}\rangle_+ = |0, 1\rangle + |0, -1\rangle , \quad (2.7)$$

for the L_n^+ algebra, and

$$|\frac{1}{2}\rangle_- = |0, 1\rangle - |0, -1\rangle , \quad (2.8)$$

for the L_n^- algebra.

The primary fields corresponding to the above highest-weight states are $\cos Q(z)$ and $\sin Q(z)$ for Eqs. (2.7) and (2.8) respectively. This corresponds to the commutation relations

$$[L_n^+, \cos Q(z)] = z^{-n} [z \partial_z + \frac{n+1}{2}] \cos Q(z), \quad (2.9)$$

$$[L_n^-, \sin Q(z)] = z^{-n} [z \partial_z + \frac{n+1}{2}] \sin Q(z). \quad (2.10)$$

Furthermore, one finds

$$[L_n^+, \sin Q(z)] = [L_n^-, \cos Q(z)] = 0. \quad (2.11)$$

Hence, $\cos Q(z)$ and $\sin Q(z)$ are laddering operators for L_n^- and L_n^+ respectively. This means, acting on any highest-weight state, that they yield another highest-weight state with the same weight. For example, acting on $|0,0\rangle$ with $\cos Q(z)$, we obtain in the limit $z \rightarrow 0$ the state $|\frac{1}{2}\rangle_+$, and indeed, one easily verifies that

$$L_{n \gg 0}^- |\frac{1}{2}\rangle_+ = 0. \quad (2.12)$$

Highest-weight states with weight $1/16$ are $|0, \alpha_0 = \pm \frac{1}{2}\rangle$ for both L_n^+ and L_n^- . Others can of course be found through the action of the laddering operators. As for the corresponding primary field, it has no simple closed expression analogous to $\sin Q(z)$. We note that the simple primary fields corresponding to weights 0 and $\frac{1}{2}$ are also primary fields for L_n^{naive} , with the same weights. We shall find the same phenomenon occurring for the supersymmetry generator, with dimension $3/2$, in the conformal theory with $c = 7/10$, in Section 5.

In the next section, we shall also consider the Z_2 twisted sector, corresponding to a $P(z)$ with half-integer modes. In this sector, the vacuum has weight $1/16$.

3. - CONSTRUCTION OF VIRASORO GENERATORS

We consider the generators

$$\begin{aligned} L_n &= \frac{1}{2i\pi} \oint \frac{dz}{z} z^{-n} \left[\sum_i \frac{1}{2} A_i P_i^2(z) + z^{\frac{1}{2}e_r^2} \sum_r B_r \cos e_r Q(z) \right]: \\ &\equiv \frac{1}{2i\pi} \oint \frac{dz}{z} z^{n+2} l(z) \end{aligned} \quad (3.1)$$

where $\pm e_r$ are vectors in an r -dimensional space. Clearly, $e_r \rightarrow -e_r$ leaves L_n inert

and we sum only over e_r and not $-e_r$. The problem is then to find restrictions on A_1 and B_r such that

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n,-m}, \quad (3.2)$$

where c are the possible central charges.

For reasons that will become apparent as we carry out the closure, we will take

$$\begin{aligned} e_r \cdot e_s &= 0, \pm 2 & \text{for } r \neq s, \\ e_r \cdot e_r &= 4 & \forall r. \end{aligned} \quad (3.3)$$

We evaluate the commutator of the L_n 's using standard techniques in conformal field theory; namely, we write

$$\begin{aligned} [L_n, L_m] &= -\frac{1}{4\pi^2} \oint_{|z|=b} \frac{dz}{z} z^{n+2} \oint_{|\zeta|=a} \frac{d\zeta}{\zeta} \zeta^{m+2} l(z) l(\zeta) \\ &\quad + \frac{1}{4\pi^2} \oint_{|z|=a} \frac{dz}{z} z^{n+2} \oint_{|\zeta|=b} \frac{d\zeta}{\zeta} \zeta^{m+2} l(\zeta) l(z), \end{aligned} \quad (3.4)$$

where $b > a$ in order that the expressions be defined, and we use the operator product expansions for $l(z)l(\zeta)$.

What we will require are the relations [6]

$$\begin{aligned} : \exp i k \alpha(z) : &: \exp i k' \alpha(z') : \\ &= (z-z')^{kk'} : \exp [i k \alpha(z) + i k' \alpha(z')] : , |z| > |z'|, \end{aligned} \quad (3.5)$$

$$\frac{P_i(z)}{z} \frac{P_j(z')}{z'} = \frac{\delta_{ij}}{(z-z')^2} + : \frac{P_i(z)}{z} \frac{P_j(z')}{z'} : , \quad (3.6)$$

$$\frac{P_i(z)}{z} : \exp i k \alpha(z') : = : \frac{P_i(z)}{z} e^{i k \alpha(z')} : + \frac{k_i}{z-z'} : e^{i k \alpha(z')} : , \quad (3.7)$$

$$: \exp i k Q(z) : \frac{P_i(z)}{z'} = : \exp i k Q(z) : \frac{P_i(z')}{z'} - \frac{k_i}{z-z'} : \exp i k Q(z) : , \quad (3.8)$$

which are only valid provided $|z| > |z'|$. The normal-ordered terms have no singularities as $z \rightarrow z'$ and so cancel from the first and second terms in Eq. (3.4). Left are the terms with poles at $z = z'$ which contribute to the integral [6]. Equations (3.5)-(3.8) can be verified by simply pushing the creation and destruction operators past each other in the usual way.

As an example, let us evaluate the $B_r B_s$ terms in the $L_n L_m$ commutator:

$$\begin{aligned} & \sum_{r,s} B_r B_s \left[\oint_{|z|=b} \frac{dz}{z} z^{n+2} \oint_{|\zeta|=a} \frac{d\zeta}{\zeta} \zeta^{m+2} : \exp e_r Q(z) : : \exp e_s Q(\zeta) : \right. \\ & \quad \left. - \oint_{|z|=a} \frac{dz}{z} z^{n+2} \oint_{|\zeta|=b} \frac{d\zeta}{\zeta} \zeta^{m+2} : \exp e_s Q(\zeta) : : \exp e_r Q(z) : \right] \\ &= \frac{1}{2} \sum_{r,s} B_r B_s \oint_{z=0} \frac{dz}{z} z^{n+2} \oint_{\zeta=z} \frac{d\zeta}{\zeta} \zeta^{m+2} \\ & \quad \times \left\{ (z-\zeta)^{e_r e_s} : \exp [e_r Q(z) + e_s Q(\zeta)] : \right. \\ & \quad \left. + (z-\zeta)^{-e_r e_s} : \exp [e_r Q(z) - e_s Q(\zeta)] : \right\} . \end{aligned} \quad (3.9)$$

We may divide the sum over r and s into terms that are diagonal (i.e., B_r^2) and off-diagonal. The diagonal terms are of the form

$$\begin{aligned} & \frac{1}{2} \sum_r B_r^2 \oint_{z=0} \frac{dz}{z} z^{n+2} \oint_{\zeta=z} \frac{d\zeta}{\zeta} \zeta^{m+2} \\ & \quad \times \frac{1}{(z-\zeta)^4} : \exp e_r [Q(z) - Q(\zeta)] : \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_n B_n^2 \oint_{\gamma=0} \frac{d\gamma}{\gamma} \gamma^{n+2} \oint_{\xi=\gamma} \frac{d\xi}{\xi} \xi^{n+2} \frac{1}{(\gamma-\xi)^4} \\
 &\quad \times \left[1 + \frac{1}{2} \left(\frac{\gamma-\xi}{\gamma+\xi} \right)^2 : [e_2 P(\frac{\gamma+\xi}{2})]^2 : + O(\gamma-\xi)^4 \right] \\
 &= (n-m) i\pi \sum_n \frac{B_n^2}{4} \oint_{\gamma=0} \frac{d\gamma}{\gamma} \gamma^{n+m} : [e_2 \cdot P(\gamma)]^2 : \\
 &\quad - 4\pi^2 \sum_n B_n^2 \delta_{n,-m} \frac{n(n^2-1)}{12} . \tag{3.10}
 \end{aligned}$$

On the other hand, for the off-diagonal terms we have

$$\begin{aligned}
 &-\frac{1}{4\pi^2} \sum_{r+s} \frac{1}{2} B_r B_s \oint_{\gamma=0} \frac{d\gamma}{\gamma} \gamma^{n+2} \oint_{\xi=\gamma} \frac{d\xi}{\xi} \xi^{m+2} \\
 &\quad \times \left\{ \delta(e_r e_s - 2) \cos[e_r Q(\gamma) - e_s Q(\xi)] \right. \\
 &\quad \left. + (e_r \rightarrow e_r, e_s \rightarrow -e_s) \right\} \\
 &= \frac{n-m}{2i\pi} \sum_{r+s} \frac{B_r B_s}{2} \oint_{\gamma=0} \frac{d\gamma}{\gamma} \gamma^{n+m+2} \left[\delta(e_r e_s - 2) \cos(e_r - e_s) Q(\gamma) \right. \\
 &\quad \left. + \delta(e_r e_s + 2) \cos(e_r + e_s) Q(\gamma) \right] . \tag{3.11}
 \end{aligned}$$

The reason for the restriction of Eq. (3.3) is that it allows the algebra to close. One could contemplate other values for the scalar products, in particular odd integers, but we have not examined if this works in general.

Examining Eq. (3.11), we find we have generated the vectors $e_r \pm e_s$ [or $-(e_r \pm e_s)$]. Hence we require that

$$\begin{aligned}
 &\pm (e_r - e_s) \quad \text{if } e_r \cdot e_s = +2 \\
 &\text{and } \pm (e_r + e_s) \quad \text{if } e_r \cdot e_s = -2 \tag{3.12}
 \end{aligned}$$

be included in the original choice of vectors $\pm e_r$.

One way of arriving at such a set of vectors is to take the roots of a simply-laced group which normally have length squared 2 and scale them by a factor of $\sqrt{2}$ to give them length squared 4. An interesting question that we will not answer is what the most general set of vectors e_r is such that the L_n algebra closes, and

what structures correspond to these sets of vectors.

Carrying out a similar manoeuvre on the other terms, we find that

$$[L_n, L_m] = (n-m) L_{n+m}^T + \frac{c^T}{12} (n^3-n) \delta_{n,-m}, \quad (3.13)$$

where

$$L_p^T = \frac{1}{24\pi} \oint \frac{dz}{z} z^p \left[\frac{1}{2} \sum_{i,j} A_{ij} :P_i(z) P_j(z): + \sum_{\alpha} B_{\alpha}^T z^{\alpha} e_{\alpha} Q(z) : \right] \quad (3.14)$$

where

$$A_{ij} = \delta_{ij} (A_i)^2 + \sum_{\alpha} \frac{1}{4} B_{\alpha}^2 e_{\alpha}^i e_{\alpha}^j \quad (3.15)$$

$$B_{\alpha}^T = \frac{1}{2} \sum_i A_i B_{\alpha} (e_{\alpha}^i)^2 + \frac{1}{2} \sum_{\alpha, \beta} B_{\alpha} B_{\beta} \left[\delta(e_{\alpha} e_{\beta} - 2) \delta(e_{\alpha} + (e_{\beta} - e_{\alpha})) + \delta(e_{\alpha} e_{\beta} + 2) \delta(e_{\alpha} - (e_{\beta} + e_{\alpha})) \right] \quad (3.16)$$

$$c^T = \sum_i A_i^2 + \sum_{\alpha} B_{\alpha}^2. \quad (3.17)$$

We conclude that the L_n 's form a closed algebra if the cross terms in $P_i P_j$ vanish and we recover A_1 for the diagonal terms

$$\delta_{ij} A_i^2 + \frac{1}{4} \sum_{\alpha} B_{\alpha}^2 e_{\alpha}^i e_{\alpha}^j = \delta_{ij} A_i. \quad (3.18)$$

One could have started with an L_n which has arbitrary cross-terms in $P_i P_j$, in which case no further restriction would be necessary. Summing on i in Eq. (3.18), we find that we may rewrite

$$c = \sum_i A_i. \quad (3.19)$$

Let us get a feel for this system of equations. Let us take some root systems, beginning with $SO(2N)$, and find the possible values of c .

SO(2N)

The scaled roots of SO(2N) are

$$\pm e_i \pm e_j, \quad i \neq j, \quad (3.20)$$

where e_i are N vectors in an N-dimensional space which have $e_i \cdot e_j = \delta_{ij}$. We may take their components to be $(e_i)^k = \sqrt{2} \delta_i^k$. For the sum in L_n , we take

$$e_i + e_j \quad i < j \quad \text{to be labelled by } B_{ij}$$

and

$$e_i - e_j \quad i < j \quad \text{to be labelled by } C_{ij}$$

Equation (3.18) then becomes

$$A_i \delta_{ij} P_i^2 = A_i^2 \delta_{ij} P_i^2 + \frac{1}{2} \sum_{j \neq i} [B_{ij}^2 (P_i + P_j)^2 + C_{ij}^2 (P_i - P_j)^2]. \quad (3.21)$$

There are no cross terms if

$$B_{ij} = \pm C_{ij}, \quad \forall i < j, \quad (3.22)$$

and we are left with

$$A_i = A_i^2 + \sum_{j > i} B_{ij}^2 + \sum_{j < i} B_{ji}^2. \quad (3.23)$$

Equation (3.16), on the other hand, becomes

$$B_{ij} = B_{ij} (A_i + A_j) + \frac{1}{2} \sum_{i < k < j} (B_{ik} C_{kj} + C_{ik} B_{kj}) \\ + \frac{1}{2} \sum_{k < i} (C_{ki} B_{ij} + B_{ki} C_{ij}) + \frac{1}{2} \sum_{k > j} (B_{ik} C_{jk} + C_{ik} B_{jk}), \quad (3.24)$$

$$C_{ij} = C_{ij} (A_i + A_j) + \frac{1}{2} \sum_{i < k < j} (B_{ik} B_{kj} + C_{ik} C_{kj}) \\ + \frac{1}{2} \sum_{k < i} (B_{ki} B_{ij} + C_{ki} C_{ij}) + \frac{1}{2} \sum_{k > j} (B_{ik} B_{jk} + C_{ik} C_{jk}) \quad (3.25)$$

When $B_{ij} = \pm C_{ij}$, Eqs. (3.24) and (3.25) coincide.

To solve these equations, we introduce the symmetric matrix

$$T_{ij} = \pm B_{ij} + \delta_{ij} A_i, \quad (3.26)$$

where $B_{ij} = B_{ji}$, $i \neq j$ and $B_{ii} = 0$. The upper and lower signs correspond to the signs of Eq. (3.22). Equations (3.23)-(3.25) can then be written as the matrix equation

$$T^2 = T. \quad (3.27)$$

As T is a symmetric matrix, we may diagonalize it by an orthogonal matrix O , $T = OTO^{-1}$:

$$\hat{T}_{ij} = \delta_{ij} \lambda_i, \quad (3.28)$$

and Eq. (3.27) becomes

$$\lambda_i (\lambda_i - 1) = 0, \quad (3.29)$$

and hence $\lambda_i = 0$ or 1 . The central charge c

$$c = \sum_i A_i = \text{tr } T = \sum_i \lambda_i. \quad (3.30)$$

Hence, it can take the values

$$c = 0, 1, 2, \dots, N. \quad (3.31)$$

SU(N)

Let us now consider the case of $SU(N)$ where the results are much more interesting. The scaled roots are of the form

$$e_{ij} = v^i - v^j$$

where v^i are N vectors in an $(N-1)$ -dimensional space which satisfy $\sum_i v^i = 0$. We will use the set

$$\begin{aligned}
 v^1 &= \left(1, \frac{\sqrt{2}}{\sqrt{2 \cdot 3}}, \frac{\sqrt{2}}{\sqrt{3 \cdot 4}}, \dots, \frac{\sqrt{2}}{\sqrt{m(m+1)}}, \dots, \frac{\sqrt{2}}{\sqrt{N(N-1)}} \right) \\
 v^2 &= \left(-1, \frac{\sqrt{2}}{\sqrt{2 \cdot 3}}, \frac{\sqrt{2}}{\sqrt{3 \cdot 4}}, \dots, \frac{\sqrt{2}}{\sqrt{m(m+1)}}, \dots, \frac{\sqrt{2}}{\sqrt{N(N-1)}} \right) \\
 v^3 &= \left(0, -\frac{2}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3 \cdot 4}}, \dots, \frac{\sqrt{2}}{\sqrt{m(m+1)}}, \dots, \frac{\sqrt{2}}{\sqrt{N(N-1)}} \right) \\
 &\vdots \\
 v^m &= \left(0, \dots, 0, -\frac{(m-1)\sqrt{2}}{\sqrt{(m-1)m}}, \dots, \frac{\sqrt{2}}{\sqrt{N(N-1)}} \right) \\
 &\vdots \\
 v^N &= \left(0, \dots, 0, \dots, -\frac{(N-1)\sqrt{2}}{\sqrt{(N-1)N}} \right),
 \end{aligned}$$

(3.32)

the first non-zero component in v^m being the $(m-1)^{\text{th}}$ component. These vectors have the properties

$$v^i v^j = 2 \delta^{ij} - \frac{2}{N},$$

$$\frac{1}{2} \sum_{i=1}^N v_m^i v_n^i = \delta_{m,n}$$

We take in the sum in L_n , $e_{ij} = v_i - v_j$, $i < j$ and label them by B_{ij} . Equations (3.18) and (3.16) become

$$A_k \delta_{kl} = A_k^2 \delta_{kl} + \frac{1}{4} \sum_{i < j} (v_i^k - v_j^k)(v_i^l - v_j^l) B_{ij}^2, \quad (3.33)$$

$$\begin{aligned}
 B_{ij} &= \sum_{m=1}^{N-1} \frac{1}{2} B_{ij} (v_i^m - v_j^m)^2 A_m \\
 &+ \frac{1}{2} \sum_{i < k < j} B_{ik} B_{kj} + \frac{1}{2} \sum_{j < k < i} B_{jk} B_{ki} + \frac{1}{2} \sum_{k < i} B_{ki} B_{kj}, \\
 &\quad i < j.
 \end{aligned} \quad (3.34)$$

As we shall see, there are many solutions to these equations and we begin with some examples. Suppose $B_{ij} = B$, $\forall_{i,j}$ and $A_m = A \forall_m$. Then the above equations become

$$B = 2AB + \frac{1}{2}(N-2)B^2, \quad A = A^2 + \frac{1}{2}NB^2, \quad (3.35)$$

which have the solutions

$$A = B = \frac{2}{N+2}, \quad (3.36)$$

or

$$A = B+1 = \frac{N}{N+2}, \quad (3.37)$$

with corresponding values of c

$$c = 2 \frac{N-1}{N+2}, \quad \text{or } c = N \frac{N-1}{N+2}. \quad (3.38)$$

The first possibility was previously found in Refs. [3] and [7] using parafermions with a $Z_N \times Z_N$ symmetry [3]. In fact, for solutions of Eq. (3.36) and (3.37), the conformal generators have a $Z_N \times Z_N$ symmetry. This is a consequence of the fact that this is the symmetry of the weights of $SU(N)$.

Next, consider setting

$$B_{ij} = b, \quad i < j < N, \quad B_{iN} = b_N, \quad i < N, \quad (3.39)$$

and

$$A_i = \begin{cases} a, & i = 1, \dots, N-2, \\ a_{N-1}, & i = N-1. \end{cases} \quad (3.40)$$

Equations (3.18) and (3.16) become

$$a = a^2 + \frac{1}{2}(N-1)b^2 + \frac{1}{2}b_N^2, \quad (3.41)$$

$$a_{N-1} = a_{N-1}^2 + \frac{1}{2}N + b_N^2, \quad (3.42)$$

and

$$b = 2ab + \frac{1}{2}(N-3)b^2 + \frac{1}{2}b_N^2,$$

$$b_N = 2ab_N + \frac{N}{N-1}(a_{N-1}-a)b_N + \frac{1}{2}bb_N(N-2).$$

(3.43)

Subtracting (3.41) from (3.43), we find that either $a = b$ or $a = 1+b$. For all four solutions, $b_N = \pm 2/(N+2)$. Two of the solutions are those of Eqs. (3.36) and (3.37), while the remaining two solutions with their central charges are:

Either

Or

$$a = 1+b = 1 - \frac{2}{(N+2)(N+1)},$$

$$a = b = \frac{2}{(N+2)(N+1)},$$

$$a_{N-1} = \frac{2}{N+2},$$

$$a_{N-1} = \frac{N}{N+2},$$

$$C = (N-2) - \frac{6}{(N+2)(N+1)}.$$

$$C = 1 - \frac{6}{(N+2)(N+1)}.$$

(3.44)

In the latter solution, we recognize the well-known minimal unitary series of BPZ [1,2]. The above construction not only provides an explicit unitary representation for the conformal generators, but also gives a one-to-one correspondence between the series and the groups $SU(N)$.

Before finding the general solution for $SU(N)$, let us give the most general solution for $SU(3)$. The condition for no off-diagonal $P_i P_j$ terms is $B_{13} = \pm B_{23}$. As a result, Eqs. (3.33) and (3.34) become

$$A_1 = A_1^2 + B_{12}^2 + \frac{1}{2} B_{13}^2,$$

$$A_2 = A_2^2 + \frac{3}{2} B_{13}^2,$$

$$B_{12} = 2 A_1 B_{12} + \frac{1}{2} B_{13} B_{23} ,$$

$$B_{13} = \frac{1}{2} B_{13} (A_1 + 3 A_2) + \frac{1}{2} B_{12} B_{23} ,$$

$$B_{23} = \frac{1}{2} B_{23} (A_1 + 3 A_2) + \frac{1}{2} B_{12} B_{23} .$$

(3.45)

Given any solution $(A_1, A_2, B_{12}, B_{13}, B_{23})$, then we also have a solution for the same A_1 and A_2 but with the values $-B_{12}, B_{13}, -B_{23}$ or $B_{12}, -B_{13}, -B_{23}$ or $-B_{12}, -B_{13}, B_{23}$. In other words, if we change the sign of two of the B's, we also have a solution. Solutions with the same values of A_1 and A_2 have the same c . Analysis shows that there are 16 solutions, only four of which are not related in the above sense. The four non-related solutions can be taken to be

$$\left(\frac{1}{10}, \frac{3}{5}, -\frac{1}{10}, \frac{2}{5}, -\frac{2}{5} \right) ,$$

with $c = 7/10$,

$$\left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5} \right) ,$$

with $c = 4/5$,

$$\left(\frac{3}{5}, \frac{3}{5}, -\frac{2}{5}, \frac{2}{5}, \frac{2}{5} \right) ,$$

with $c = 6/5$,

$$\left(\frac{9}{10}, \frac{2}{5}, -\frac{1}{10}, \frac{2}{5}, \frac{2}{5} \right) ,$$

with $c = 13/10$.

(3.46)

The first step in finding the most general solution for $SU(N)$ is to ensure the absence of off-diagonal $P_i P_j$, $i \neq j$ terms. That is,

$$\frac{1}{4} \sum_{i < j} B_{ij}^2 (\nu_i^k - \nu_j^k) (\nu_i^l - \nu_j^l) = \delta^{kl} c_k . \quad (3.47)$$

Taking the scalar product with $(\nu_s)^l$ and defining $B_{ij} = B_{ji}$ with $B_{ii} = 0$, the

equation becomes

$$\frac{1}{2} \sum_{j=1}^N B_{sj}^2 (v_s^k - v_j^k) = v_s^k c_k. \quad (3.48)$$

Analysing this equation and using the explicit form for the (v_j) 's of Eq. (3.32), we find it implies

$$B_{sj}^2 = b_j^2 \quad \forall s = 1, \dots, j-1. \quad (3.49)$$

In fact, one can show that this condition is sufficient for Eq. (3.47) to hold. By taking appropriate values of s , j and k in Eq. (3.48), we find that

$$2c_{s-1} = s b_s^2 + \sum_{j=s+1}^N b_j^2. \quad (3.50)$$

Consider first the case $B_{sj} = b_j$ $s = 1, \dots, j-1$. Equations (3.33) and (3.34) become

$$\begin{aligned} 2b_j = 2b_j \left[\frac{i-1}{i} A_{i-1} + \sum_{k=i}^{j-2} \frac{A_k}{k(k+1)} + \frac{j}{j-1} A_{j-1} \right] \\ + b_j \sum_{k=i+1}^{j-1} b_k + \sum_{k=j+1}^N b_k^2 + (i-1) b_i b_j, \quad i < j, \end{aligned} \quad (3.51)$$

$$A_i = A_i^2 + c_i, \quad (3.52)$$

Subtracting Eq. (3.51) from itself for two values of i related by one, we find

$$2(A_{i-1} - A_i) + i(b_i - b_{i+1}) = 0, \quad i = 2, \dots, N-2. \quad (3.53)$$

For $j = N$, Eq. (3.51) becomes

$$(N-1) = (N-2) A_{N-2} + N A_{N-1} + \frac{1}{2} (N-1)(N-2) b_{N-1}, \quad (3.54)$$

provided $b_N \neq 0$.

Iterating Eq. (3.53), we have

$$A_i = A_{j-1} + \frac{1}{2}(j-1)b_j - \frac{1}{2} \sum_{k=i+2}^{j-1} b_k - \frac{1}{2}(i+1)b_{i+1}, \quad i < j-1. \quad (3.55)$$

Using this result, we now eliminate in Eq. (3.51) all A_i 's in terms of A_{j-1} 's to obtain the result

$$2b_j = 4b_j A_{j-1} + (j-2)b_j^2 + \sum_{k=j+1}^N b_k^2, \quad j=2, \dots, N-1. \quad (3.56)$$

Subtracting (3.52) from (3.56), we find that either of the two following possibilities must hold.

$$(a) \quad b_j = A_{j-1},$$

or

$$(b) \quad b_j - A_{j-1} = -1, \quad j=2, \dots, N-1. \quad (3.57)$$

It remains to find a restriction on b_N and A_{N-1} . Using Eq. (3.57) we have either

$$(a) \quad A_{N-2} = b_{N-1},$$

or

$$(b) \quad A_{N-2} = b_{N-1} + 1. \quad (3.58)$$

Substituting (a) or (b) into Eq. (3.54), we can eliminate A_{N-2} to find a relation between b_{N-1} and A_{N-1} . Repeating the exercise with Eq. (3.52) for $i = N-2$, we find a relation between b_{N-1} and b_N . On the other hand, Eq. (3.52) for $i = N-1$ provides a relation between A_{N-1} and b_N . Eliminating b_{N-1} , we find for both possibilities (a) and (b) that

$$b_N^2 = \frac{4}{N+2}, \quad (3.59)$$

and

$$A_{N-1} = \frac{N}{N+2}, \quad \text{or} \quad A_{N-1} = \frac{2}{N+2}. \quad (3.60)$$

We give b_N^2 , as it is only b_N^2 that occurs in the equations. Corresponding to the two possibilities in Eq. (3.60), we have for possibility (a):

$$A_{N-2} = \ell_{N-1} = \frac{2}{(N+2)(N+1)} , \quad \approx \quad \frac{2}{N+2} , \quad (3.61)$$

while for possibility (b)

$$\ell_{N-1} = A_{N-2} - 1 = -\frac{2}{N+2} , \quad \approx \quad -\frac{2}{(N+1)(N+2)} . \quad (3.62)$$

The most general solution has now been found. Starting from the four possibilities in Eqs. (3.61) and (3.62), we may find lower A_i 's and b_i 's by using possibility (a) or (b) in Eq. (3.57) at every step in conjunction with Eq. (3.53). For example, suppose we have arrived at an A_j and b_{j+1} . This could have occurred either by using possibility (a), i.e., $A_j = b_{j+1}$, denoted in Fig. 1 by a straight arrow, or by possibility (b), i.e., $A_j = b_{j+1} + 1$, denoted in Fig. 1 by a wiggly arrow. In the former case,

$$A_j = \ell_j + \frac{2}{j} (A_{j-1} - A_j) , \quad (3.63)$$

or in the latter case, using Eq. (3.53),

$$(j+2)A_j - j = j\ell_j + 2A_{j-1} . \quad (3.64)$$

To descend to the next step, we can use possibility (a) or (b) for each of the above two possibilities. In the $A_j = b_{j+1}$ case, we have

$$\text{for (a)} \quad A_j = A_{j-1} , \quad \ell_j = \ell_{j+1} , \quad (3.65)$$

$$\text{for (b)} \quad A_{j-1} = A_j + \frac{j}{j+2} , \quad \ell_j = \ell_{j+1} - \frac{2}{j+2} , \quad (3.66)$$

while in the case $A_j = b_{j+1} + 1$, we have

$$\text{for (a)} \quad A_{j-1} = A_j - \frac{j}{j+2} , \quad \ell_j = \ell_{j+1} + \frac{2}{j+2} , \quad (3.67)$$

and for (b)

$$A_j = A_{j-1} , \quad \ell_j = \ell_{j+1} . \quad (3.68)$$

We realize that $A_j = A_{j-1}$, $b_j = b_{j+1}$ or $A_{j-1} = A_j \pm e_j$; $b_j = b_{j+1} \mp (1-e_j)$ where

$e_j \equiv j/(j+2)$. This process is illustrated in Fig. 1, where the change in A_j is indicated, the corresponding change in b_j then being given as above.

The values of A_{N-1} , A_{N-2} and b_N^2 , b_{N-1} of Eqs. (3.59)-(3.62) can also be arrived at by the above procedure. We imagine that there is a step even before A_{N-1} in which a mythical A_N can have value 0 or 1. From 0 we must use only possibility (b) while from 1 we must use only possibility (a) with $e_N = N/(N+2)$. The values of A_{N-2} , b_{N-1} are then arrived at by the above process - see Fig. 2.

We may use the above rules to arrive at all possible values of A_j , starting from either $A_{N-1} = e_N$ or $A_{N-1} = 1 - e_N$. One realizes that the above rules mean that we descend a step by adding $\pm e_j$ or 0. However, we must do this in such a way that the signs of non-zero contributions alternate. Hence, starting from $A_{N-1} = e_N$, we find that the possible values for A_j are

$$A_j = \sum_{n=0}^m (-1)^n e_{N-(p_1+\dots+p_n)} , \quad (3.69)$$

where $p_1, \dots, p_m$ are any set of positive integers such that

$$\sum_{n=1}^m p_n \leq N-j . \quad (3.70)$$

Starting from $A_{N-1} = 1 - e_N$, we find one minus all the above possibilities. This means that we have, in all, 2^{N-1} possible solutions.

The central charge is given by $c = \sum_{i=1}^{N-1} A_i$ and one easily sees that the above solutions lead to the possible values of c given by

$$\begin{aligned} c &= c_N - c_{N-p_1} + c_{N-p_1-p_2} - c_{N-p_1-p_2-p_3} + \dots \\ &= \sum_{n=0}^m (-1)^n c_{N-(p_1+\dots+p_n)} , \end{aligned} \quad (3.71)$$

where $(p_1, \dots, p_m)$ are positive integers such that

$$\sum_{i=1}^m p_i \leq N-2 , \quad (3.72)$$

and

$$c_q = q \frac{q-1}{q+2} , \quad (3.73)$$

as well as $(N-1)$ minus all the above values of c .

The most general solution has a simple interpretation. Looking at Eq. (3.37), we recognize e_j as the value of A for $SU(j)$ when all the A 's are equal. If we denote this conformal operator by $\mathcal{L}^{(j)}$, then the most general conformal operator which has a closing algebra for $SU(N)$ is of the form

$$\mathcal{L}_q = \sum_{n=0}^m (-1)^n \mathcal{L}_q^{(N-(p_1+\dots+p_n))}, \quad (3.74)$$

where $(p_1, \dots, p_m)$ are positive integers which obey Eq. (3.72), or else it is of the form

$$L_q^{\text{naive}} - \mathcal{L}_q, \quad (3.75)$$

where $\mathcal{L}_q$ is any of the above and L_q^{naive} is the conformal generator with $A_i = 1 \forall i$, $B_{ij} = 0$.

We recover the central charges of Eq. (3.71) provided

$$[\mathcal{L}_n^{(M)}, \mathcal{L}_m^{(M)} - \mathcal{L}_m^{(N)}] = 0, \quad \forall n, m, \quad M < N, \quad (3.76)$$

which one may explicitly verify to be true. This is not surprising, since we know that for some special cases we are dealing with a bosonized form of the GKO construction.

The reader will readily find the four possible central charges of Eq. (3.46) for $SU(3)$, and find that the possible values for $SU(4)$ are

$$\frac{13}{10}, \quad \frac{4}{5}, \quad \frac{3}{2} \quad \text{and} \quad 2, \quad (3.77)$$

as well as 3 minus the above. Looking at $SU(5)$, etc., one realizes that one only finds one value of $c < 1$. The exception is $SU(3)$, where there are two values of $c < 1$.

In deriving the above results, one took $B_{ij} = b_j \quad i < j$; however, one can take $B_{ij} = \varepsilon_{ij} b_j \quad i < j$ with $\varepsilon_{ij} = \pm 1$. It is straightforward to find what solutions can be generated in this way by examining the equations upon making this change.

Rather than find a closed Virasoro algebra based on ordinary bosonic strings, as above, we can use twisted strings. We now consider Z_2 twisted strings [9] which have a conformal field of weight 0 given by

$$R^i(z) = i \sum_{r \in Z + \frac{1}{2}} \frac{c_r^i}{2} z^{-2}, \quad (3.78)$$

and one of weight 1 given by

$$S^i(z) = i z R^i(z). \quad (3.79)$$

The oscillators c_r^i , $r \in Z + \frac{1}{2}$ obey the relations $[c_r^i, c_s^j] = r \delta_{r+s,0} \delta^{ij}$. The usual Virasoro operators are

$$L_n = \frac{1}{2} \frac{1}{2\pi i} \oint \frac{dz}{z} z^{2n} : \sum_i S_i^2(z^2) : - \frac{D}{16} \delta_{n,0}, \quad (3.80)$$

where D is the number of components of c_r^i (i.e., $i = 1, \dots, D$). They obey the algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{D}{12} (n^3 - n) \delta_{n+m,0}. \quad (3.81)$$

The last term in Eq. (3.80) is required in order for the central term to come out as above. This is most easily seen by evaluating $\langle [L_1, L_{-1}] \rangle = \langle 2L_0 \rangle$.

We now consider the conformal operators

$$\begin{aligned} L_n = & \frac{1}{2\pi i} \oint \frac{dz}{z} z^{2n} \left[\sum_i \frac{1}{2} A_i : S_i^2(z^2) : \right. \\ & \left. + \frac{1}{16} \sum_r B_r z^2 : \exp i e_r R(z^2) : \right] \\ & + \frac{1}{2} \delta_{n,0} \end{aligned} \quad (3.82)$$

This expression is invariant under $e_r \rightarrow -e_r$, so as before we sum only over e_r with a given sign. Demanding that these operators close on the normal conformal algebra of Eq. (3.2) proceeds rather similarly to the untwisted case, and we will only give the results. Use is made of the formula

$$\begin{aligned} & : \exp i k R(y^2) : : \exp i q R(\zeta^2) : \\ & = \left(\frac{3-\zeta}{3+\zeta} \right)^{kq} : \exp [i k R(y^2) + i q R(\zeta^2)] : . \end{aligned} \quad (3.83)$$

Taking the e_r 's to satisfy Eqs. (3.3) and (3.12), one finds that the algebra closes if A_i and B_r satisfy the same equations as for the untwisted case, i.e., Eqs. (3.15) and (3.16). The central term is again given by

$$c = \sum_i A_i^2 + \sum_r B_r^2 = \sum_i A_i^2, \quad (3.84)$$

and is of the correct form, provided we take

$$f = \frac{1}{16} \sum_i A_i^2. \quad (3.85)$$

A special case of the above Z_2 twisted construction was studied in Ref. [9].

4. - HIGHEST-WEIGHT STATES AND PRIMARY FIELDS

The purpose of this section is to construct the highest-weight states and their primary fields corresponding to the conformal generators found in the previous section.

Let us first consider the untwisted $SU(N)$ case. The conformal operators contain vertex operators and these, acting on states of given momenta, will in general lead to new states which have their momenta changed by $\pm e_r$. Very loosely therefore, it will be useful to consider our highest-weight states as containing momenta which are related by $\pm e_r$. As such, it is natural to consider states that are associated with the weight lattice or a weight lattice shifted by half a root. For completeness, we recall that a highest-weight state $|h\rangle$ is one for which

$$L_0 |h\rangle = h |h\rangle, \quad (4.1)$$

$$L_n |h\rangle = 0, \quad n \geq 1 \quad (4.2)$$

For the unitary minimal series which has

$$c = 1 - \frac{6}{(N+1)(N+2)} \quad (4.3)$$

the allowed values of h are

$$h = \frac{[p(N+2) - q(N+1)]^2 - 1}{4(N+1)(N+2)}, \quad p=1,2,\dots,N, \quad q=1,2,\dots,p. \quad (4.4)$$

A useful formula is

$$:e^{ikQ(z)}: = e^{ikQ_<} e^{ikz} z^{kp} e^{ikQ_>} , \quad (4.5)$$

where

$$iQ_<(z) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^n \quad \text{and} \quad iQ_>(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n z^{-n}. \quad (4.6)$$

Hence, we find that

$$:e^{ikQ(z)}: |0, p\rangle = e^{ikQ_>} z^{kp} |0, p+k\rangle. \quad (4.7)$$

As a first example, let us consider the states of the form

$$\alpha_{-i}^i |0, 0\rangle, \quad i=1, \dots, N-1, \quad (4.8)$$

where $|0, 0\rangle$ satisfies $\alpha_n^i |0, 0\rangle = 0$, $n \geq 1$, and has zero momentum. Such states, which are associated with the roots have automatically satisfy Eq. (4.2) and

$$h = A_i. \quad (4.9)$$

If we consider the minimal series, for example, we find from Eq. (3.44) that the two possible values of h are

$$h = \frac{N}{N+2} \quad \text{or} \quad \frac{2}{(N+2)(N+1)}. \quad (4.10)$$

Let us consider highest-weight states of the form

$$|\psi\rangle = \sum_{\lambda} c_{\lambda} |0, \omega_{\lambda}\rangle, \quad (4.11)$$

where we assume $\omega_s \cdot e_r = 0, \pm 2$. These states satisfy Eq. (4.2) and will obey Eq. (4.1) provided

$$\sum_i A_i c_s \omega_s^i + \sum_t \sum_n B_n c_t [\delta(\omega_t + e_n - \omega_s) \delta(\omega_t e_n + 2) + \delta(\omega_t - e_n - \omega_s) \delta(\omega_t e_n - 2)] = h c_s, \forall s. \quad (4.12)$$

If we take $\omega_s = v_s$, the weights of Eq. (3.32), then the above equation becomes

$$T C = 2h C, \quad (4.13)$$

where C is the column vector with components $(c_1, c_2, \dots, c_N)$ and T is the matrix

$$T_{ij} = B_{ij} + \delta_{ij} \sum_k A_k [v_i^k]^2, \quad (4.14)$$

where $B_{ij} = B_{ji}$ for $j > i$ with $B_{ii} = 0$.

To find the possible values of h , we consider $\det(T - 2h) = 0$. Despite the apparent complexity of T , one can evaluate this determinant for the general solution for A_j and $B_{ij} = b_j$ for all $i < j$ given in the previous section. Subtracting the first row from the second row and then the first column from the second column, and using Eq. (3.53), one finds that the allowed values of h are

$$\sum_{k=1}^{N-1} A_k (v_j^k)^2 - b_j = 2h, \quad j = 2, \dots, N-2,$$

or

$$\left(\frac{N-1}{N} A_N - h\right) \left(\frac{N-2}{N-1} A_{N-2} + \frac{A_{N-1}}{N(N-1)} + \frac{N-2}{2} b_{N-1} - h\right) = \frac{N-1}{(N+2)^2}. \quad (4.15)$$

If we consider the unitary minimal series, then we find the allowed values of h are

$$0, \frac{N}{N+2}, \frac{2}{(N+2)(N+1)}. \quad (4.16)$$

We notice that the naive conformal weight of the states, that is, the weight for $L_n^{\text{naive}} = \frac{1}{2} : \sum_m \alpha_{n-m}^i \alpha_m^i :$, is not the same as the weight for the L_n 's we are considering.

For example, those of Eqs. (4.8) and (4.11) have a naive weight of 1 and $(\omega_s)^2/2$ respectively. Also apparent is the fact that the construction of highest-weight states is not unique.

We can continue searching for highest-weight states by taking any set of vectors and examining the conditions for closure. However, a natural set to consider is obtained by shifting the roots or weights by a given vector, for example by half a root. The resulting vectors do not all have the same length, and so result in states with different L_0^{naive} . Since L_0^{naive} and L_0 commute, we may diagonalize them both and consider a set in which L_0^{naive} has a fixed value, and search for highest-weight states. For example, let us shift the roots by $\pm e_s/2$. The resulting vectors are $e_r \pm e_s/2$ and they have scalar products 1 if $r = s$ and 5 ± 2 or 5 otherwise. If we consider the set with $L_0^{\text{naive}} = \frac{1}{2}$, that is those vectors of length 1, we have

$$|\psi\rangle = c_+ | \frac{e_s}{2} \rangle + c_- | -\frac{e_s}{2} \rangle , \quad (4.17)$$

which has a value of h of

$$\sum_i A_i \frac{e_s^{i2}}{4} \pm \frac{1}{2} B_s . \quad (4.18)$$

One may modify this procedure by applying α_{-n}^1 to states to increase their value of L_0^{naive} and include them in the set with the same value of L_0^{naive} . Shifting the roots by half a root as above, and looking at states with L_0^{naive} of $3/2$, we can arrive at states of the form

$$|\psi\rangle = \sum_{j,s} c_s^j \alpha_{-j}^s |\mu_s\rangle + \sum_t D_t |\omega_t\rangle , \quad (4.19)$$

where the vectors have the possible scalar products

$$\begin{aligned} e_t \omega_s &= 0 \text{ or } \pm 3 , & e_s \mu_s &= 0, \pm 1, \pm 2 , \\ \omega_s^2 &= 3 , & \mu_s^2 &= 1 \end{aligned} \quad (4.20)$$

Corresponding to these scalar products we require

$$\begin{aligned}
 e_2 \pm \omega_t &\in \{ \nu_s \} & \text{if } e_2 \cdot \omega_t = \mp 3, \\
 e_2 \pm \mu_s &\in \{ \nu_s \} & \text{if } e_2 \cdot \mu_s = \mp 2, \\
 e_2 \pm \mu_s &\in \{ \omega_t \} & \text{if } e_2 \cdot \nu_s = \mp 1.
 \end{aligned} \tag{4.21}$$

One finds that Eq. (4.2) places constraints on C_s^j and D_t . Having solved these constraints, we find that Eq. (4.1) reduces to a matrix condition on the C_s^j and D_t , the determinant of which gives the allowed values of h .

In the next section we will illustrate the above procedure for the group $SU(3)$, i.e., $c = 7/10$. We will label the above states by their value of L_0^{naive} and refer to them as shifted roots of $L_0^{\text{naive}} = \frac{1}{2}$ and $L_0^{\text{naive}} = 3/2$ corresponding to Eqs. (4.17) and (4.19).

We may repeat the above trick for the fundamental weights ν_s to obtain vectors of the form $\omega_s = \nu_s \pm (e_r/2)$. These vectors have length squared $1/3$, $7/3$ or $13/3$ and their scalar products with e_r are $0, \pm 1, \pm 3$ or ± 4 . Taking the vectors with an L_0^{naive} of $1/6$, we have states of the form

$$|\psi\rangle = |\omega_s\rangle, \tag{4.22}$$

where

$$\omega_s^2 = \frac{1}{3} \quad \text{and} \quad e_r \cdot \omega_s = 0, \pm 1. \tag{4.23}$$

These states automatically satisfy Eq. (4.2) and have

$$h = \frac{1}{2} \sum_i (\omega_s^i)^2 A_i. \tag{4.24}$$

One can repeat the exercise for the vectors of length $7/3$ including α_{-1}^i action on the vectors of Eq. (4.22).

Clearly, we can carry out the procedure for the weight corresponding to any representation and then consider the states obtained from the weight by shifting by half a root and decorating with α_{-n}^i if necessary.

Let us now turn to the highest-weight states in the Z_2 twisted sector. Consider first the vacuum state $|0\rangle$. It satisfies Eq. (4.2) and has

$$h = \frac{1}{16} \left(\sum_i A_i + \sum_n B_n \right). \quad (4.25)$$

We can also examine states of the form

$$|\psi\rangle = \sum_i d_i c_{-1/2}^i |0\rangle \quad (4.26)$$

These automatically satisfy Eq. (4.2) and Eq. (4.1) reduces to the equation

$$S d = h d,$$

where S is the matrix

$$S_{ij} = \frac{1}{2} (A_i + h_0) \delta_{ij} - \frac{1}{8} \sum_n B_n e_n^i e_n^j, \quad (4.27)$$

where $h_0 = 1/16 (\sum_i A_i + \sum_n B_n)$, the allowed values of h being given by

$$\det(S - h) = 0. \quad (4.28)$$

In the next section we give examples of such highest-weight states. Clearly there are many other such states one could examine.

Let us now turn to the construction of primary fields associated with the above highest-weight states. We recall that a primary field of dimension h is one such that

$$[L_n, \phi(z)] = z^n [z \partial_z + (n+1)h] \phi(z). \quad (4.29)$$

Given such a field, the state

$$|\psi\rangle = \lim_{z \rightarrow 0} \phi(z) |0,0\rangle \quad (4.30)$$

is a highest-weight state of dimension h . Consider then the state of Eq. (4.8). One may, at first sight, think that the corresponding primary field was

$$\frac{p^i(z)}{z} \quad (4.31)$$

However, using Eqs. (3.7) and (3.8), one readily verifies that

$$\begin{aligned} [L_n, \frac{P^i(z)}{z}] = & z^{-n} \left\{ A_i z \frac{\partial}{\partial z} \frac{P^i(z)}{z} + (n+1) A_i \frac{P^i(z)}{z} \right. \\ & \left. - \sum_n \frac{1}{2} B_n e_i^i z [e^{ie_n Q(z)} - e^{-ie_n Q(z)}] \right\}, \end{aligned} \quad (4.32)$$

which is not the same as Eq. (4.29). However, we observe that

$$[L_{n-L_0}, P^i(1)] = n A_i P^i(1), \quad (4.33)$$

and, as a result,

$$z^{L_0} \phi(1) z^{-L_0} z^{-h} \quad (4.34)$$

does satisfy Eq. (4.29) and so is a primary field.

Using the primary field of Eq. (4.34) to define a highest-weight state through Eq. (4.30) does not yield the state one might expect. We find

$$\begin{aligned} \lim_{z \rightarrow 0} \phi(z) |0,0\rangle &= \lim_{z \rightarrow 0} e^{L_0 \ln z} \phi(1) |0,0\rangle z^{-h} \\ &= \lim_{z \rightarrow 0} e^{L_0 \ln z} e^{L_{-1}^{\text{naive}}} |h\rangle z^{-h} \\ &= \lim_{z \rightarrow 0} e^{L_0 \ln z} e^{L_{-1}^{\text{naive}}} e^{-L_0 \ln z} |h\rangle \\ &= \exp(L_{-1}^{\text{naive}} - L_1) |h\rangle \end{aligned}$$

where $|h\rangle$ is the highest-weight state from which one "extracted" $\phi(z)$. We have used the identity

$$e^{L_{-1}^{\text{naive}}} |0,h\rangle = \exp h \sum_n \frac{1}{n} \alpha_{-n} |0,h\rangle$$

Since $L_{-1}^{\text{naive}} - L_1$ commutes with L_n , this is indeed a highest-weight state.

The situation is the same for the states of Eq. (4.11) when $\omega_s = \nu_s$. The field

$$\sum_s c_s \exp i \omega_s \alpha(z) \quad (4.35)$$

is in general not a primary field. Demanding the $(n+1)z^n \phi$ term in Eq. (4.29) leads to Eq. (4.12) while the first term requires that

$$\begin{aligned} A: c_s \omega_s^i + \sum_{n,t} \frac{1}{2} B_n c_t e_n^i \left[\delta(\omega_t + e_n - \omega_s) \delta(\omega_t e_n + 2) \right. \\ \left. - \delta(\omega_t - e_n - \omega_s) \delta(\omega_t e_n - 2) \right] = c_s \omega_s^i. \end{aligned} \quad (4.36)$$

Multiplying Eq. (4.36) by (ω_s^i) , summing on i and using Eq. (4.12), we find that $h = \frac{1}{2}(\omega_s)^2$. In other words, Eq. (4.35) can only be a primary field if its dimension coincides with its naive dimension. In general, this is not the case, although there are examples when this happens. Following the same argument as in the earlier case, we find that provided Eq. (4.12) is satisfied, a primary field of dimension h is given by

$$z^L \sum_s c_s \exp i \omega_s \alpha(z) z^{-L} z^{-h}. \quad (4.37)$$

These cases illustrate the general procedure. One writes down the naive primary field which one would expect at first sight from examining the highest weight, takes $z = 1$ and boosts to arbitrary z with L_0 . In the event that the naive and actual conformal weights coincide, one may be able to take any z , and no boosting is necessary.

Although there would seem to be many ways of obtaining a given highest-weight state, this situation would be more understandable if there were operators which commuted with the L_n and laddered between states of the same weight. We now show the existence of operators that commute with L_n . For any $SU(N)$ solution, one can show that

$$[L_n^{\text{naive}}, L_m] = (n-m) L_{n+m} + \frac{1}{12}(n^3-n) \delta_{n,-m} \sum_i A_i, \quad ,$$

and as a result

$$L_n - L_n^{\text{naive}}$$

commutes with L_n . If we specialize to the unitary minimal solution, since $L_n = L_n^{\text{SU}(N)} - L_n^{\text{SU}(N-1)}$, L_m commutes with $L_n^{\text{SU}(N-1)}$. We may search for other operators by demanding that Eq. (4.36) hold without a right-hand side and Eq. (4.12) hold with $h = 0$. For the minimal series with $B_{ij} = b_j$ for all $i < j$, then

$$\sum_z e^{i\nu_z Q(z)}$$

commutes with all the L_n for any z . Given these three operators which commute with L_n , we may generate more by taking their commutators or by taking the operator product expansions. For the $\text{SU}(3)$ example, we will find further operators that commute with the L_n . This occurrence of operators which commute with all the Virasoro generators has previously been noted in Ref. [11].

5. - THE $\text{SU}(3)$ CASE

We now discuss in detail the case of $\text{SU}(3)$ and focus in particular on $c = 7/10$. To be concrete, we will take the solution [see Eq. (3.46)]

$$A_1 = \frac{1}{10}, \quad A_2 = \frac{3}{5}, \quad B_{12} = \frac{-1}{10}, \quad B_{13} = -B_{23} = \frac{2}{5}. \quad (5.1)$$

The possible values of the highest-weight states are given by Eq. (4.4) to be

$$h = 0, \frac{3}{80}, \frac{1}{10}, \frac{7}{16}, \frac{3}{5} \text{ and } \frac{3}{2}, \quad (5.2)$$

and we will construct all of them using the states given in the previous section.

To introduce some order we will label the states by their h value and L_0^{naive} and refer to them as root, weight, shifted-root and shifted-weight. We also recall that the operators

$$L_n - L_n^{\text{naive}}, \quad \forall n, \quad (5.3)$$

$$L_n^{\text{SU}(2)} = \frac{1}{2\pi} \oint \frac{dz}{z} \left[\frac{1}{4} : P_1^2(z) : - \frac{1}{4} : z^2 \cos 2Q_1(z) : \right] \quad (5.4)$$

and

$$\begin{aligned}
 S_2 &= \frac{1}{2i\pi} \oint \frac{dz}{z} z^2 z^{\frac{2}{3}} : e^{v_1 Q(z)} - e^{v_2 Q(z)} - e^{v_3 Q(z)} : \\
 &= \frac{1}{2i\pi} \frac{1}{3} \oint \frac{dt}{t} t^{3/2} t^2 : e^{v_1 Q(t^3)} - e^{v_2 Q(t^3)} - e^{v_3 Q(t^3)} : \quad (5.5)
 \end{aligned}$$

all commute with L_n . Another expression that commutes with L_n is

$$e^{iQ_1(z)} + e^{-iQ_1(z)} \quad (5.6)$$

and hence so does

$$T_2 = \frac{1}{2i\pi} \oint \frac{dz}{z} z^2 z^{1/2} [e^{iQ_1(z)} + e^{-iQ_1(z)}] \quad (5.7)$$

We note that $[T_r, T_s] = 2\delta_{r+s,0}$.

Let us begin with the root states of Eq. (4.8) which have an $L_0^{\text{naive}} = 1$:

$$|\frac{1}{10}, 1\rangle = \alpha_{-1}^{(1)} |0, 0\rangle, \quad |\frac{3}{5}, 1\rangle = \alpha_{-1}^{(2)} |0, 0\rangle. \quad (5.8)$$

We also have the vacuum

$$|0, 0\rangle, \quad (5.9)$$

which, as the above equation says, has $h = 0$ and $L_0^{\text{naive}} = 0$.

The weight states of Eq. (4.11) are given by

$$\begin{aligned}
 |0, \frac{2}{3}\rangle &= |v_1\rangle - |v_2\rangle - |v_3\rangle \sim S_{-\frac{2}{3}} |0, 0\rangle \\
 |\frac{1}{10}, \frac{2}{3}\rangle &= |v_1 + v_2\rangle \sim S_{\frac{1}{3}} |\frac{1}{10}, 1\rangle \\
 |\frac{3}{5}, \frac{2}{3}\rangle &= |v_1\rangle - |v_2\rangle + 2|v_3\rangle \sim S_{\frac{1}{3}} |\frac{1}{10}, 1\rangle \quad (5.10)
 \end{aligned}$$

where the second equation explains how they are created from the states of Eqs. (5.8) and (5.9) using the S_v of Eq. (5.5).

Next let us consider the shifted roots. A shift by $e_1/2$ yields the states of Eq. (4.17) which are

$$|0, \frac{1}{2}\rangle = |1, 0\rangle + |-1, 0\rangle \sim T_{1/2} |0, 0\rangle,$$

and

$$|\frac{1}{10}, \frac{1}{2}\rangle = |1, 0\rangle - |-1, 0\rangle \sim T_{-1/2} |\frac{1}{10}, 1\rangle. \quad (5.11)$$

Shifting by $e_2/2$, on the other hand, we find

$$\begin{aligned} |\frac{3}{80}, \frac{1}{2}\rangle &= |\frac{1}{2}, \frac{1}{2}\sqrt{3}\rangle + |-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\rangle, \\ |\frac{7}{10}, \frac{1}{2}\rangle &= |\frac{1}{2}, \frac{1}{2}\sqrt{3}\rangle - |-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\rangle. \end{aligned} \quad (5.12)$$

Shifting by $e_1/2$ and looking at the states with $L_0^{\text{naive}} = 3/2$, we find

$$\begin{aligned} |\frac{1}{10}, \frac{3}{2}\rangle &= 2\alpha_{-1}^2 (|1, 0\rangle - |-1, 0\rangle) \\ &\quad + \sqrt{3} (|0, \sqrt{3}\rangle + |0, -\sqrt{3}\rangle), \\ |\frac{3}{5}, \frac{3}{2}\rangle &= \alpha_{-1}^2 (|1, 0\rangle + |-1, 0\rangle) \sim T_{-1/2} |\frac{3}{5}, 1\rangle, \\ |\frac{3}{2}, \frac{3}{2}\rangle &= \alpha_{-1}^2 \sqrt{3} (|1, 0\rangle - |-1, 0\rangle) \\ &\quad - 2 (|0, \sqrt{3}\rangle + |0, -\sqrt{3}\rangle). \end{aligned} \quad (5.13)$$

Hence we have indeed found all the highest-weight states.

Let us turn to the shifted weights of Eq. (4.22). If we shift by $+e_1/2$ and consider states with $L_0^{\text{naive}} = 1/6$, we have

$$|\frac{1}{10}, \frac{1}{6}\rangle = |0, \frac{1}{\sqrt{3}}\rangle \sim T_{1/2} |\frac{1}{10}, \frac{2}{3}\rangle, \quad (5.14)$$

while, repeating the procedure with $e_2/2$, we have

$$|\frac{3}{80}, \frac{1}{6}\rangle = |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \quad (5.15)$$

We note that

$$|\frac{1}{10}, \frac{3}{2}\rangle \sim \bar{S}_{-\frac{4}{3}} |\frac{1}{10}, \frac{1}{6}\rangle + S_{-\frac{4}{3}} \overline{|\frac{1}{10}, \frac{1}{6}\rangle}, \quad (5.16)$$

where $\bar{S}_r = S_r^\dagger$, while

$$|\frac{3}{80}, \frac{1}{2}\rangle \sim S_{-\frac{1}{3}} |\frac{3}{80}, \frac{1}{6}\rangle + \bar{S}_{-\frac{1}{3}} \overline{|\frac{3}{80}, \frac{1}{6}\rangle}. \quad (5.17)$$

The shifted weights obtained by a $\pm e_1/2$ shift with $L_0^{\text{naive}} = 7/6$ are given by

$$\begin{aligned} T_{-\frac{1}{2}} |0, \frac{2}{3}\rangle, (L_{-1} - L_{-1}^{\text{naive}}) |\frac{1}{10}, \frac{1}{6}\rangle, \\ \text{and } T_{\frac{1}{2}} |\frac{3}{5}, \frac{2}{3}\rangle. \end{aligned} \quad (5.18)$$

The reader will notice that, given the states

$$|0,0\rangle, |\frac{1}{10}, 1\rangle, |\frac{3}{5}, 1\rangle, |\frac{3}{80}, \frac{1}{6}\rangle, |\frac{7}{16}, \frac{1}{2}\rangle, \quad (5.19)$$

then all the other above representations of highest-weight states can be obtained from these by the action of the operators of Eqs. (5.3) to (5.7). This would indicate that any highest-weight state corresponding to the weight or shifted weight of any representation can be obtained from those of Eq. (5.19) by the application of these operators. We also note that the actual weight of a state is always less than, or equal to, the naive weight. While this was apparent from the formulae in the previous sections for the cases considered, it is probably also a general result. This explains why only one state with $h = 3/2$ was among the above list. In the twisted sector, using Eq. (4.25), the vacuum has $h = 3/80$. From the states of the form of Eq. (4.26), we find that $h = 3/80$ or $7/16$.

The primary fields for the above highest-weight states are obtained as discussed in the previous section. The case of $|3/2, 3/2\rangle$ is different, as the naive and actual weights of the state coincide. Examining Eq. (5.13), we extract the primary field to be

$$J(z) = \frac{1}{\sqrt{30}} \left[\sqrt{3} \frac{P_2(z)}{z} : e^{iQ_1(z)} \eta + \eta^\dagger e^{-iQ_1(z)} : \right. \\ \left. - 2 : e^{i\sqrt{3}Q_2(z)} + e^{-i\sqrt{3}Q_2(z)} : \right], \quad (5.20)$$

where

$$\eta = \exp i\pi (Q_1 + \sqrt{3} Q_2). \quad (5.21)$$

The factor η is necessary in order to have a $J(z)$ which is Hermitian and plays the rôle of a cocycle. It is straightforward to verify that it satisfies the correct operator product expansion, i.e.,

$$J(z) J(z') = \frac{2}{z-z'} L(z) + \frac{7}{15(z-z')^3}, \quad (5.22)$$

where

$$L(z) = \frac{1}{20} \frac{P_1^2(z)}{z^2} + \frac{3}{10} \frac{P_2^2(z)}{z^2} - \frac{1}{10} : \omega Q_1(z) : \\ + \frac{2}{5} \eta : \omega [Q_1(z) + \sqrt{3} Q_2(z)] - \omega [Q_1(z) - \sqrt{3} Q_2(z)] :. \quad (5.23)$$

Acting on weights and roots, η is +1 and we recover the $L(z)$ we are using. On shifted weights and roots, $\eta = -1$. This explains some of the signs that occur for these latter highest-weight states.

As is well known [2,12], the existence of $J(z)$ implies a supersymmetry and one can indeed verify this on the states. For example, we have

$$J_{-1/2} \left| \frac{1}{10}, 1 \right\rangle = \left| \frac{3}{5}, \frac{3}{2} \right\rangle, \\ J_{1/2} \left| \frac{3}{5}, 1 \right\rangle = \left| \frac{1}{10}, \frac{1}{2} \right\rangle. \quad (5.24)$$

Given the above machinery, one may attempt to construct the Green functions. As with the method of Fateev and Dotsenko [12], one must be careful that momentum conservation does not render the result zero. In fact, one must make a judicious

choice of primary fields and states. Let us first consider the fusion rules or three-point functions. We choose the states and primary fields

$$\begin{aligned}
 | \frac{1}{10}, \frac{1}{6} \rangle &= | 0, \frac{1}{\sqrt{3}} \rangle + | 0, -\frac{1}{\sqrt{3}} \rangle, \\
 \phi_{\frac{1}{10}}(1) &= e^{\frac{i}{\sqrt{3}} Q_2(1)} + e^{-\frac{i}{\sqrt{3}} Q_2(1)}, \\
 | \frac{3}{5}, \frac{2}{3} \rangle &= | \nu_1 \rangle - | \nu_2 \rangle + 2 | \nu_3 \rangle + (\nu_2 \rightarrow -\nu_2), \\
 \phi_{\frac{3}{5}}(1) &= e^{i\nu_1 Q(1)} - e^{i\nu_2 Q(1)} + 2e^{i\nu_3 Q(1)} + (\nu_2 \rightarrow -\nu_2),
 \end{aligned}
 \tag{5.25}$$

and $|3/2, 3/2\rangle$ as in Eq. (5.13) with its primary field in Eq. (5.20). One can verify that the three-point function

$$\begin{aligned}
 \langle 0 | \phi_{\frac{1}{10}}(z_1) \phi_{\frac{3}{5}}(z_2) \phi_{\frac{3}{5}}(z_3) | 0 \rangle \\
 \sim \langle h_1 | \phi_{\frac{1}{10}}(1) | h_3 \rangle
 \end{aligned}
 \tag{5.26}$$

does indeed give the correct fusion rules for this model.

Similarly, one could go to the next model of the minimal series, based on $SU(4)$ and $c = 4/5$, and construct explicitly the W algebra [12], involving operators of dimension three, out of vertex operators, just as we have done with Eq. (5.20) for the superconformal generators.

6. - OUTLOOK

One of the problems of conformal theories is a general method for constructing or computing arbitrary expectation values in a manifestly unitary framework. The only method known so far involves solving differential equations [1,13]. In our construction, one could contemplate computing expectation values of products of conformal operators as defined in the previous sections. For example, in the $c = 7/10$ case, one could construct the four-point function for four fields of weight $1/10$, given in Eq. (5.25) [i.e., $\langle 1/10 | \phi_{1/10}(1) Z^{L_0} \phi_{1/10}(1) Z^{-L_0} | 1/10 \rangle$]. Expectation values constructed in this way inevitably satisfy the relevant differential equations. However, it is not clear that these expressions are well defined, as the summation over intermediate states may involve multiple and even infinite counting due to the existence of the degeneracy created by the laddering operators.

Presumably, after removal of this degeneracy by some form of projection, the answer would come out finite, and in agreement with known results. This is analogous to the insertion of the Brink-Olive projection operator between fermion-emission vertices in the old dual model, which was necessary to give the correct four-fermion amplitude. Put another way, the construction envisaged above has more degrees of freedom, i.e., more oscillators, than that of Dotsenko and Fateev. One would imagine that the latter is analogous to a light-cone formulation, while the former is a gauge-invariant formulation.

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FIGURE CAPTIONS

Fig. 1 : Descent from level j to level $j-1$.

Fig. 2 : Descent from level $N-1$ to level $N-2$.

NOTE ADDED

After this paper had been typed, we became aware of the work of G.V. Dunne, I.G. Halliday and P. Suranyi [Phys. Lett. B213 (1988) 139], where the conformal algebras are also constructed with vertex operators, and where the relation to the GKO construction is discussed.

Fig 1

Fig 2