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RIEMANN MONODROMY PROBLEM AND CONFORMAL FIELD THEORIES

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Abstract

A systematic analysis of the use of the Riemann monodromy problem for determining correlators (conformal blocks) on the sphere is presented. The monodromy data is constructed in terms of the braid matrices and give a constraint on the non-integer part of the conformal dimensions of the primary fields. To determine the conformal blocks we need to know the order of singularities. We establish a criterium which tells us when the knowledge of the conformal dimensions of primary fields suffices to determine the blocks. When zero modes of the extended algebra are present the analysis is more difficult. In this case we give a conjecture that works for the $SU(2)$ WZW case.

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1. Introduction

The problem of classification of all two-dimensional conformal field theories (CFT) has attracted a lot of attention. It is of interest both to critical phenomena in two dimensions and to string theory. A systematic study of CFT was initiated by Polyakov, Belavin and Zamolodchikov [1]. Unitary theories with $c < 1$ have been completely classified and their structure is known [2]. For theories with $c \geq 1$, a lot of effort has been made to understand them in terms of extended (chiral) algebras and GKO construction [3], [4], [5], [6], [7], [8].

Recently, a new approach to the conformal bootstrap problem was suggested in the works of several groups [9], [10], [11]. The key observation is that the conformal field theories are characterized by the monodromy properties of the conformal blocks or alternatively by their crossing (braiding) properties. The whole approach is based on the equivalence between the associativity of the operator product algebra (OPA) and duality [9,10]. It follows that conformal blocks in different channels (s, t, u for the 4-point function case) transform into each other by the duality transformations and are connected by linear transformations

$$G_p^{s(ijkl)}(z) = \sum_{p'} F \begin{pmatrix} j & k \\ i & l \end{pmatrix}_{pp'} G_{p'}^{t(ijk)}(1-z)$$

$$G_p^{s(ijkl)}(z) = \sum_q B \begin{pmatrix} j & k \\ i & l \end{pmatrix}_{pq} G_q^{u(ikjl)}(1/z)$$
(1.1)

where the i, j, k, l refer to external legs and p, p', q to internal legs (see fig.1 for precise notation). The functions $G_p^s, G_{p'}^t, G_q^u$ refer to conformal blocks in the (s,t,u) channels respectively. The crossing matrices F and B are constant matrices. It was found in [10] that these matrices obey polynomial equations which have the form of the Yang-Baxter equation and pentagonal relation. Those relations are equivalent to the associativity of the operator algebra and have a nice algebraic interpretation as the consistency condition associated with the Friedan-Shenker linear bundle [12].

It was suggested [10], but not rigorously proven, that classifying all possible solutions to these polynomial equations amounts to classifying all CFT. This may very well be true,

but it still leaves us with the following question. Given a set of fusion rules and the corresponding polynomial equations, and given a solution of these equations how could we determine the Hilbert space structure of the theory? How could we learn about the possible existence of an extended algebra? Alternatively, we would like to be able to calculate the correlation functions of the theory. In particular, the correlation functions of primary fields are of utmost interest being observable. Moreover, once these correlation functions are known we can determine the structure constants of the underlying CFT.

The purpose of this paper is to explore a systematic method which allows one to determine the correlation functions (or conformal blocks) of a CFT from the knowledge of the crossing matrices F and B . As it turns out this information is not sufficient and one needs to know the singular behaviour of the correlators near the singularities. This information is usually encoded in the conformal dimensions of the primary fields (including the integer part!). However when zero modes are present the situation is more delicate. Fortunately for a large class of coset models we know the dimensions of the fields and this extra information is enough to determine (in principle) uniquely the conformal blocks. The main tool of our approach will be the theory of the Riemann monodromy problem [13], [14], [15], [16]. The relevance of this problem to the questions we are interested in was mentioned in [9] for the case of WZW and minimal models. However, to the best of our knowledge, no general attempt to develop and exploit this approach for an arbitrary CFT has yet been made. In particular we would like to apply this approach to the rational CFT.

The paper is organized as follows: in section 2 we describe the classical Riemann problem and discuss its relevance to CFT. We stress those aspects of the Riemann problem that are relevant to our investigation. We discuss the constraints imposed on the CFT. For example, constraint equations on the non-integer part of the conformal dimensions arise from the monodromy problem initial data. Section 3 is devoted to the problem of finding the physical conformal blocks from the solutions of the Riemann problem. The challenge is to find these rational linear combinations of solutions which correspond to the blocks of some underlying conformal field theory. This demands extra information, i.e. the singular behaviour of the blocks near the singular points. When no zero modes are present this extra information is given by the conformal dimensions (including the integer part which is not constraint by the monodromy initial data) of the primary fields. We formulate a necessary condition for the conformal dimensions of the primary fields to determine uniquely the singular behaviour of the blocks. In section 4 we very briefly present the solution to the Riemann monodromy problem constructed by Sato, Miwa and Jimbo [16]. This solution

is given in terms of an infinite series of contour integrals. The series are known to converge when the eigenvalues of the monodromy matrices are small enough. This section includes no new material and is presented just for the sake of completeness. In section 5 we use the formalism developed in section 3 to investigate the important case of coset models. Although some of the analysis holds for general G/H cosets, we will mainly be interested in those cases for which we have the knowledge of the conformal dimensions (including the integer part). These include e.g. $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ and parafermions. The basic underlying assumption is that the crossing (braid) matrices factorize. Although we do not give a full proof to this effect, we do give some compelling arguments. Section 6 is devoted to the investigation of these cases in which zero modes are present. We discuss the constraints on the physical blocks and make several conjectures on how we can go about determining them. We check some of our ideas in the case of the WZW model. In section 7 we summarize our results, conjectures and discuss possible directions to continue the present work.

2. Riemann Monodromy Problem and Conformal Field Theories

The classical Riemann monodromy problem is formulated as follows [14,16]. For a given set of s points $a_1, \dots, a_s$ and (possibly) infinity in the complex plane and a set of $m \times m$ matrices $M_1, \dots, M_s, M_\infty$ find a system of m functions $f_1(z), \dots, f_m(z)$ such that for any closed path γ in $C - \{a_1, \dots, a_s\}$ the monodromy will be given by those matrices. The analytic continuation $F_\gamma(z)$ of the functions $F(z) = (f_1(z), \dots, f_m(z))$ along the path γ induces the linear transformation

$$F(z) \longrightarrow F_\gamma(z) = F(z)M_\gamma \quad (2.1)$$

where M_γ is a constant matrix depending on γ through its homotopy class $[\gamma]$. Since $\pi_1(C - \{a_1, \dots, a_s\})$ is a group with s generators associated with paths $\gamma_1, \dots, \gamma_s$ around each of the s points $a_1, \dots, a_s$, the monodromy group is generated by s matrices $M_1, \dots, M_s$ which we have preassigned. This corresponds to

$$F(z) \longrightarrow F_{\gamma_i}(z) = F(z)M_i \quad (2.2)$$

where γ_i is the path surrounding a_i .

The functions $F(z)$ must satisfy the following additional properties:

- (i) $F(z)$ is analytic but for the isolated points $a_1, \dots, a_s, \infty$
- (ii) Near the point a_ν $F(z)$ has the form

$$F(z) \sim \tilde{F}_\nu(z)(z - a_\nu)^{L_\nu} \quad (2.3)$$

where $\tilde{F}_\nu(z) = 1 + c_1 z + \dots$ are analytic functions and

$$M_\nu = \exp(2\pi i L_\nu) \quad (2.4)$$

Considerable efforts have been made to solve the problem (see [16] for a detailed bibliography) and the following results were established:

1. For sufficiently small $\|L_i\|$ the Riemann problem has m independent solutions $f_i^{(j)}$ ($i, j = 1, \dots, m$) called fundamental solutions. Every other solution is given as a linear combination of these solutions with rational coefficients. By sufficiently small we mean that the eigenvalues of the matrices L_i are small enough to ensure convergence in the infinite series construction of the solutions. When no exact convergence is guaranteed, the m independent solutions can be still understood in terms of analytic continuations. Note that the fundamental solutions $f_i^{(j)}$ form an $m \times m$ matrix F (each row corresponds to one fundamental solution) such that (in matrix notation)

$$F_{\gamma_i}(z) = F(z)M_i \quad (2.5)$$

2. The matrix of fundamental solutions $F(z)$ satisfies a system of Fuchsian type of linear differential equations which have the form [14, 16]

$$\frac{dF}{dz} = \sum_{i=1}^s \frac{A_i}{z - a_i} F \quad (2.6)$$

where the $A_i = A_i(a_1, \dots, a_s)$ are constant matrices (i.e. independent of z), which do depend on the points $a_1, \dots, a_s$. Note that the matrices A_i in eq. (2.6) act on the upper index j of the matrix elements $f_i^{(j)}$ of the matrix F . This index refers to the different sets of solutions of the Riemann monodromy problem. Monodromy matrices M_ν act on the lower index i that refers to the different functions in the same solution of the Riemann problem.

3. As functions of the parameters $a_1, \dots, a_s$ the matrices A_i satisfy the following system of Schlesinger differential equations [17] .

$$\begin{aligned} \frac{dA_i}{da_j} &= \sum_{k(\neq i)} [A_k, A_i] \left(\frac{1}{a_i - a_k} - \frac{1}{x_0 - a_k} \right) & i = j \\ \frac{dA_i}{da_j} &= [A_j, A_i] \left(-\frac{1}{a_i - a_j} + \frac{1}{x_0 - a_j} \right) & i \neq j \\ \frac{dA_i}{dx_0} &= \sum_{j(\neq i)} [A_j, A_i] \frac{1}{x_0 - a_j} \end{aligned} \quad (2.7)$$

Here x_0 is a normalisation point $F(x_0)=1$. (Actually one can choose x_0 as one of the a_k 's. This modifies and simplifies the set of equations [16]).

The system of Schlesinger equations (2.7) expresses the condition that the monodromy is preserved while we vary the position of the singular points $a_1, \dots, a_s$ and x_0 .

The Riemann monodromy problem can therefore be expressed in either one of the following equivalent forms :

- a. Find a multivalued matrix $F = F(z)$ with monodromy data $a_1, \dots, a_s$ and $M_1, \dots, M_s$.
- b. Solve the linear set of differential equations of eq. (2.6) with the singularity data $a_1, \dots, a_s$ and $A_1, \dots, A_s$.

Before further discussing and proving some of the properties and uniqueness of the solutions to the Riemann problem we would like to address the question of the relevance of this problem to CFT. It is quite obvious and was already mentioned in [9] that the Riemann monodromy problem is relevant for determining correlators in CFT. In equation (1.1) we have denoted by F and B the crossing matrices for the (s-t) and (s-u) duality of 4-point correlators. These two sets of matrices are connected by phase transformations where the phases are associated with the conformal dimensions [10] . To the effect that we know (the non-integer part of) the dimensions, we can work with just the F 's or just the B 's. The conformal blocks $G_p^{(ijkl)}(z)$ undergo the following transformation as we go around the points 1, 0 and ∞ respectively (see figs 1, 2 for notations)

$$\begin{aligned}
G_p^{(ijkl)} &\sim \left[F \begin{pmatrix} j & k \\ i & l \end{pmatrix} T^{(1)} F^{-1} \begin{pmatrix} j & k \\ i & l \end{pmatrix} \right]_{pq} G_q^{(ijkl)} \\
G_p^{(ijkl)} &\sim (T^{(0)})_{pq} G_q^{(ijkl)} \\
G_p^{(ijkl)} &\sim \left[B \begin{pmatrix} j & k \\ i & l \end{pmatrix} T^{(\infty)} B^{-1} \begin{pmatrix} j & k \\ i & l \end{pmatrix} \right]_{pq} G_q^{(ijkl)}
\end{aligned} \tag{2.8}$$

Here $T^{(0)}$, $T^{(1)}$ and $T^{(\infty)}$ are the corresponding matrices of exponents of anomalous dimensions. Recall that the conformal blocks are functions of the variable $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ due to projective invariance. The matrices $T^{(m)}$ are diagonal matrices ($m = 0, 1, \infty$) with

$$\begin{aligned}
m = 0, 1 \quad (T^{(ab)})_{pq} &= \delta_{pq} \exp(2\pi i(-\Delta_a - \Delta_b + \Delta_c)) \\
m = \infty \quad (T^{(ab)})_{pq} &= \delta_{pq} \exp(2\pi i(\Delta_a + \Delta_c - \Delta_b))
\end{aligned} \tag{2.9}$$

The conformal blocks are defined through the 4-point correlator $\langle \Phi_1(0)\Phi_2(z)\Phi_3(1)\Phi_4(\infty) \rangle$, with Φ_c appearing in the operator product of $\Phi_a * \Phi_b$. Hence,

$$\begin{aligned}
T^{(0)} &= T^{(12)} \\
T^{(1)} &= T^{(23)} \\
T^{(\infty)} &= T^{(24)}
\end{aligned} \tag{2.10}$$

Recall that the conformal blocks are actually projections of the correlation function on different primary fields that appear in the operator product.

In view of eq. (2.8) we conclude that the 4-point conformal blocks are specified as solutions of the Riemann monodromy problem with three singular points ($0, 1, \infty$) and with monodromy matrices which are built out of the crossing matrices and anomalous dimensions. For the simplest case when there are only two conformal blocks (e.g. the Ising model), the corresponding equation (2.6) reduces to the equation defining the hypergeometric function [1]. The conformal blocks are then the independent solutions of this equation.

The case of the n -point conformal blocks can be dealt in a similar way i.e. it can be mapped into the Riemann monodromy problem with $n-1$ singular points $a_1, \dots, a_{n-1}$ and well-defined monodromy matrices built out of the crossing matrices and the anomalous dimension T -matrices. To see how it works let us consider the blocks associated with the 5-point correlation functions. The appropriate monodromy matrices are

$$\begin{aligned}
(M_1)_{q_1 q_2}^{p_1 p_2} &= (T^{(12)})_{p_1 q_1} \delta_{p_2 q_2} = \exp(2\pi i(-\Delta_1 - \Delta_2 + \Delta_{p_1})) \delta_{p_1 q_1} \delta_{p_2 q_2} \\
(M_2)_{q_1 q_2}^{p_1 p_2} &= \sum_l B \begin{pmatrix} i_3 & i_2 \\ p_2 & i_1 \end{pmatrix}_{p_1 l} (T^{(13)})_{ll} B^{-1} \begin{pmatrix} i_3 & i_2 \\ p_2 & i_1 \end{pmatrix}_{l q_1} \delta_{p_2 q_2} \\
(M_3)_{q_1 q_2}^{p_1 p_2} &= \sum_{l_1 l_2} B \begin{pmatrix} i_4 & i_3 \\ i_5 & p_1 \end{pmatrix}_{p_2 l_2} B \begin{pmatrix} i_4 & i_2 \\ l_2 & i_1 \end{pmatrix}_{p_1 l_1} (T^{(14)})_{l_1 l_1} \\
&\quad B^{-1} \begin{pmatrix} i_4 & i_2 \\ l_2 & i_1 \end{pmatrix}_{l_1 q_1} B^{-1} \begin{pmatrix} i_4 & i_3 \\ i_5 & q_1 \end{pmatrix}_{l_2 q_2}
\end{aligned} \tag{2.11}$$

The derivation of equation (2.11) is graphically depicted in fig. 3. The matrices M_i are the monodromy matrices associated with z going around the singular point a_i . The indices $i_1, \dots, i_5$ refer to external legs while p_1, p_2, q_1, q_2 refer to internal legs (see fig. 3 for notation). In addition there is the monodromy matrix associated with going around the last point a_4 (which always can be chosen to be at infinity) which we have not written down. This monodromy matrix has a similar structure but we have to use also the F -type matrices. Moreover, as we will comment shortly, the monodromy matrix around infinity can be always expressed in terms of all other monodromy matrices. When z goes around a_i

$$G_{p_1 p_2}^{(i_1, \dots, i_5)} \sim (M_i)_{p_1 p_2}^{l_1 l_2} G_{l_1 l_2}^{(i_1, \dots, i_5)} \tag{2.12}$$

The content of equation (2.11) is quite simple. We have chosen a basis of conformal blocks such that each of them is only multiplied by a phase when we go around the point a_1 (see fig.3). We will refer to this basis as the canonical basis around the point a_1 . In order to find the monodromy associated with going around the point a_k we first have to move the i_{k-1} leg attached to it to be near the point z . This involves applications of the matrices B . Then we can go with z around a_k using the matrix $T^{(1k)}$. Then we return to our original

canonical block basis by using B^{-1} . Equation (2.12) tells us that the 5-point blocks viewed as functions of z are solutions to the Riemann problem with 4 singular points.

The generalization to higher n -point blocks is obvious and goes along the same lines. Those blocks are solution to Riemann monodromy problem with $n-1$ singular points. The monodromy matrices are constructed in a similar way using the crossing matrices and the T matrices associated with the conformal dimensions.

From the structure of the monodromy matrices (see e.g. equations (2.8) and (2.11)), it is clear that the matrices L_ν defined in equations (2.3) and (2.4) are matrices whose eigenvalues are associated with (non-integer part of) conformal dimensions. Also the eigenvalues of the matrices A_i appearing in the differential equations of equation (2.6) are associated with these conformal dimensions. This can be easily established by considering the differential equation near each of the singular points and integrating it out obtaining a power behaviour with the power controlled by the eigenvalues of the A_i 's. This behaviour can be compared to equation (2.3). Moreover, the monodromy matrices M_ν should satisfy the following constraint

$$M_1 \cdot M_2 \cdot \dots \cdot M_s \cdot M_\infty = I \quad (2.13)$$

This equation restricts the conformal dimensions. For the 4-point correlator $\langle \Phi_{i_1}(0)\Phi_{i_2}(z)\Phi_{i_3}(1)\Phi_{i_4}(\infty) \rangle$ the constraint reads

$$\sum_i (\Delta_{i_1} + \Delta_{i_2} - \Delta_i) + \sum_j (\Delta_{i_2} + \Delta_{i_3} - \Delta_j) + \sum_k (-\Delta_{i_1} + \Delta_{i_3} - \Delta_k) = 0 \text{ mod}(Z) \quad (2.14)$$

This set of equations fixes the non-integer part of the dimensions while the integer part remains undetermined. The monodromy properties do not fix the integer part of the dimensions. Clearly, multiplying a function with given monodromy properties by a polynomial (or even a rational function) does not change the monodromy. The constraint on the dimensions implied by equation (2.14) is analogous to those found using different approaches in refs. [18] , [19] , [20] .

At this point we would like to go back and investigate the general properties of the solutions to the Riemann problem. For the general problem which we have formulated in the beginning of this section the initial data consists of s singular points $a_1, \dots, a_s$ and infinity and $m \times m$ matrices $M_1, \dots, M_s$ and M_∞ (subject to the constraint of equation (2.13)). As we remarked this problem has m independent solutions in the sense that any other solution can be written as a linear combination of the fundamental solutions

with coefficients which are rational functions. For small values of the eigenvalues (to ensure some convergence properties) this has been shown explicitly in [14]. Actually if we formulate the problem in terms of the differential equations of eq. (2.6), there are always m independent integrals (i.e. solutions) which, if necessary, can always be defined through analytic continuation. The crucial problem that we are facing is, therefore, how to choose the linear combinations which correspond to physical cases i.e. to blocks of an underlying conformal field theories. Clearly some more information is needed. This extra information is contained in the integer parts of the dimensions which do not affect the monodromy. In what follows we will qualify and elaborate on this statement.

Returning to our set of differential equations (equations (2.6)), it has the form

$$\frac{df_i}{dz} = C_{ij} f_j \quad i, j = 1, \dots, m \quad (2.15)$$

with

$$C_{ij} = \sum_{k=1}^s \frac{A_{(ij)}^{(k)}}{z - a_k}$$

By taking consecutive derivatives of this equation and using (2.15) repeatedly we obtain the following set of equations:

$$\begin{aligned} \frac{df_i}{dz} &= C_{ij} f_j \\ \frac{d^2 f_i}{dz^2} &= C_{ij}^{(2)} f_j \end{aligned} \quad (2.16)$$

$$\frac{d^m f_i}{dz^m} = C_{ij}^{(m)} f_j$$

where the $C_{ij}^{(k)}$ are obtained from the C_{ij} of equation (2.15). We can use the last $(m-1)$ equations to solve for the f_j 's ($j \neq i$) in terms of $\frac{d^2 f_i}{dz^2}, \dots, \frac{d^m f_i}{dz^m}$ and f_i . Putting this solution into the i -th equation we obtain a single m -th order differential equation for any one of the

functions f_i e. g. f_1 . The structure of this equation is

$$f^{(m)} + u_{m-1}f^{(m-1)} + \dots + u_0f = 0 \quad (2.17)$$

with the u_l well-defined functions built from the $C_{ij}^{(k)}$ in equation (2.16). This equation is of the Fuchs type i.e. an equation all of whose integrals are analytic functions with regular singularities. The most general form of the Fuchs equation is [21], [22], [23]

$$f^{(m)} + \frac{P_{(s-1)}(z)f^{(m-1)}}{Q(z)} + \frac{P_{2(s-1)}(z)f^{(m-2)}}{Q^2(z)} + \dots = 0 \quad (2.18)$$

where $P_{k(s-1)}$ is a polynom of order $k(s-1)$ and $Q(z) = \prod_{i=1}^s (z - a_i)$. This equation is known to have m independent solutions having only regular (i.e. power behaved) singularities provided the so-called Fuchs condition is satisfied [21,22,23]

$$\sum_{j=1}^m \sum_{k=1}^s \rho_j^{(k)} = (s-1) \frac{m(m-1)}{2} \quad (2.19)$$

where s is the number of singular points besides infinity. (Note that zeroes of $f_j^{(i)}$ are also included in this number). The $\rho_j^{(k)}$ are, respectively, the orders of the singularities of the m independent solutions of eq. (2.18) near the point a_k that form a canonical set for this point. If we denote by f_j those solutions, then near a_k

$$f_j \sim (z - a_k)^{\rho_j^{(k)}} (b_{0j}^{(k)} + b_{1j}^{(k)}(z - a_k) + \dots) \quad (2.20)$$

with $b_{0j}^{(k)} \neq 0$. Since f in equations (2.18) can be any of the functions $f_1, \dots, f_m$, we have here a matrix of solutions which corresponds to the matrix F in equation (2.6). (For example if we take f to be f_1 then the f_j in equation (2.20) are f_{1j} i.e. correspond to the first row of the matrix F). Recall also that we should look for regular singular solutions since conformal blocks of a CFT have such a behaviour.

In Appendix A we consider, as an illustration, the case of $m=2$ and three singular points which can be taken to be $a_1 = 0, a_2 = 1, a_3 = \infty$. This case is of importance for the consideration of e.g. minimal models. Equation (2.17) is reduced after a substitution

$f \longrightarrow z^{\rho_1^{(1)}}(z-1)^{\rho_1^{(2)}} w(z)$ to the usual hypergeometric equation:

$$z(1-z)w'' + (\gamma - (1+\alpha+\beta)z)w' - \alpha\beta w = 0 \quad (2.21)$$

where

$$\alpha = \rho_1^{(\infty)} + \rho_1^{(1)} + \rho_1^{(2)}$$

$$\beta = \rho_2^{(\infty)} + \rho_1^{(1)} + \rho_1^{(2)}$$

$$1 - \gamma = \rho_2^{(1)} - \rho_1^{(1)}$$

and we have used the Fuchs condition (eq.(2.19)) which for this case gives ($m=2, s=2$) $\sum_{i=1}^2 \sum_{k=1}^2 \rho_i^{(k)} = 1$. This hypergeometric equation is known to have two independent solutions [24]

$$\begin{aligned} w_1 &= F(\alpha, \beta, \gamma, z) \\ w_2 &= z^{(1-\gamma)} F(1+\alpha-\gamma, \beta-\gamma+1, 2-\gamma, z) \end{aligned} \quad (2.22)$$

Next we would like to discuss the uniqueness properties of the solutions to the Riemann problem [13] . Suppose we have two sets of solutions given by the matrices F and G respectively, then

$$F = RG \quad (2.23)$$

where R is a matrix of rational functions. To prove equation (2.23) it is enough to note that FG^{-1} must be a matrix of analytic functions with (at most) poles at $a_1, \dots, a_s, \infty$ as their only singularities. Hence FG^{-1} must be a matrix of rational functions. Hence the Riemann problem admits m rationally independent solutions. Any other solution is a rational linear combination of those solutions. Clearly, by taking a linear rational combination we do not change the monodromy. It is clear, therefore, that the initial data of the Riemann monodromy problem does not fix the integer part of the singularities of the solutions (see also eqs. (2.13), (2.14) and the discussion which follows).

The next question is how to determine the rational functions. In appendix B we show that these are uniquely determined (up to a multiplication by a constant) once we prescribe the orders of the singularities $\rho_j^{(k)}$. We conclude that there is a unique (up to a multiplication by the constants) solution to a Riemann problem if the leading behaviour (including the integer part) near the singularities $\rho_j^{(k)}$ is given. This proves the statement we made that in order to determine completely the conformal blocks of the CFT the extra data we need, on top of the monodromy data, is the integer part of the dimensions.

In appendix A we prove these statements explicitly for the simple case of the hypergeometric functions. The example of the hypergeometric functions is important also from a physical point of view. It arises in the Knizhnik - Zamolodchikov solution for 4- point correlators of fields belonging to the fundamental representation of $SU(N)$ in the WZW model [4] . (Actually, there are an equal number of fields in the representations N and $\bar{N}$). In this case there are two basic solutions to the Riemann problem $(F_2^{(0)}, F_2^{(1)})$ and $(F_1^{(0)}, F_1^{(1)})$, where we have used here the notations of ref. [4] . (In our notations, see Appendix A, $F_1^{(0)}$ and $F_1^{(1)}$ correspond to (ω_1, ω_2) and $F_2^{(0)}$ and $F_2^{(1)}$ correspond to $(\tilde{\omega}_1, \tilde{\omega}_2)$.)

$$F_1^{(0)} = z^{-2\Delta}(1-z)^{\Delta_1-2\Delta} F(-\frac{1}{2\bar{k}}, \frac{1}{2\bar{k}}, 1 + \frac{N}{2\bar{k}}, z) \quad (2.24)$$

$$F_1^{(1)} = z^{\Delta_1-2\Delta}(1-z)^{\Delta_1-2\Delta} F(-\frac{N-1}{2\bar{k}}, -\frac{N+1}{2\bar{k}}, 1 - \frac{N}{2\bar{k}}, z)$$

and

$$F_2^{(0)} = -(2k+N)z^{1-2\Delta}(1-z)^{\Delta_1-2\Delta} F(1 - \frac{1}{2\bar{k}}, 1 + \frac{1}{2\bar{k}}, 2 + \frac{N}{2\bar{k}}, z) \quad (2.25)$$

$$F_2^{(1)} = z^{\Delta_1-2\Delta}(1-z)^{\Delta_1-2\Delta} F(-\frac{N-1}{2\bar{k}}, -\frac{N+1}{2\bar{k}}, -\frac{N}{2\bar{k}}, z)$$

where $\bar{k} = -\frac{1}{2}(N+k)$, $\Delta = \frac{N^2-1}{2N(N+k)}$, $\Delta_1 = \frac{N}{N+k}$ with k being the level. The four functions given in eqs. (2.24) and (2.25) are just the correlators of fields associated with the fundamental representation of $SU(N)$ in the $SU(N)_k$ WZW model. It is easy to check that the order of singularities of $F_1^{(0)}$ and $F_2^{(0)}$ differ by integer as well as those of $F_1^{(1)}$ and $F_2^{(1)}$.

3. Physical Solutions of the Monodromy Problem and the Problem of Zero Modes

In the previous section we discussed the relevance of the Riemann problem to CFT in general and the problem of calculating the conformal blocks in particular. The Riemann problem characterized by the monodromy data of eq. (2.2) or alternatively eq. (2.6), has

m independent solutions. Our task is to determine those rational linear combinations of solutions which are physical i.e. correspond to blocks of a conformal field theory. We have seen that this demands the knowledge of the behaviour of the blocks, as functions of z , near the singularities. Once this behaviour is known the rational linear combination is uniquely determined. Note that even though we have m independent solutions, it does not mean that we have m physical solutions. The situation is clearly demonstrated in the case of minimal models. The Riemann problem associated with the correlator $\langle \Phi_{2,1}(0)\Phi_{2,1}(z)\Phi_{2,1}(1)\Phi_{2,1}(\infty) \rangle$ in the minimal model has only one physical solution given essentially by eq. (A.7). The other independent solution eq. (A.8) is absent having the wrong order of singularities. On the other hand in the WZW case both solutions given in eqs. (2.24) and (2.25) materialize, i. e. correspond to conformal blocks.

Returning to our problem, we need to know the behaviour near the singularities. We can use the knowledge of the conformal dimensions and OPE

$$\Phi_1(z) * \Phi_2(a) \sim \sum_i \frac{\Phi_3^i}{(z-a)^{\Delta_1+\Delta_2-\Delta_3^i}} (b_0^i + b_1^i(z-a) + \dots) \quad (3.1)$$

$$(b_0^i \neq 0)$$

to argue that the leading singular behaviour of the blocks associated with the correlator $\langle \Phi_1(a)\Phi_2(z)\dots \rangle$ (where the dots stand for other primary fields) is

$$F_i(z) = \frac{b_0^i}{(z-a)^{\Delta_1+\Delta_2-\Delta_3^i}} \quad (3.2)$$

Again as a simple demonstration let us take the minimal models and look at the 4-point correlator $\langle \Phi_{1,2}(0)\Phi_{1,2}(z)\Phi_{1,2}(1)\Phi_{1,2}(\infty) \rangle$. In this case we have two conformal blocks corresponding to the OPE

$$\Phi_{1,2}(z) * \Phi_{1,2}(0) \sim \frac{I}{z^{2\Delta}} + \frac{\Phi_{1,3}(0)}{z^{2\Delta-\Delta_1}} \quad (3.3)$$

where we have written only the leading term associated with each field on the r.h.s.. We have denoted by Δ the dimension of $\Phi_{1,2}$ and by Δ_1 that of $\Phi_{1,3}$. Then we expect the two

blocks to behave near $z \approx 0$ in the following way

$$\begin{aligned} F_1(z) &\sim \frac{1}{z^{2\Delta}} \\ F_2(z) &\sim \frac{1}{z^{2\Delta-\Delta_1}} \end{aligned} \quad (3.4)$$

In a similar way we can determine the leading behaviour of F_1 and F_2 for $z \rightarrow 1, \infty$. Then the physical solution is uniquely determined and we find the Dotsenko-Fateev result [25] (see also eq. (A.7))

$$\begin{aligned} F_1(z) &= \text{const} \cdot z^{1+2a} (z(1-z))^{2\alpha_{1,2}^2} F(-a, 1+a, 2+2a, z) \\ F_2(z) &= \text{const}' \cdot (z(1-z))^{2\alpha_{1,2}^2} F(-a, -1-3a, -2a, z) \end{aligned} \quad (3.5)$$

where $a = 2\alpha_+ \alpha_{1,2} = -\alpha_+^2$, $\alpha_+^2 = \frac{p+1}{p}$ and $c = 1 - \frac{6}{p(p+1)}$. (This result for the conformal blocks was recently derived in ref. [26] in a somewhat different context.)

Thus, it seems that the conformal dimensions and the operator product algebra determine completely the leading behaviour of the blocks and therefore uniquely determine the physical rational combination of the solutions to the Riemann problem. This is certainly the case for the minimal models. However, this statement must be qualified when we have zero modes in the chiral algebra of the underlying CFT. In this case there is a set of fields $\Phi_i^{(m)}$ whose dimensions equal that of the primary (with respect to the extended algebra) Φ_i . The fields $\Phi_i^{(m)}$ are obtained by applying the zero mode operators on the primary field Φ_i (the index m is associated with the zero modes). As an example take the WZW $SU(2)$ model where all fields $\Phi_l^{(m)}$ ($m = -l, -l+2, \dots, l$) have the same dimension as the primary field $\Phi_l^{(l)}$. The fields which are obtained by application of zero modes are not primary with respect to the extended algebra although they are with respect to the Virasoro algebra. They certainly all have the same eigenvalue of L_0 . When we consider the conformal blocks we refer to blocks with definite symmetry i.e. we consider correlators of all these fields $\Phi_l^{(m)}$. When zero modes are present the leading singularity of the blocks can differ from the ones which are naively determined from the OPE of the primary fields. An example is provided by the $SU(N)$ WZW model discussed at the end of the previous section. Naively,

based on the OPE

$$\Phi_N(z) * \Phi_{\tilde{N}}(0) \sim \frac{I}{z^{2\Delta}} + \dots \quad (3.6)$$

we would expect both $F_1^{(0)}$ and $F_2^{(0)}$ to behave as $z^{-2\Delta}$, in contradiction to the behaviour depicted in eq. (2.25) where $F_2^{(0)}$ is shown to behave as $z^{1-2\Delta}$. Note that we deviate from the naive expectation by an integer. This result can be understood in the following way. When no zero modes are present we can carry the analysis based on the OPE of eq. (3.1) and determine the leading singular behaviour. However when zero modes are present we have to consider all fields $\Phi_i^{(m)}$ and therefore the O.P.E.

$$\Phi_1^{(m)}(z) * \Phi_2^{(k)}(a) \sim \sum_{i,l} \frac{\Phi_3^{i(l)}}{(z-a)^{\Delta_1+\Delta_2-\Delta_3^i}} (b_0^{i(mkl)} + (z-a)b_1^{i(mkl)} + \dots) \quad (3.7)$$

where m, k, l are indices related to the zero modes.

While the b_0^i associated with the OPE of primary fields of the extended algebra differ from zero, it may be very well that some of the $b_0^{i(mkl)}$ in eq. (3.7) do vanish. Hence the leading singularity of the blocks ρ does not always coincide with the singularity $\bar{\rho}$ determined from the O.P.E. of the primary fields. All that we know is

$$\rho \geq \bar{\rho} \quad (3.8)$$

When no zero modes are present we have an equality in eq. (3.8) and the knowledge of the conformal dimensions and fusion rules uniquely determine the physical blocks. This may not be the case if there are zero modes and as a result for some of the blocks $\rho > \bar{\rho}$. The knowledge of conformal dimensions and fusion rules is not sufficient and we have to know which of the various values of $b_0^{i(mkl)}$ vanish. This information is equivalent to the knowledge of the full structure of the extended algebra. In section 6 we will return to this more difficult case and suggest a procedure which works at least for the WZW case.

We can now formulate a condition under which the knowledge of the fusion algebra and conformal dimensions of primary fields uniquely determine the physical conformal blocks. Suppose F_j is a set of independent solutions to a given Riemann problem. Using the notation of eq. (2.19), we know that their order of leading singularities $\rho_j^{(i)}$ are constrained by the Fuchs condition (eq. (2.19)). We will refer to this quantity as the second index of

the problem

$$Ind_2 = \sum_{i=1}^s \sum_{j=1}^m \rho_j^{(i)} = (s-1) \frac{m(m-1)}{2} \quad (3.9)$$

Note that this index depends only on the initial data i. e. the number of singular points (s) and the number of functions (m). Next we can perform the same sum for the order of singularities determined naively from the knowledge of the fusion algebra and conformal dimensions (disregarding any complications that may arise from the existence of zero modes). We'll refer to this quantity as the first index of the problem.

$$Ind_1 = \sum_{i=1}^s \sum_{j=1}^m \bar{\rho}_j^{(i)} \quad (3.10)$$

From equation (3.8) we obtain the inequality

$$Ind_1 \leq Ind_2 \quad (3.11)$$

The equality holds if $\rho_j^{(i)} = \bar{\rho}_j^{(i)}$ i. e. the order of singularities of conformal blocks coincide with those obtained by using the fusion algebra and conformal dimensions of primary fields of the extended algebra. Hence, whenever

$$Ind_1 = Ind_2 \quad (3.12)$$

we can determine uniquely the physical blocks. In this case the conformal blocks are the rational linear combinations of the independent solutions of eq. (2.6). Alternatively we can work with eq. (2.17). Among theories for which eq. (3.12) holds we find the minimal models, superconformal models and $Z(N)$ parafermionic theories. The minimal and superconformal models are particular examples of more general coset models which are discussed in section 5. A simple example related to parafermionic theories is given at the end of section 6.

Whenever

$$Ind_1 < Ind_2 \quad (3.13)$$

the situation is more complicated, zero modes are present and we need extra information. Note, however, that there are cases in which we have zero modes and still equation (3.12) holds. Such is the case of the superconformal models.

As a simple example for the case in which eq. (3.13) holds, take the $\langle \Phi_{1/2}(0)\Phi_{1/2}(z)\Phi_{1/2}(1)\Phi_{1/2}(\infty) \rangle$ correlator in the SU(2) WZW model. In this case (see eqs. (2.24), (2.25))

$$\begin{aligned}\bar{\rho}_1^{(1)} &= -2\Delta & \bar{\rho}_2^{(1)} &= \Delta_1 - 2\Delta & (z \rightarrow 0) \\ \bar{\rho}_1^{(2)} &= -2\Delta & \bar{\rho}_2^{(2)} &= \Delta_1 - 2\Delta & (z \rightarrow 1) \\ \bar{\rho}_1^{(\infty)} &= 0 & \bar{\rho}_2^{(\infty)} &= \Delta_1 & (z \rightarrow \infty)\end{aligned}\tag{3.14}$$

and it is easy to check that $Ind_1 = 0$. On the other hand $Ind_2 = 1$ ($s=2$, $m=2$). Hence the two indices differ and the zero modes are indeed present. We shall return to this case in section 6.

4. Formal Solutions of the Riemann Problem

In this section we present the formal solution of the Riemann problem given by Jimbo, Miwa and Sato [16]. The solution is given as an infinite series of contour integrals which can be shown to converge provided the eigenvalues of the monodromy matrices are small. This section does not include any new material and is presented just for the sake of completeness.

The solution given in [16] for the matrix $F(z)$ (eqs. (2.5), (2.6)) has the form

$$F(x_0, z) = 1 + 2\pi i(z - x_0) \sum_{\mu, \nu=1}^s Z_{\mu\nu}(x_0, z)\tag{4.1}$$

where

$$\begin{aligned}Z_{\mu\nu}(x_0, z) &= \delta_{\mu\nu} \int_{-\infty}^0 \int_{-\infty}^0 dx_1 dx_2 \frac{1}{2\pi} \frac{1}{x_0 - x_1 - a_\mu} R(x_1, x_2; L_\nu) \frac{1}{2\pi} \frac{1}{x_2 + a_\nu - z} \\ &+ \sum_{l=1}^{\infty} \sum_{\nu_1, \dots, \nu_{l-1}=1}^s \int_{-\infty}^0 \dots \int_{-\infty}^0 dx_1 \dots dx_{2l+2} \frac{1}{2\pi} \frac{1}{x_0 - x_1 - a_\mu} \\ &R(x_1, x_2; L_\mu) A_{\mu\nu_1}(x_2, x_3) \dots R(x_{2l-1}, x_{2l}; L_{\nu_{l-1}}) \\ &A_{\nu_{l-1}\nu}(x_{2l}, x_{2l+1}) R(x_{2l+1}, x_{2l+2}; L_\nu) \frac{1}{2\pi} \frac{1}{x_{2l+2} + a_\nu - z}\end{aligned}\tag{4.2}$$

and

$$A_{\mu\nu}(x, x') = \frac{1}{2\pi}(1 - \delta_{\mu\nu}) \frac{i}{x + a_\mu - x' - a_\nu - i\epsilon_{\mu\nu}} \quad (4.3)$$

$$R(x, x'; L) = -2i \sin(\pi L) x_-^L x'^L \frac{1}{2\pi} \left(\frac{i}{x - x' + i\epsilon} \exp(-\pi i L) + \frac{-i}{x - x' - i\epsilon} \exp(\pi i L) \right)$$

where $x_-^L = 0$ if $x > 0$ and $|x|^L$ if $x < 0$; $\epsilon_{\mu\nu} = \text{sign}(\mu - \nu)$. For details see [16]. It is interesting to note that the various terms appearing in this solution can be understood as correlation functions of some quantum field theory [16]

$$F(z)_{ij} \sim \langle \psi_i^*(x_0)(z) \varphi(a_1, L_1) \dots \varphi(a_s, L_s) \psi_j(z) \rangle$$

where ψ is an m -component free fermion field and

$$\varphi(a; L) =: \exp\left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dx' \psi(x)^t R(x - a, x' - a; L) \psi^*(x')\right) : \quad (4.4)$$

This observation is quite intriguing. For the time being we lack a deeper understanding of this relationship. For the simple case of the Ising model the appearance of the fermionic field $\psi(z)$ can be associated with the fermionic description of the model.

Another representation of the matrix of solutions $F(z)$ in terms of crossing matrices can be found in [27].

An important property of the solution given in eq. (4.1) is its projective invariance

$$F(h(z); h(a_1) \dots h(a_s)) = F(z, a_1, \dots, a_s) \quad (4.5)$$

where

$$h(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, C)$$

The matrices A_μ which appear in the Fuchsian set of differential equations (2.6) are also projectively invariant as can be easily checked from the following relation:

$$A_\mu = \lim_{z \rightarrow a_\mu} \frac{F^{-1}(z) F'(z)}{z - a_\mu} = \Phi_\mu(a_\mu) L_\mu \Phi_\mu(a_\mu)^{-1} \quad (4.6)$$

where

$$F(z) \xrightarrow{z \rightarrow a_\mu} \Phi_\mu(a_\mu) (z - a_\mu)^{L_\mu}$$

Unfortunately, this formal solution is not a convenient one to work with, involving an infinite series of contour integrals. For the important case of coset models, and in

particular $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$, we shall derive in the next section a more compact representation of the physical solutions in terms of the solutions associated with the various factors.

5. Coset Models

For coset models G/H [3] we are going to assume that the crossing (braid) matrices and, therefore, also the monodromy matrices built out of them factorize. This means that the crossing (braid) matrices are tensor product of the matrices associated with the corresponding G and H factors

$$B = B_G \otimes B_H^+ \quad (5.1)$$

(up to a possible common phase which does not affect monodromy properties). The fact that in eq. (5.1) B_H^+ appears rather than B_H , is due to the fact that the H factor is the one appearing in the denominator of the coset G/H . For the important case of $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ which we are going to consider in this section, the crossing (braid) matrices are tensor products of the matrices associated with the WZW factor

$$B = B_1 \otimes B_2 \otimes B_3^+ \quad (5.2)$$

The following arguments support the factorization assumption for coset models:

1. For minimal models that correspond to the coset $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ it has been shown that there exists factorization between the two energy subalgebras [9,25]. The crossing (braid) matrices associated with the conformal blocks of the correlation functions $G(z)$ of the fields $\Phi_{(n,m)}$ (using the notation of ref. 25)

$$G(z) = \langle \Phi_{(n_1,m_1)}(z_1) \Phi_{(n_2,m_2)}(z_2) \Phi_{(n_3,m_3)}(z_3) \Phi_{(n_4,m_4)}(z_4) \rangle \quad (5.3)$$

factorize in the following way

$$B \begin{pmatrix} (n_1, m_1) & (n_3, m_3) \\ (n_2, m_2) & (n_4, m_4) \end{pmatrix} = B \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} \otimes B \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \quad (5.4)$$

This factorization associated with the energy subalgebras is related (though not identical) to the factorization between $SU(2)_k$ and $SU(2)_{k+1}$ factors of the coset (for fields with integer pseudospins with respect to these factors the $SU(2)_1$ factor is acting trivially).

2. Also for superconformal models, based on the coset $SU(2)_k \times SU(2)_2 / SU(2)_{k+2}$ there is a factorization between the two energy subalgebras [28] .

3. We can compare the Dotsenko-Fateev solution [25] to the Christe-Flume solution [29] for the $SU(2)$ WZW model to establish the factorization of the monodromy matrices. This is done by comparing the non-integer powers of the respective integrands.

4. It is well known that the modular transformation S associated with a coset factorizes [30]

$$S = S_G \otimes S_H^+ \quad (5.5)$$

The well-known Verlinde relation [18] allows us to express the elements A_{ijk} of the fusion algebra of a CFT

$$\Phi_i * \Phi_j = A_{ijk} \Phi_k \quad (5.6)$$

in terms of the elements of the modular matrix, S . To the extent that the A_{ijk} can be expressed in terms of crossing (or monodromy) matrices [31] , [32] , [33] we get a relation between the crossing matrices and the modular matrix S . This relation tells us that the assumption of factorization of the crossing matrices is consistent with the factorization in eq. (5.5).

5. In ref. [34] some arguments were given for the following linear relation between operators of the G , H and G/H models

$$g_i^\Lambda(z) = h_i^\Lambda(z) \Phi_\Lambda^\Lambda(z) \quad (5.7)$$

where g and h are specific fields in the G representation Λ and the H representation λ respectively. This relation implies the following relation between G, H and G/H blocks

$$\langle g_1 g_2 g_3 g_4 \rangle_i = \sum_j C_j \langle h_1 h_2 h_3 h_4 \rangle_j \langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle_{i,j} \quad (5.8)$$

Equation (5.8) implies the factorization of the crossing matrices (eq. (5.1)).

All of these arguments give us confidence in our factorization assumption.

For the coset models based on $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ we know the dimensions (including the integer part) of all branching functions

$$\Delta_{pq}^r = \frac{p(p+2)}{4k(k+2)} + \frac{q(q+2)}{4l(l+2)} - \frac{r(r+2)}{4(k+l)(k+l+2)} + I \quad (5.9)$$

where I is the integer part given in ref. [35]. Moreover, for all those cases in which exact results are known (i.e. correlators of minimal models and of superconformal models) we could check that (see eqs. (3.9),(3.10))

$$Ind_1 = Ind_2 \quad (5.10)$$

We have checked eq. (5.10) for a number of other examples. We are going to assume that equation (5.10) holds for all models based on the $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ coset. Let us note that eq. (5.10) also holds for the $Z(k)$ parafermionic theories based on the coset $SU(2)_k / U(1)$. Thus, according to the general formalism discussed in section 3, the conformal blocks are uniquely determined by our formalism. We can get an explicit representation for the coset blocks in terms of the blocks associated with the various $SU(2)$ WZW factors. As an example let us consider conformal blocks associated with the correlator $\langle \Phi_{p_1 q_1}^{r_1}(0) \Phi_{p_2 q_2}^{r_2}(z) \Phi_{p_3 q_3}^{r_3}(1) \Phi_{p_4 q_4}^{r_4}(\infty) \rangle$. Let us denote as follows the blocks associated with the corresponding blocks of the various $SU(2)$ factors

$$\begin{aligned} f_i^{(a)} &= \langle \Phi_{p_1}(0) \Phi_{p_2}(z) \Phi_{p_3}(1) \Phi_{p_4}(\infty) \rangle_i \\ g_j^{(b)} &= \langle \Phi_{q_1}(0) \Phi_{q_2}(z) \Phi_{q_3}(1) \Phi_{q_4}(\infty) \rangle_j \\ h_k^{(c)} &= \langle \Phi_{r_1}(0) \Phi_{r_2}(z) \Phi_{r_3}(1) \Phi_{r_4}(\infty) \rangle_k \end{aligned} \quad (5.11)$$

where a,b,c refer to definite isospin symmetry and i, j, k enumerate the different blocks associated with the O. P. E. of the primary fields. (The notation $\langle \Phi_{p_1} \Phi_{p_2} \Phi_{p_3} \Phi_{p_4} \rangle_i$ means that the only intermediate states we are considering are descendants of i in the O.P.E. $\Phi_{p_1} * \Phi_{p_2}$ and $\Phi_{p_3} * \Phi_{p_4}$). In the language of the formalism developed in section 2 the indices a,b,c go over the independent physical solutions of the corresponding Riemann problem while the indices i, j, k refer to the different functions forming each solution. Recall that each solution is a set of m functions according to the terminology of section

2. The complete set of solutions to the Riemann problem associated with the coset is, therefore, given by

$$G_{ijk}^{abc} = f_i^{(a)} g^{(b)} \tilde{h}_k^{(c)} \quad (5.12)$$

where $\tilde{h}_k^{(c)}$ is obtained from the expression for $h_k^{(c)}$ 29 by means of the formal substitution $r_i \longrightarrow -r_i$ and $r_i r_j \longrightarrow -r_i r_j$. This is related to the fact that the correlator h is associated with the denominator factor of the coset and in the factorization equation (5.2) this factor comes with B^+ . Since the monodromy properties of G_{ijk}^{abc} in eq. (5.12) are given by eq. (5.2) it follows from the general formalism developed in section 2 that the most general solution can be written in terms of rational linear combinations of these solutions

$$F_{ijk} = \sum_{a,b,c} R^{abc} G_{ijk}^{abc} \quad (5.13)$$

with R^{abc} rational functions. Clearly a similar result will hold for any coset model G/H for which $Ind_1 = Ind_2$. Since we know for the case under study the exact dimensions eq. (5.9), and since eq. (5.10) is assumed to hold, we are guaranteed that the physical blocks are uniquely determined by the monodromy initial data and the dimensions. The representation given in eq. (5.13) constitute the main result of this section. It is a new type of representation, different from the Coulomb gas representation of ref. [25] .

It is also straightforward to show that the F_{ijk} defined in eq. (5.13) satisfy a Fuchsian set of equations (see eq. (2.6)). From the knowledge of the corresponding systems for the different $SU(2)$ WZW factors we can get the Fuchsian differential equations for F_{ijk} . This is obtained by differentiating eq. (5.13) and using eq. (2.6), (2.15) for the various factors

$$\begin{aligned} \frac{dF_{ijk}}{dz} = & \sum_{abc} \left(\frac{dr^{abc}}{dz} G_{ijk}^{abc} + \right. \\ & \left. + (C_{ip}^{(f)} \delta_{js} \delta_{km} + C_{js}^{(g)} \delta_{ip} \delta_{km} + C_{km}^{(h)} \delta_{ip} \delta_{js}) G_{psm}^{abc} r^{abc} \right) \end{aligned} \quad (5.14)$$

Let us illustrate our main result, eq. (5.13), in a particular simple case of the minimal models associated with the coset $SU(2)_{p-2} \times SU(2)_1 / SU(2)_{p-1}$. For this case we consider the blocks associated with the correlator

$\langle \Phi_{11}^0(0)\Phi_{11}^0(z)\Phi_{11}^0(1)\Phi_{11}^0(\infty) \rangle$. There are two blocks associated with the O.P.E.

$$\Phi_{11}^0(z) * \Phi_{11}^0(0) \sim \frac{\Phi_{00}^0}{z^{2\Delta}} + \frac{\Phi_{20}^0}{z^{2\Delta-\Delta_1}} \quad (5.15)$$

Hence using eq. (3.5) the two blocks are given by

$$\begin{aligned} F_1(z) &\sim z^{-1/2-3/2p}(1-z)^{1/2+1/2p} F(1+1/p, -1/p, -2/p, z) \\ F_2(z) &\sim (z(1-z))^{1/2+1/2p} F(1+1/p, 2+3/p, 2+2/p, z) \end{aligned} \quad (5.16)$$

Note that in the notation of Dotsenko and Fateev [25]

$$\Phi_{11}^0 = \Phi_{1,2} \quad \Phi_{20}^0 = \Phi_{1,3} \quad (5.17)$$

and we are really considering fields associated with the energy subalgebra.

According to our result eq. (5.13) we should be able to express F_1 and F_2 in terms of the correlators associated with $\langle \Phi_{1/2}(0)\Phi_{1/2}(z)\Phi_{1/2}(1)\Phi_{1/2}(\infty) \rangle$ of the WZW model given in eqs. (2.24), (2.25). First, let us consider the case of level $k=1$ WZW model $SU(2)_1$. For this case we have only one conformal block (up to a multiplication by a rational function)

$$G(z) \sim \frac{1}{(z(z-1))^{1/2}} \quad (5.18)$$

Next we consider the case $k=p-2$. There are four blocks given by eqs. (2.24), (2.25)

$$F_1^{(0)}(z) = z^{-3/2p}(1-z)^{1/2p} F(1/p, -1/p, 1-2/p, z) \quad (5.19)$$

$$F_1^{(1)}(z) = z^{1/2p}(1-z)^{1/2p} F(1/p, 3/p, 1+2/p, z)$$

and

$$F_2^{(0)}(z) = z^{1-3/2p}(1-z)^{1/2p} F(1+1/p, 1-1/p, 2-2/p, z) \quad (5.20)$$

$$F_2^{(1)}(z) = z^{1/2p}(1-z)^{1/2p} F(1/p, 3/p, 2/p, z)$$

where now $\Delta = 3/4p$, $\Delta_1 = 2/p$ and $2\bar{k} = -(2+k) = -p$.

Note that the non-integer part of the parameters of the hypergeometric functions appearing in $F_1(z)$ and $F_2(z)$ (eq.(5.16)) is equal to the non-integer part of the parameters appearing in $(F_1^{(0)}, F_2^{(0)})$ and $(F_1^{(1)}, F_2^{(1)})$ respectively (eqs. (5.19) and (5.20)). Hence, we can use the Gauss theorem [24] to express the hypergeometric functions appearing in eq. (5.16) in terms of those appearing in eqs. (5.19),(5.20)

$$\begin{aligned} z^{-3/2p}(1-z)^{1/2p} F_1(1+1/p, -1/p, -2/p, z) = \\ = R_1(z)F_1^{(0)}(z) + R_2(z)F_2^{(0)}(z) \end{aligned} \quad (5.21)$$

and

$$\begin{aligned} (z(1-z))^{1/2p} F(1+1/p, 2+3/p, 2+2/p, z) = \\ = \tilde{R}_1 F_1^{(1)}(z) + \tilde{R}_2(z) F_2^{(1)}(z) \end{aligned} \quad (5.22)$$

These relations allow us to express $F_1(z)$ and $F_2(z)$ in the following way.

$$\begin{aligned} F_1(z) &= r_1(z)G(z)F_1^{(0)}(z) + r_2(z)G(z)F_2^{(0)}(z) \\ F_2(z) &= r_3(z)G(z)F_1^{(1)}(z) + r_4(z)G(z)F_2^{(1)}(z) \end{aligned} \quad (5.23)$$

where $r_i(z)$ are rational functions. Equation (5.23) is just the representation of eq. (5.13) for this simple case. Another simple example, associated with the correlator $\langle \Phi_{01}^1 \Phi_{01}^1 \Phi_{01}^1 \Phi_{01}^1 \rangle$ which involves the denominator factor of the coset, is given in Appendix C.

Let us stress the fact that our representation (eq. (5.13)) allows us to reduce the task of computing the correlators (conformal blocks) to that of finding the rational function coefficients. This amounts to solving a system of linear algebraic equations where the unknowns are the constant coefficients appearing in the parametrization we take for the rational functions. The algebraic equations follow by matching the behaviour near the various singularities in equation (5.13).

6. The Case of Zero Modes.

In this section we shall discuss the situation in which zero modes are present and (see eqs. (3.9), (3.10))

$$Ind_1 \neq Ind_2 \quad (6.1)$$

As we have seen in section 3, in this case the knowledge of the dimensions of the primary fields of the extended chiral algebra does not suffice to give us the additional information needed to determine the physical blocks. We need also to know the order of singularities in the O.P.E. of fields created by applying the zero modes on primary fields (see eq. (3.7) and the discussion which followed). This amounts to the knowledge of the chiral algebra itself. Clearly this is a much more difficult situation for which we do not have a full answer. As a result some of the following discussion would be incomplete and more conjectural.

Let $F_j^{(i)}$ be the matrix of fundamental rationally-independent solutions of a given Riemann problem. Under a monodromy transformation around the singular point a_l

$$F_j^{(i)} = M_{jk}^{(l)} F_k^{(i)} \quad (6.2)$$

Correlators of primary fields of the extended algebra are rational linear combinations of these solutions. This holds also for the zero mode descendants which are known to have the same monodromy data. The data encoded in the braid matrices contains the monodromy data. It contains, however, more information. First consider the complete set of physical blocks G_j canonical for the point $a=0$ (i.e. going around $z \approx 0$ results just in a pure phase and no mixing). Now suppose we consider the 4-point correlator $\langle \Phi_1(0)\Phi_2(z)\Phi_3(1)\Phi_4(\infty) \rangle$ and take the special case that the fields Φ_1 and Φ_3 coincide. Then

$$G_i(1-z) = F_{ij} G_j(z) \quad (6.3)$$

where F_{ij} is the corresponding s-t crossing matrix. Next suppose we are dealing with a case in which zero modes are present, such as the WZW models. The blocks have an extra index associated with the zero modes (see discussion in section 3). Equation (6.3) is replaced by

$$G_i^{(a)}(1-z) = \sigma^{(ab)} F_{ij} G_j^{(b)}(z) \quad (6.4)$$

where σ is a matrix which induces the interchange, under duality, of the appropriate tensor structures of the symmetry group. From the point of view of the general formulation of

the Riemann problem given in section 2, the matrix σ acts on the (i) index in eq. (6.2) i.e. on the index which goes over the independent sets of solutions. The matrix σ allows us to relate one set of solutions to another. The σ matrix does not affect the monodromy properties provided $\sigma^2 = 1$. As an example, take the $SU(2)$ WZW case discussed in section 2 (eqs. (2.24), (2.25)) and appendix A. It is straightforward to check using the results of Knizhnik and Zamolodchikov [4] that

$$F_a^{(i)}(1-z) = \sigma_{(ab)} F_b^{ij} F_b^{(j)} \quad (6.5)$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (6.6)$$

(We have used the notation of ref [4] which differs somewhat from ours. See the comment just before eq. (2.24)). Equations (6.3) and (6.4) give a selfconsistency relation which the physical blocks should obey.

Can we go any further in the case when zero modes are present? In what follows we shall present an approach which works for the $SU(2)_k$ WZW case. We believe this approach can be applied also to other cases, although we have not proven it and leave it as a conjecture. The main idea is to try and lift the degeneracy associated with the zero modes thus obtaining a CFT for which $Ind_1 = Ind_2$, the analysis of section 3 holds and the blocks are uniquely determined. Of course we want to lift the degeneracy in such a way that we shall be able to go back and obtain eventually the correlators associated with the original CFT we are interested in.

We start with a CFT with an extended chiral algebra which admits zero modes. Let us denote by A_i the zero mode operators that by acting on a given primary field Φ give its zero mode descendant $\Phi^{(i)}$,

$$\Phi^{(i)} = A_i \Phi \quad (6.7)$$

The operators A_i form an algebra

$$A_i * A_j = \sum \alpha_{ijk} A_k \quad (6.8)$$

The O.P.E. associated with the zero mode descendants is given by

$$A_i \Phi_1(z) * A_j \Phi_2(0) = \sum_{k,p} \alpha_{ijk} A_k \Phi_p(0) / z^\delta \quad (6.9)$$

where p labels all quantum numbers other than those associated with the algebra of eq.

(6.8), and Φ_p are primary fields of the extended algebra. In writing eq. (6.9) we have made the implicit important assumption that the fusion rules over the quantum number associated with the algebra of eq. (6.8) decouple from the fusion rules associated with all other quantum numbers. This means that the dependence on the indices i,j,k in eq. (6.9) is the same, irrespective of the primary fields we start with. This assumption is equivalent to the assumption that the crossing (braid) matrices factorize into $\sigma \otimes B$ as in eq. (6.4).

Next, starting with our original CFT which we denote as $(CFT)_I$ (I stands here for initial), we would like to construct a new CFT whose fusion rules are in one-to-one correspondence with the algebra of eq. (6.8). By this we mean that the primary fields Q_i of this $(CFT)'$ are labelled by the same index i which is associated with the quantum numbers of the algebra in eq. (6.8). Their fusion rules are given, therefore, by eq. (6.8). To the extent that $(CFT)'$ really exists and we can identify it, we can now construct yet another conformal field theory which we shall denote as $(CFT)_F$ (F stands for final). The primary fields of $(CFT)_F$ are

$$\varphi_p^i(z) = \Phi_p^{(i)}(z)Q_i(z) \quad (6.10)$$

The corresponding correlation functions have the factorized form

$$\langle \varphi_{p_1}^{i_1} \dots \varphi_{p_n}^{i_n} \rangle = \langle \Phi_{p_1}^{(i_1)} \dots \Phi_{p_n}^{(i_n)} \rangle \langle Q_{i_1} \dots Q_{i_n} \rangle \quad (6.11)$$

We would like to construct $(CFT)'$ in such a way that both $(CFT)'$ and $(CFT)_F$ do not have zero modes and, therefore, $Ind_1 = Ind_2$ for both of them. Hence, as far as these two theories are concerned we are back in the situation discussed in section 3 and the corresponding correlators are uniquely determined. If the whole program works throughout, then eq. (6.11) allows us to obtain the correlators of $(CFT)_I$ from those of $(CFT)_F$. Let us stress that we have not proven that the whole construction really works. In particular we have not proven that it is always possible to identify and to prove the existence of $(CFT)'$.

We would like now to show that the whole procedure works for the $SU(2)$ WZW case. We start with $(CFT)_I$ which is the $SU(2)$ WZW model. The primary fields are $[5]$, $[36]$ Φ_l^i where the lower index refer to the pseudospin and the upper index to its projection on the third axis. The operators A^i are

$$A^i = (J_0^-)^i$$

(the $(-)$ index indicates that we are looking at the lowering pseudospin operator, and the 0 index that we are looking at the zero mode component of the current). The zero mode

descendant fields are

$$\Phi_l^{(m)} = (J_0^-)^m \Phi_l^0 \quad (6.12)$$

The fusion algebra associated with the quantum number m , the pseudospin projection on the third axis, is

$$\Phi_{l_1}^{(m_1)} \times \Phi_{l_2}^{(m_2)} = \sum_k \Phi_{l_k}^{(m_1+m_2)} \quad (6.13)$$

Indeed in this case the fusion rules associated with the quantum number m decouple from the fusion rules associated with l . (See eq. (6.9) and the discussion which followed).

Next we should construct $(CFT)^l$, with primary fields Q^m which have the fusion algebra

$$Q^{m_1} \times Q^{m_2} = Q^{m_1+m_2} \quad (6.14)$$

But this is just the fusion algebra of a free bosonic theory [5,36]

$$Q^m(z) \sim \exp(-im\chi(z)/\sqrt{k}) \quad (6.15)$$

with χ a free bosonic field. Hence in this case $(CFT)^l$ is a simple theory and it is straightforward to calculate its correlators.

Now we can identify our $(CFT)_F$ whose primary fields are

$$\varphi_l^m = \Phi_l^{(m)} Q^m \quad (6.16)$$

and which does not have zero modes. This is nothing but the parafermionic $Z(k)$ theory. This construction of the parafermion theory is, of course, closely related to the fact that the $Z(k)$ parafermionic theory can be constructed as the coset $SU(2)_k/U(1)$ [5,36].

As a simple example consider the correlator $\langle \sigma_1^+(0) \sigma_1(z) \sigma_1(1) \sigma_1^+(\infty) \rangle$. In the notation of ref. [36]

$$\sigma_1 = \Phi_1^1 \quad \sigma_1^+ = \Phi_1^{-1} \quad (6.17)$$

and we have the following fusion algebra

$$\begin{aligned} \sigma_1(z) * \sigma_1^+(0) &\sim \Phi_2^0 + \Phi_0^0 \\ \sigma_1(z) * \sigma_1(0) &\sim \Phi_0^2 + \Phi_2^2 \end{aligned} \quad (6.18)$$

The degree of singularity, denoted by $\bar{\rho}$ in section 3, can be easily computed using the formulae for the dimensions given in ref. [5,36]. We get the following matrices whose eigenvalues determine $\bar{\rho}$.

$$\begin{aligned}
z \rightarrow 0 & \quad \begin{pmatrix} -2h_1 & 0 \\ 0 & h_2^0 - 2h_1 \end{pmatrix} \\
z \rightarrow 1 & \quad \begin{pmatrix} h_0^2 - 2h_1 & 0 \\ 0 & h_2^2 - 2h_1 \end{pmatrix} \\
z \rightarrow \infty & \quad \begin{pmatrix} 0 & 0 \\ 0 & h_2^0 \end{pmatrix}
\end{aligned} \tag{6.19}$$

where h_j^i is the conformal dimension of the field Φ_j^i

$$\begin{aligned}
h_0^0 &= 0 \\
h_1 &= h_1^1 = h_1^{-1} = \frac{k-1}{k(k+2)} \\
h_2^2 &= \frac{k-2}{k(k+2)} \\
h_0^2 &= 1 - 1/k \\
h_2^0 &= \frac{2}{k+2}
\end{aligned} \tag{6.20}$$

It is straightforward to check that

$$Ind_1 = Ind_2 = 1 \quad (s=2, m=2) \tag{6.21}$$

Hence, we are indeed in a situation for which our analysis of section 3 holds. It is easy to see that the solution of the corresponding Riemann problem is given by the hypergeometric functions (see eq. (5.20) of ref. [5]). Then using eq. (6.11) (which in our case corresponds to eq. (5.12) of ref.[5]), we can reconstruct the corresponding SU(2) WZW correlators.

7. Summary and conclusions

In the present paper we address the question to what extent the knowledge of the crossing (braid) matrices associated with a given fusion algebra determines the correlators (conformal blocks) of the theory on the sphere. The main mathematical tool we use is the Riemann monodromy problem on the sphere. The knowledge of the crossing (braid) matrices defines the monodromy data associated with the n -point correlators (blocks) of the theory. The monodromy data depends only on the non-integer part of the conformal dimensions. Moreover, by applying this approach to CFT we get right away constraint equations on the non-integer part of the conformal dimensions (see eq.(2.14)). The Riemann monodromy problem with m unknown functions usually has m sets of rationally independent solutions. Actually the theory of the Riemann monodromy problem assures us of the existence of m independent solutions only if all eigenvalues of the monodromy matrices do not differ by integers. Otherwise some of the solutions may turn out to be linearly dependent and the complete set involves solutions with logarithmic singularities on top of those with the power behaved singularities. This makes e.g. the calculation of correlators of primary fields whose dimensions differ by integers a much more difficult task [25] .

Once we have the m fundamental sets of solutions the main problem we face is to find those rational linear combinations which correspond to physical blocks. The extra information needed is the singular behaviour of the blocks near the singular points. Naively we expect to obtain this information from the knowledge of the conformal dimensions (including integer part!) of the primary fields. This is indeed the case for theories for which the underlying chiral algebra does not have zero modes. Once zero modes are present the situation is usually much more complicated. We have established a criterium i. e. $Ind_1 = Ind_2$ (see eqs. (3.9), (3.10)) which determine whether the information contained in the dimensions of the primary fields is sufficient for the unique determination of the blocks. (Note that $Ind_1 = Ind_2$ can hold also when zero modes are present as in the Ramond sector of the superconformal models). The set of theories for which $Ind_1 = Ind_2$ for all correlators includes minimal models, superconformal models and parafermionic models. The first two sets of theories are particular cases of coset models based on the coset $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ for which we believe the criterium holds and for which we know the exact dimensions of all branching functions. We have given several arguments to support our assumption that the crossing (braid) matrices for a coset model G/H factorize i.e. $B_G \otimes B_H^+$. To the extent that this factorization holds, we can give for these coset

models G/H which satisfy $Ind_1 = Ind_2$ a representation of the correlators (blocks) in terms of the rational linear combinations of correlators (blocks) associated with the G and H factors (see eq. (5.13)). The rational function coefficients are uniquely determined from the knowledge of the conformal dimensions. Now the problem of determining the correlators (blocks) for the coset models reduces to finding a solution to a set of linear algebraic equations, provided that the correlators (blocks) associated with the G and H factors are known. These linear algebraic equations arise when we determine the rational coefficient functions by matching the correct order of singularities on both sides of eq. (5.13). The constant coefficients in the parametrization of the rational functions are the unknowns in the resulting set of algebraic equations.

When $Ind_1 \neq Ind_2$ zero modes are present and the situation is much more complicated. We have made a conjecture (section 6) that in these cases one can always go from the initial $(CFT)_I$ to some final $(CFT)_F$ for which $Ind_1 = Ind_2$ and our analysis holds. If the construction we propose holds, then the knowledge of the correlators of $(CFT)_F$ and of $(CFT)'$ allows us to determine the correlators of $(CFT)_I$. The idea is to do some "coset construction" where the numerator is associated with $(CFT)_I$ and the denominator associated with the algebra satisfied by the zero mode "lowering operators", i. e. all zero mode operators that by acting on a given primary field give all its zero mode descendants. We have shown that this procedure works for the $SU(2)_k$ WZW model where $(CFT)_F$ is found to be the $Z(k)$ parafermionic theory. It remains an open question how general this approach is and whether it (or some generalization of it) can be applied to other cases.

We would like to end this paper by recapitulating some of the open problems we left in our analysis and further questions for future investigations.

1. Give a complete proof for the factorization of crossing (braid) matrices for coset models. In view of all the arguments we presented in section 5 we believe that the factorization really holds and probably it is not too difficult to prove it.

2. Can we always apply the approach of section 6 (or some modification of it) to handle the more difficult case when zero modes are present? Although our approach works for the $SU(2)_k$ WZW models, we do not have any proof for its applicability in general.

3. The general theory of the Riemann monodromy problem allows us, as we have seen in section 2, to rephrase the problem in terms of a set of differential equations (eq. (2.6)). We would like to understand the role of these equations better. In particular, can we learn from them directly about the existence of some extended algebra? Are those differential equations related in some way to the constraint equations associated with the

existence of null vectors? What is the relation (if any) between these equations and the differential equations related to the Ward identities which were discussed in [20] , [37] . Clearly having a better understanding of the underlying chiral algebra will enable us to construct the whole Hilbert space.

4. Can we extend the approach of this paper and apply it to correlators on higher genus Riemann surfaces? This implies that we should be able to solve the Riemann problem on higher genus surfaces. (For the mathematical literature on this item see [15]).

5. We have not used modular invariance in all our approach. Clearly modular invariance will play a role once we generalize the approach to higher genus surfaces. As far as the sphere is concerned we have not relied directly on modular invariance. Maybe the role of modular invariance is to fix the integer part of the dimension which is outside the monodromy data.

Appendix A

In this Appendix we consider the case of the Riemann monodromy problem with 3 singular points (a_1, a_2, a_3) which can be chosen to be $(0, 1, \infty)$, and two unknown functions. In the terminology of section 2, $s=2$ and $m=2$. Note that the number of free parameters in this case is $m^2(s-1)+1=5$. (We can always diagonalize one of the monodromy matrices M_i and make all entries in the first row of another matrix equal. See ref. [22] for details). In this particular case this is also the number of independent $\rho_j^{(i)}$ (recall that the $\rho_j^{(i)}$ satisfy the Fuchs condition (eq. (2.19)) which in our case gives 1).

We expect from the general theory of the Riemann problem to find two independent set of solutions with each set having two functions whose monodromy we prescribe. For the case we consider eq. (2.8) has the following form

$$w'' + \left(\sum_{i=1}^3 \frac{A_i}{z - a_i} \right) w' + \left(\sum_{i=1}^3 \frac{B_i}{z - a_i} \right) \frac{1}{\prod_{i=1}^3 (z - a_i)} w = 0 \quad (A.1)$$

where

$$\begin{aligned} A_i &= 1 - (\rho_1^{(i)} + \rho_2^{(i)}) \\ B_i &= \rho_1^{(i)} \rho_2^{(i)} (a_i - a_{i-1})(a_i - a_{i+1}) \end{aligned} \quad (A.2)$$

The $\rho_j^{(i)}$ are the orders of the singularities at the point a_i . Taking $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$ and making the substitution

$$w = z^{\rho_1^{(1)}} (z - 1)^{\rho_1^{(2)}} \hat{w} \quad (A.3)$$

we obtain the following equation for $\hat{w}$

$$z(1-z)\hat{w}'' + (\gamma - (1+\alpha+\beta)z)\hat{w}' - \alpha\beta\hat{w} = 0 \quad (A.4)$$

with α, β, γ determined in terms of the $\rho_j^{(i)}$

$$\begin{aligned} \alpha &= \rho_1^{(\infty)} + \rho_1^{(1)} + \rho_1^{(2)} \\ \beta &= \rho_2^{(\infty)} + \rho_1^{(1)} + \rho_1^{(2)} \\ 1 - \gamma &= \rho_2^{(1)} - \rho_1^{(1)} \end{aligned} \quad (A.5)$$

and where we have used the Fuchs condition (eq. (2.19)).

Equation (A.4) is the hypergeometric equation and is known to have two rationally independent solutions

$$\begin{aligned}\hat{w}_1 &= F(\alpha, \beta, \gamma, z) \\ \hat{w}_2 &= z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)\end{aligned}\tag{A.6}$$

Using eq. (A.3) we find two independent solutions of eq. (A.1)

$$\begin{aligned}w_1 &= z^{\rho_1^{(1)}} (z-1)^{\rho_1^{(2)}} F(\alpha, \beta, \gamma, z) \\ w_2 &= z^{\rho_2^{(1)}} (z-1)^{\rho_2^{(2)}} F(1-\alpha, 1-\beta, 2-\gamma, z)\end{aligned}\tag{A.7}$$

where, in order to derive the last equation, we have used well-known properties of hypergeometric functions [24]. These are solutions to the Riemann monodromy problem. The corresponding expressions for the monodromy matrices can be found in refs. [21, 22].

We can write down another two solutions which have the same monodromy properties as those of eq. (A.7), but are rationally independent.

Take $\tilde{\rho}_1^{(\infty)} = \rho_1^{(\infty)} + 1$ and $\tilde{\rho}_2^{(\infty)} = \rho_2^{(\infty)} - 1$. Then we have

$$\begin{aligned}\tilde{w}_1 &= z^{\tilde{\rho}_1^{(1)}} (z-1)^{\tilde{\rho}_1^{(2)}} F(\alpha+1, \beta-1, \gamma, z) \\ \tilde{w}_2 &= z^{\tilde{\rho}_2^{(1)}} (z-1)^{\tilde{\rho}_2^{(2)}} F(-\alpha, 2-\beta, 2-\gamma, z)\end{aligned}\tag{A.8}$$

Any other system of functions $(\bar{w}_1, \bar{w}_2)$ which has the same monodromy properties can be expressed as rational combinations of the solutions given in eqs. (A.7), (A.8), i.e.

$$\begin{aligned}\bar{w}_1 &= R_1 w_1 + \tilde{R}_1 \tilde{w}_1 \\ \bar{w}_2 &= R_2 w_2 + \tilde{R}_2 \tilde{w}_2\end{aligned}\tag{A.9}$$

where $R_1, R_2, \tilde{R}_1, \tilde{R}_2$ are rational functions. The proof of eq. (A.9) is straightforward and relies on the Gauss theorem [21] which tells us that all hypergeometric functions which have the form $F(a+m, b+n, c+l, z)$ with m, n, l integers can be expressed as rational combinations of any two of them. Clearly all solutions with the same monodromy have the same non-integer part of $\rho_j^{(i)}$ and hence the corresponding α, β, γ can differ only by integers.

Appendix B

In this Appendix we give a proof that the knowledge of the order of singularities $\rho_j^{(i)}$ of the solution to the Riemann monodromy problem determine the rational combination of fundamental solutions uniquely.

Suppose f_j and g_j are two sets of solutions with exactly the same $\rho_j^{(i)}$ data. The $\rho_j^{(i)}$ obey the Fuchs condition (eq. (2.19))

$$\sum \rho_j^{(i)} = (s-1) \frac{m(m-1)}{2} \quad (B.1)$$

Note that the r.h.s. of eq. (B.1) depends only on s and m . Near the point a_1

$$\begin{aligned} f_j &\sim (z - a_1)^{\rho_j^{(1)}} (c_0^j + c_1^j(z - a_1) + \dots) \\ g_j &\sim (z - a_1)^{\rho_j^{(1)}} (b_0^j + b_1^j(z - a_1) + \dots) \end{aligned} \quad (B.2)$$

where c_0^j, b_0^j are constants different from zero. Then it is clear that the set of functions

$$h_j = b_0^j f_j - c_0^j g_j \quad (\text{no summation on } j) \quad (B.3)$$

is also a solution to the Riemann monodromy problem.

However,

$$h_j \sim (z - a_1)^{\rho_j^{(1)}+1} (d_1^j + \dots) \quad (B.4)$$

Hence the order of singularity ρ^h of h_j satisfies

$$\rho_j^{h(1)} \geq \rho_j^{(1)} + 1 \quad (B.5)$$

and similarly

$$\rho_j^{h(i)} \geq \rho_j^{(i)} \quad (B.6)$$

for all other singular points a_j . Thus

$$\sum_{j=1}^m \sum_{i=1}^s \rho_j^{h(i)} > \sum_{j=1}^m \sum_{i=1}^s \rho_j^{(i)} \quad (B.7)$$

But the two sides of eq. (B.7) are given by the Fuchs condition (eq. (2.19)) which depends just on s and m . Therefore, the two sides of eq. (B.1) should be equal. The contradiction implies that $h_j=0$. This means that g_j and f_j are proportional.

Appendix C

Consider the 4-point correlator

$$\langle \Phi_{01}^1(0) \Phi_{01}^1(z) \Phi_{01}^1(1) \Phi_{01}^1(\infty) \rangle$$

(or $\langle \Phi_{2,1} \Phi_{2,1} \Phi_{2,1} \Phi_{2,1} \rangle$ in the notation of ref. [25]). Recall that the fields entering this correlator have pseudospins $p_i=0$, $q_i=r_i=1/2$ with respect to the $SU(2)_{p-2}$, $SU(2)_{p-1}$ and $SU(2)_1$ factors which appear in the GKO coset construction $SU(2)_{p-2} \times SU(2)_1 / SU(2)_{p-1}$ of the minimal models. Note that the fields that enter this correlation function have non-zero pseudospin with respect to the denominator factor. We give this example in addition to the example given in section 5 to illustrate further the general formalism developed there.

There are two conformal blocks that contribute to this correlation function [25]:

$$F_1(z) = z^{-1/2+3/2(p+1)} (1-z)^{1/2-1/2(p+1)} F\left(1 - \frac{1}{p+1}, \frac{1}{p+1}, 1 + \frac{2}{p+1}, z\right) \quad (C.1)$$

$$F_2(z) = z^{1/2-1/2(p+1)} (1-z)^{1/2-1/2(p+1)} F\left(1 - \frac{1}{p+1}, 2 - \frac{3}{p+1}, 2 - \frac{2}{p+1}, z\right)$$

On the other hand we know the expressions for the conformal blocks related to the correlator $\langle \Phi_{1/2} \Phi_{1/2} \Phi_{1/2} \Phi_{1/2} \rangle$ of the $SU(2)_{p-1}$ WZW model (see eqs. (5.19), (5.20)). We use their integral representations [29] :

$$\begin{aligned} F_1^{(0)}(z) &= z^{\frac{2j_1 j_2}{p+1}} (1-z)^{\frac{2j_1 j_2}{p+1}} \int_0^z dt t^{-\frac{2j_1}{p+1}-1} (t-1)^{\frac{-2j_1}{p+1}} (t-z)^{\frac{-2j_2}{p+1}} \\ F_1^{(1)}(z) &= z^{\frac{2j_1 j_2}{p+1}} (1-z)^{\frac{2j_1 j_2}{p+1}} \int_1^\infty dt t^{-\frac{2j_1}{p+1}-1} (t-1)^{\frac{-2j_1}{p+1}} (t-z)^{\frac{-2j_2}{p+1}} \end{aligned} \quad (C.2)$$

and

$$\begin{aligned} F_2^{(0)}(z) &= z^{\frac{2j_1 j_2}{p+1}} (1-z)^{\frac{2j_1 j_2}{p+1}} \int_0^z dt t^{-\frac{2j_1}{p+1}} (t-1)^{\frac{-2j_2}{p+1}-1} (t-z)^{\frac{-2j_1}{p+1}} \\ F_2^{(1)}(z) &= z^{\frac{2j_1 j_2}{p+1}} (1-z)^{\frac{2j_1 j_2}{p+1}} \int_1^\infty dt t^{-\frac{2j_1}{p+1}} (t-1)^{\frac{-2j_2}{p+1}-1} (t-z)^{\frac{-2j_1}{p+1}} \end{aligned} \quad (C.3)$$

where $j_i = 1/2$ ($i = 1, 2, 3, 4$) for the case under study.

We have to modify these expressions to get the building blocks for the coset correlation functions. As it was described in section 5 this modification amounts to the change $j_i \rightarrow -j_i$, $j_i j_k \rightarrow -j_i j_k$ in the integral representations of ref. 29. The resulting expressions can be rewritten easily in terms of the hypergeometric functions.

$$\begin{aligned}\tilde{F}_1^{(0)} &= z^{3/2(p+1)}(1-z)^{-1/2(p+1)} F\left(\frac{-1}{p+1}, \frac{1}{p+1}, 1 + \frac{2}{p+1}, z\right) \\ \tilde{F}_1^{(1)}(z) &= z^{-1/2(p+1)}(1-z)^{-1/2(p+1)} F\left(\frac{-1}{p+1}, \frac{-3}{p+1}, 1 - \frac{2}{p+1}, z\right)\end{aligned}\tag{C.4}$$

and

$$\begin{aligned}\tilde{F}_2^{(0)} &= z^{1+3/2(p+1)}(1-z)^{-1/2(p+1)} F\left(1 - \frac{1}{p+1}, 1 + \frac{1}{p+1}, 2 + \frac{2}{p+1}, z\right) \\ \tilde{F}_2^{(1)} &= z^{1-1/2(p+1)}(1-z)^{-1/2(p+1)} F\left(\frac{-1}{p+1}, \frac{-3}{p+1}, -\frac{2}{p+1}, z\right)\end{aligned}\tag{C.5}$$

Comparing eqs. (C.4) and (C.5) with eq. (C.1) and using the Gauss theorem [24] we immediately convince ourselves that $F_1(z)$ and $F_2(z)$ are rational linear combinations of $G(z)\tilde{F}_j^{(i)}$ where $G(z)$ is associated with the $SU(2)_1$ factor in the coset construction and is given by eq. (5.18):

$$\begin{aligned}F_1(z) &= G(z)R_1(z)\tilde{F}_1^{(0)}(z) + G(z)R_2(z)\tilde{F}_2^{(0)}(z) \\ F_2(z) &= G(z)R_3(z)\tilde{F}_1^{(1)}(z) + G(z)R_4(z)\tilde{F}_2^{(1)}(z)\end{aligned}\tag{C.6}$$

This is just the representation (5.13) for the correlation function

$$\langle \Phi_{01}^1 \Phi_{01}^1 \Phi_{01}^1 \Phi_{01}^1 \rangle.$$

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FIGURE CAPTIONS

Fig. 1 : Crossing operations for the 4-point conformal blocks.

Fig. 2 : The notation for the 4-point correlation function.

Fig. 3 : Monodromy transformation of the 5-point conformal block.

$$\begin{array}{c} j \\ | \\ i \text{---} \text{---} p \text{---} \text{---} k \\ | \\ p \end{array} \quad \ell = \sum_q F_{pq} \left[\begin{array}{cc} j & k \\ i & p \end{array} \right] \quad \begin{array}{c} j & k \\ & \diagdown \quad \diagup \\ & \text{---} q \text{---} \\ | \\ i \text{---} \text{---} p \end{array}$$

$$\begin{array}{c} j \\ | \\ i \text{---} \text{---} p \text{---} \text{---} k \\ | \\ p \end{array} \quad \ell = \sum_{p'} B_{pp'} \left[\begin{array}{cc} j & k \\ i & p \end{array} \right] \quad \begin{array}{c} k & j \\ | & | \\ i \text{---} \text{---} p' \text{---} \text{---} p \end{array}$$

Fig. 1

Fig. 2

Fig. 3