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STATUS OF LATTICE QCD AT FINITE TEMPERATURES

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ABSTRACT

The status of lattice QCD studies at finite temperatures as of fall 1988 is reviewed with emphasis on the chiral phase transition with light dynamical quarks. Developments over the past year concerning the order and the critical temperature of the chiral transition as well as the dynamics of the high temperature phase are surveyed. The problem of the finite baryon number density and the recently reopened issue of the order of the pure gauge deconfinement transition are summarized.

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results and the degree of improvement needed to reach reliable quantitative predictions.

Studies of the finite temperature behavior cannot be separated from those at zero temperature. For example, the value of the lattice spacing, important for setting the physical scale, requires hadron mass calculations at zero temperature. Progress also depends on the technical aspects such as simulation algorithms and computer technologies. We cannot hope to cover these areas in any depth in this review. For the recent progress on such matters and also in lattice QCD in general, the proceedings of the annual meeting on lattice gauge theories which was held this year at FNAL^{10,11} just before this conference provide an important source of up-to-date reviews and references. There are also detailed review articles^{12,13} including the basic techniques of lattice QCD.

We start our survey with the phase diagram in Sec.2. This is followed by a discussion on the physics characteristics of the high temperature plasma phase (Sec. 3) and a brief comment on the problem of finite baryon number density (Sec. 4). In Sec.5, we summarize the recent controversy on the order of the deconfinement transition in the pure gauge theory. We conclude with a brief summary in Sec. 6.

We close the introduction with a few technical words. The lattice size will be denoted as $N_s \times N_t$ with N_s and N_t the number of lattice sites in the spatial and temporal directions, and the lattice spacing as a . The physical temperature is given by $T = 1/(N_t a)$. With $\beta = 6/g^2$ the inverse (bare) coupling constant squared of QCD, the continuum limit is $\beta \rightarrow \infty$ simultaneously with $a \rightarrow 0$, keeping T fixed. At a fixed N_t , a weaker coupling (i.e., larger β) corresponds to a higher temperature. The quark mass (in physical units) is written as m_q , and the number of degenerate flavors as N_f . Unless specified explicitly, we only discuss studies using the Kogut-Susskind form¹⁴ of the quark action. The results^{15,16} with the Wilson quark action¹⁷ are problematical and not well understood.

2. Phase Diagram.

2.1 Qualitative characteristics.

The phase structure of QCD as a function of the temperature (or of β) and the quark mass m_q has been examined extensively. Theoretical considerations¹⁸ indicate that there is a first order deconfining phase transition in the pure gauge limit $m_q = \infty$ which, however, weakens¹⁹ as the quark mass decreases. Bulk of the pure gauge simulations support a first order deconfinement, and its weakening has been clearly seen in several full QCD simulations^{2,7,20}. (The first order nature of the pure gauge transition has recently been questioned²¹. We shall come back to this point in Sec. 5).

The weakening continues down to the quark mass of order $m_q \sim T$. Further decreasing the quark mass, however, the transition has been found to turn stronger again^{2-8,22-24} for $N_f \geq 2$. An important characteristic of the transition is an increasingly discontinuous drop of the chiral order parameter $\langle \bar{\chi}\chi \rangle$ across the transition for smaller m_q . Its extrapolation along the high temperature side toward the chiral limit is consistent with $\langle \bar{\chi}\chi \rangle = 0$ at

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Abstract

The status of lattice QCD studies at finite temperatures as of fall 1988 is reviewed with emphasis on the chiral phase transition with light dynamical quarks. Developments over the past year concerning the order and the critical temperature of the chiral transition as well as the dynamics of the high temperature phase are surveyed. The problem of the finite baryon number density and the recently reopened issue of the order of the pure gauge deconfinement transition are summarized.

1. Introduction.

In his review¹ last summer at Quark Matter 87, Frithjof Karsch summarized the lattice QCD studies at finite temperatures over the previous year as a period of transition in which the focus had shifted from the pure gauge system to full QCD including light dynamical quarks. Due to the development of efficient dynamical fermion algorithms and wider accessibility of supercomputer facilities, the progress in full QCD calculations had indeed been quite rapid over this period, and many suggestive results had been obtained: To list a few, there were

- evidences²⁻⁸ for a first-order chiral phase transition, including the physically relevant case of two light flavors with the critical temperature perhaps as low as 150 MeV^{6,7},
- indications^{7,5} for a large energy density just above the critical point, and a non-ideal gas behavior of the pressure⁷,
- suggestions⁹ for possible existence of heavy modes with hadronic quantum numbers.

Compared to the excitement over this period, the development during the past year has been more sober. In fact it has mainly been spent in attempting to consolidate and refine our understanding through simulations on a larger lattice with better statistics and more quantitative methods of analyses. New results have been obtained in the course of these studies and, as often happens, more problems to be resolved have been exposed.

In this review we survey the main features of the lattice results on finite temperature QCD in the light of the recent developments, placing emphasis on full QCD with light dynamical quarks. In particular we shall try to examine the limitations of current simulation

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$m_q = 0$, while it stays non-vanishing on the low temperature side^{2,7,23}. This shows that the transition for small m_q is characterized by chiral symmetry and that this symmetry is restored in the high temperature phase, as expected from theoretical analyses²⁵. The mechanism of transition is therefore different from the region of heavy quark where it is characterized by the $Z(3)$ symmetry²⁶ of the gauge action and quarks only obscure it by breaking it explicitly¹⁹. The simulation results show that the two transitions, though distinct in character, are smoothly joined in the region of intermediate quark mass $m_q \sim T$. There is only a single line of transition which runs across the phase diagram as m_q is varied.

2.2 Order of the chiral transition.

Let us call the transition for small quark mass as chiral transition. We have so far been using the term "transition" in a loose sense; a rapid change of the observables as the coupling constant is varied over a short interval. An important question is whether the chiral transition is a genuine phase transition with a singularity at the critical temperature, and if so whether it is a first order one. Its effects will not stand out in reality unless it is a first order phase transition with at least a moderate discontinuity.

This used to be a controversial point. Most of the recent simulations conclude, however, that the chiral transition is first order for $N_f \geq 2$ for sufficiently small quark mass (ref. 6, 20, 22-24 for $N_f = 2$, ref. 8, 22 for $N_f = 3$, and ref. 2-5, 7, 20, 23 for $N_f = 4$). This is based on the observation of signatures for a coexistence of two distinct thermodynamic states close to the transition. The signals taken as support for first order are: i) Starting from hot or cold configurations, the values of observables remain apart for a sufficiently long simulation time. ii) Over a very long time, however, the system jumps back and forth (flip-flop behavior) between the two states with a relatively short transition time. iii) The distribution of observables show a two-peak structure corresponding to the two states. All dynamical fermion algorithms in use today have reported such evidences for the quark mass smaller than $m_q/T \sim 0.1$ for a variety of lattice sizes ranging from $4^3 \times 4$ to $12^3 \times 4$ and also $10^3 \times 6$.

There are two questions which should be clarified before the issue of the order could be considered settled: i) Does the chiral transition cease to be first order above a certain quark mass? ii) How would the first order signatures change with an increasing spatial volume V ?

On the first question the results differ depending on the algorithm: For $N_f = 2$, the hybrid algorithm²⁷ sees clear first order signatures only up to $m_q/T = 0.05^{24}$ (much less clear signal at $m_q/T = 0.4^{28}$), while the Langevin method²⁹ finds evidence^{6,20} already at $m_q/T = 0.4$. For the case of $N_f = 4$, a similar difference is found for the hybrid^{4,5} and Langevin²⁰ algorithms, and the recently developed hybrid Monte Carlo algorithm³⁰ have pushed the value as high as $m_q/T = 2.0^{31}$.

It is desirable to settle the difference for two reasons. Firstly one would like to know if the realistic light quark mass of a few MeV is still in the first order range. Secondly if a smoothing does occur, there will be a singularity-free region in the phase diagram connecting

the low and high temperature parts. The hadronic excitations of the low-temperature phase may then be analytically continued through this region into the high temperature phase, leading to a possible existence of modes with hadronic quantum numbers above the chiral phase transition⁹. At present the answer to the first point is marginally affirmative; for $N_f = 2$, with a critical temperature of $\sim 150 - 200$ MeV (see the next section), the largest value $m_q/T = 0.05$ agreed upon by all the algorithms corresponds to $m_q \sim 10$ MeV. We should remember, however, that the transition remains sharp ($\delta T_c/T_c < \text{a few \%}$) up to $m_q/T \sim 0.4$ even if some of the algorithms have not found clear first order signatures. On the second point, there is a consensus for $N_f = 2$ that no clear sign of discontinuity is seen at $m_q/T \sim 1$. An agreement is absent for $N_f = 4$.

The question ii) above originates from the necessity of turning the rather qualitative first order criteria used so far into a more quantitative analysis for establishing the order of the chiral transition. General theoretical considerations³² show that a first order transition exhibits a specific finite size scaling behavior as the spatial volume V is increased. At a finite volume, the discontinuous jumps of observables are rounded off into a sharp but continuous rise over a finite width. With $V \rightarrow \infty$, the location of this transition shifts by an amount proportional to $1/V$ and the width shrinks as $1/V$. A first-order chiral transition should quantitatively follow this pattern as one increases the spatial volume.

Several recent studies^{31,28,33} with $N_t = 4$ and $N_s = 4 - 12$ attempting such an analysis in connection with question i) have exposed a rather unexpected trend illustrated in fig. 1. With an increasing spatial volume, the two peaks of the distribution of the chiral order parameter appear to be coming closer and the valley in between becoming less deep. (The tendency is not clear for the Wilson and Polyakov loop.) This is just the opposite of a naive expectation that a first order signal would become more pronounced for a larger V . One might suspect that the two peaks may eventually merge together with a consequence that the transition is not first order. This does not necessarily follow, however, since the apparent merging of the peaks in fig. 1 could well be due to the shift of the transition point with a varying spatial volume. Further work is needed to establish that the finite size scaling explains the behavior seen in fig. 1.

The majority of simulations have been carried out with N_f degenerate quarks. In reality this applies only to the up and down quark since the strange quark is about 20-25 times heavier. Some attempts^{34,35} have been made recently to simulate the realistic case taking $m_u/m_s = m_d/m_s = 20 - 25$ and $m_u/T = 0.05 - 0.1$. This corresponds to $m_u = m_d \sim 10$ MeV and $m_s \sim 200 - 300$ MeV with $T_c \sim 150$ MeV. The result is essentially the same as in the $N_f = 2$ case with clear first-order signals and a small shift of the critical coupling. The strange quark is too heavy to affect the order of the chiral transition. It is basically driven by the chiral symmetry of up and down quarks.

The evidences for a first-order chiral transition for $N_f \geq 2$ are consistent with the prediction of the theoretical analysis²⁵ in the continuum. This analysis also predicts that

Fig. 1

The distribution of the chiral order parameter with a varying spatial volume. From an $N_f = 4$ simulation³¹ with $N_t = 4$ and $m_q/T = 0.4$.

Table I

	N_t	$a(m_\rho)$ (fm)	$T_c(m_\rho)$ (MeV)	$T_c(m_N)$ (MeV)
$N_f = 2$	4	0.347(15)	142(6)	109(6)
	6	0.201(31)	164(25)	117(14)
$N_f = 0$ (pure gauge)	6	0.141(24)	234(39)	200(18)

The chiral transition temperature for $N_f = 2$ determined by $m_\rho = 770$ MeV and $m_N = 940$ MeV on an $N_t = 4$ and 6 lattices. Also listed are the lattice spacing from m_ρ , and the pure gauge ($N_f = 0$) deconfinement temperature at $N_t = 6$. From ref. 37.

a phase transition should be absent for $N_f = 1$. There are some indications^{20,36} that this may be the case for $m_q/T < 0.2$.

2.3 Critical temperature of the chiral transition.

On a lattice with N_t temporal lattice sites, the critical temperature is given by

$$T_c = \frac{1}{N_t a(\beta_c)}, \quad (1)$$

with $a(\beta_c)$ the lattice spacing at the point $\beta = \beta_c$ where the transition takes place. A natural way to estimate $a(\beta_c)$ is to carry out calculation of hadron masses at zero temperature. Since hadron masses have a functional form

$$m_H = \frac{1}{a} f_H(\beta, m_q a), \quad (2)$$

using a few experimental values for m_H and lattice estimates of $f_H(\beta, m_q a)$ at $\beta = \beta_c$ fix $a(\beta_c)$ and the quark mass m_q . After a few preliminary works^{6,7}, a detailed calculation³⁷ has been made for $N_f = 2$ on an $N_t = 4$ and 6 lattice. The result is given in table I which also lists the pure gauge deconfinement temperature on an $N_t = 6$ lattice from the same work.

Two features stand out in table I; a substantial decrease of the chiral transition temperature from the pure gauge value, and a disparity of order 30% in T_c depending on the hadron mass used. This immediately raises the question of how reliable the hadron mass estimate is and whether the low value of $T_c \sim 150$ MeV in table I remains in the continuum limit. Both these are essentially the question of scaling, namely the smallness of magnitude of lattice artifacts coming from the finiteness of the lattice spacing.

There are several indications that this effect is still quite substantial. The disparity of T_c in table I means that the nucleon to ρ mass ratio is too high. This is a generic feature of current hadron mass calculations both in quenched and in full QCD. This is illustrated in fig. 2 which collects the recent full QCD hadron mass data in the form of mass ratios. One sees a trend that m_N/m_ρ lies higher than the experimental value of 1.22 and extrapolates toward $m_N/m_\rho \sim 1.5 - 1.6$ as m_π/m_ρ approaches the physical value 0.18. Among a variety of effects that could have led to the high value, the coarseness of the lattice is probably one of the major sources for full QCD. The lattice spacing in table I, typical of today's full QCD simulations with the Kogut-Susskind quark action, is $a \sim 0.2$ fm for the $N_t = 6$ lattice, while a large amount of data from pure gauge theory simulations indicate that the value $a \sim 0.1$ fm or less is needed before the finite lattice spacing effects begin to disappear. (For more detailed discussions and literatures on these points, see ref. 12-13.)

We would now like to estimate which way T_c would move as the lattice spacing is reduced. For this purpose we examine the scaling behavior expected in the continuum limit,

$$T_c/\Lambda_{\overline{MS}} \sim \text{constant}, \text{ as } N_t \rightarrow \infty, \quad (3)$$

where the Λ parameter

$$\Lambda_{\overline{MS}} = c(N_f) \cdot \frac{1}{a} (\beta/6b_0)^{b_1/2b_0^2} e^{-\beta/12b_0}, \quad (4)$$

with $c(N_f)$ an N_f -dependent constant³⁸ is evaluated at β_c . We show the available values in fig. 3 where the quark mass has been extrapolated to zero using the existing data of β_c for small m_q (ref. 37, 24 for $N_f = 2$ and ref. 4, 5, 7 for $N_f = 4$). We have also plotted the estimates^{39,40} for the pure gauge deconfining transition up to $N_t = 14$. For the pure gauge case, the scaling behavior is beginning to set in for $N_t \geq 10$. We note, however, that it is quite delicate to ascertain (3) in detail. On a small spatial volume used ($N_s/N_t \leq 2-3$), the data suffers from a non-negligible effect of rounding of the transition. As one sees in fig. 3, a slight change in a parameter ($r = 3/4$ or $1/2$) in the definition of β_c alters³⁹ the estimate of $T_c/\Lambda_{\overline{MS}}$ by 5% at $N_t = 14$. For the chiral transition, $N_t = 4$ and 6 are clearly before the region of scaling. Interestingly the chiral transition in full QCD and the deconfinement in the pure gauge theory have similar values of $T_c/\Lambda_{\overline{MS}}$ at each N_t in spite of the considerable difference of their origin. If this similarity is of some guide, the scaling for the chiral transition will require at least $N_t = 8-10$ with a reduction of 15-20% of $T_c/\Lambda_{\overline{MS}}$ from $N_t = 6$.

To see what this implies for the critical temperature, one needs the value of $\Lambda_{\overline{MS}}$. It is still not well known for full QCD, but collecting the existing hadron mass data^{37,41,16} we estimate $\Lambda_{\overline{MS}}(N_f = 2) \approx 130$ MeV (see ref. 12 for more detail). With a 20% reduction of the ratio $T_c/\Lambda_{\overline{MS}}$ from $N_t = 6$, this implies $T_c \approx 1.55 \Lambda_{\overline{MS}} \approx 200$ MeV as the scaling value of the chiral transition temperature for $N_f = 2$. This represents a 20% increase from the value at $N_t = 6$ (with m_ρ) in table 1. For comparison, the pure deconfinement temperature estimated by combining the T_c data for $N_t = 10-14$ ($1/2$ definition) and $\Lambda_{\overline{MS}}(N_f = 0) = 135(14)$ MeV from the quenched hadron mass data over the corresponding range of β (see ref. 13) is $T_c^{dec} = 1.79(7) \Lambda_{\overline{MS}} = 240(30)$ MeV.

Our estimate of the chiral transition temperature has a large margin of uncertainty since it involves several extrapolations of the existing data. It does illustrate, however, that the current simulations with $a \sim 0.2$ fm still suffer from a non-negligible amount of finite lattice spacing effect and that the value of T_c might increase toward the continuum limit. In this context we note that the large decrease of the chiral transition temperature from the pure gauge deconfinement value seen in table 1 should be interpreted with some care. At $N_t = 6$, the lattice spacing for full QCD and for the pure gauge theory are different. Since the scaling has not started in both cases, the magnitude of its violation is different, and this could cause an apparently large decrease of the chiral transition temperature. Indeed our estimate for the scaling value of T_c is substantially closer to the pure gauge value. We clearly need simulations with a temporal size of $N_t = 8-10$ to have some control on the stability of T_c with a decreasing lattice spacing.

Fig. 2

The nucleon to ρ mass ratio against π to ρ ratio squared from recent full QCD simulations. The open and filled symbols correspond to Kogut-Susskind and Wilson quark actions. For more details and references see ref. 11.

Fig. 3

The scaling behavior of the critical temperature as a function of N_t . The circles and squares are for $N_f = 2$ and 4 flavor chiral phase transition at $m_q = 0$. The triangles are for the pure gauge deconfinement. For $N_t \geq 8$ the normal triangles are from ref. 39 and the inverted ones from ref. 40. For the difference of $3/4$ and $1/2$, see text.

3. Physics of the high temperature phase.

3.1 Energy density and pressure.

Production of the high temperature phase and its experimental detection depend very much on the characteristics of its thermodynamic properties. A primary quantity in production is the amount of increase of the energy density across the transition. We show in fig. 4 a recent data⁴² for the temperature dependence of energy density in the presence of light dynamical quarks. The contributions coming from the gluon and the quark part of the Hamiltonian, denoted as ϵ_g and ϵ_q , are drawn separately. This is a typical behavior seen in many simulations^{4-7,24,34,35}; both ϵ_g/T^4 and ϵ_q/T^4 abruptly rise away from zero at the critical point, as expected for a first order chiral transition. For the gluon part, this is followed by a wide bump and a subsequent slow decrease, while the quark part shows only a mild increase above the transition. There is an additional characteristic that the height of the bump of the gluon part rapidly increases for a larger number of flavors and for smaller quark mass. These are much less pronounced for the quark part after normalizing by a trivial factor of N_f .

We note that the height of the bump of ϵ_g/T^4 is quite large. For the case of fig. 4, for example, it is roughly 3 times the Stefan-Boltzmann value $\epsilon_g/T^4 = 8\pi^2/15$, and 50% larger even compared with the weak-coupling prediction⁴³ to $O(g^2)$ on the same size of the lattice. In contrast, the quark part is quite similar to the weak-coupling prediction and appears to approach it smoothly from below.

The large overshoot of the gluon part of the energy density is in marked contrast⁴⁴ with its behavior in the pure gauge case (see fig. 5) where ϵ_g/T^4 does not exhibit any such sign and is consistent⁴⁵ with the weak-coupling prediction above $T/T_c \sim 1.1$, while decreasing steeply⁴⁶ toward T_c . The origin of the difference is not yet completely understood. Weak-coupling calculations⁴³ give some hint, however, that this is a lattice artifact due to a large lattice spacing in current simulations ($a \sim 0.2$ fm), enhanced by the dynamical quark contribution. For a $12^3 \times 4$ lattice with $m_q/T = 0.1$ used in fig. 4, for example, weak-coupling perturbation theory gives^{42,43}

$$\begin{aligned}\epsilon_g/T^4 &= 7.72 + (-0.842 + 0.859 \cdot N_f)g^2 + \dots, \\ \epsilon_q/T^4 &= 6.13 \cdot N_f - 0.455 \cdot N_f g^2 + \dots,\end{aligned}\quad (5)$$

where we have included the vacuum subtractions for ϵ_g . Noting that the quark mass dependence is already weak⁴⁷, we compare these with the continuum result⁴⁸ at $m_q = 0$ (only the sum is available),

$$\epsilon/T^4 = (\epsilon_g + \epsilon_q)/T^4 = \frac{8\pi^2}{15} + \frac{7\pi^2}{20} N_f - \left(\frac{1}{2} + \frac{5}{24} N_f\right)g^2 + \dots \quad (6)$$

One sees that the lattice value for the $N_f g^2$ term arising from the quark loop has a net positive sign in contrast to a small negative value in the continuum. Apparently this occurs because of a large positive $O(N_f g^2)$ term in the gluon part of the energy density. For

Fig. 4

The gluon (a) and the quark (b) part of the energy density as a function of β . The triangles and circles are for 8^3 and 12^3 spatial volume, both with $N_t = 4$. The lines are weak-coupling prediction to order g^2 on the same size of the lattice (solid for 8^3 and dashed for 12^3). For the gluon part, the two lines overlap. From ref. 42.

Fig. 5

The gluon energy density and the pressure in the pure gauge theory normalized by the lowest order lattice weak-coupling value. The ratio T/T_c takes into account empirical violation of scaling. The circles and triangles are from ref. 45 ($12^3 \times 4$), and the squares from ref. 46 ($24^3 \times 4$). The latter assumed $p = 0$ to obtain the energy density. From F. Karsch⁴⁴.

example, the simulation result for ϵ_q/T^4 shows a better agreement with the weak-coupling prediction if the negative $O(N_f g^2)$ term is included (see fig. 4). Since the positive sign of the $O(N_f g^2)$ term for ϵ_q/T^4 is consistent with the behavior seen in simulations (*e.g.*, the height of bump increases with N_f), and since its magnitude has to decrease below the corresponding term in ϵ_q/T^4 in the continuum limit, one suspects that the overshoot becomes less pronounced on a larger lattice. To check this point, one has to reduce the lattice spacing, or equivalently, has to work with a larger temporal lattice size N_t . The scaling analysis of the previous section and some weak-coupling studies⁴⁷ suggest $N_t = 8 - 10$ as a desirable value. The spatial size N_s also has to be increased proportionately (at least $N_s/N_t \sim 3$ appears needed).

The closeness of the quark part of the energy density to the weak-coupling value above the transition as well as the similar behavior of the gluon energy density in the pure gauge case indicate that there exist light modes with the quantum number of quarks and gluons. If these are the only modes, the pressure may be expected to follow approximately the ideal gas law $p = \epsilon/3$. This turned out not to be the case near the transition; the pressure is much smaller until $T/T_c \sim 1.5 - 2$ where it approaches the $p \sim \epsilon/3$ behavior. This has been seen both for the chiral transition in full QCD⁷ and in the pure gauge transition^{49,45,50,51} (for the latter, see fig. 5). Correspondingly the entropy density $s = (\epsilon + p)/T$ is also reduced close to the critical temperature. This implies that the high temperature phase is not simply a plasma of weakly interacting quarks and gluons. There might be non-perturbative heavy modes, perhaps with hadronic quantum numbers⁹, playing a significant role^{49,1,51} close to the transition.

We add a comment⁴⁴ on the strangeness production close to T_c . A recent simulation³⁵ with the up and down quark of an equal mass $m_{u,d}/T = 0.05$ and the strange quark with $m_s/T = 1$ reported a value $\epsilon_s/\epsilon_u \approx 0.5$ for the ratio of the energy density for the strange and up quark close to T_c . This represents a suppression⁴⁴ by roughly a factor of 2 compared to a free quark gas which would give 0.88 for the ratio. If verified by more detailed simulations, this might be relevant for considerations on the strangeness production as a plasma formation signature in heavy ion collisions.

3.2 Quark number susceptibility.

The behavior of energy density indicates that quarks and gluons constitute the light modes of the excitation spectrum. An interesting way to check this is provided by the quark number susceptibility⁵². For a quantum number Q , the susceptibility is generally defined by

$$\chi_Q \equiv \frac{T}{V} \frac{\partial}{\partial \mu_Q} \ln Z < Q > = \frac{1}{V} \langle Q^2 \rangle - \langle Q \rangle^2, \quad (7)$$

where μ_Q is the chemical potential for Q , set equal to zero in the final formula. This quantity measures the easiness with which modes carrying a non-vanishing Q may be produced in the plasma. If there is a light mode of mass $M (\ll T)$ with $Q \neq 0$, the susceptibility is large.

Fig. 6

The quark number susceptibility on an $8^3 \times 4$ lattice with $N_f = 2$ and $m_q/T = 0.1$. The arrow on the top-left marks the value for a free quark gas on the same lattice size. From ref. 52.

Fig. 7

The Debye mass on an $8^3 \times 4$ lattice with filled symbols for full QCD and open ones for pure gauge. The circles are from ref. 42 ($N_f = 2, m_q/T = 0.1$) and ref. 45, 57 (pure gauge). The triangles are for $N_f = 3$ with $m_q/T = 0.4$ from ref. 60. The inverted triangles are from ref. 46. The Ape result²¹ is quite consistent. All values are obtained with the power $n = 1$ except for the two noted explicitly. The shades represent the location of the transition. The solid line is the weak-coupling prediction to lowest order.

If there are only heavy modes ($M \gg T$), it will be suppressed ($\chi_Q \sim e^{-M/T}$).

In fig. 6 we show a recent result for the susceptibility for the number of quarks⁵². Across the transition, the susceptibility rapidly rises to a plateau whose value is similar to that for free quarks on the same size of the lattice. This strongly supports the existence of quarks in the light excitation spectrum as soon as the temperature is raised above the chiral transition.

3.3 Screening length of the plasma.

Debye screening length. The importance of the Debye screening length in the high temperature phase has been well emphasized in relation to the J/ψ suppression⁵³. This is the quantity ξ_D which determines the range of the quark-antiquark potential above the transition;

$$V_{q\bar{q}}(r) \sim -\frac{\alpha}{r^n} e^{-r/\xi_D}, \quad (8)$$

with n and α some constants. There are a number of calculations of the Debye mass $\mu_D \equiv 1/\xi_D$ in the pure gauge theory^{54-59,21,45,46}. Two calculations have recently been made for full QCD, one⁶⁰ at $m_q/T = 0.4$ with $N_f = 3$ and the other⁴² at $m_q/T = 0.1$ with $N_f = 2$. We have compiled the data on an $N_t = 4$ lattice in fig. 7 against $T/\Lambda_{\overline{MS}}$, taking advantage of the similarity of $T_c/\Lambda_{\overline{MS}}$ with a varying N_f .

Estimate of the Debye mass depends sensitively on the range r accessible and the power n assumed. The dependence on n is illustrated in fig. 7 (open circles at right); changing n from 1 to 2 reduces μ_D quite substantially (see below for a further discussion of this point). The dependence on the range is a question of finite size scaling. Some studies⁵⁸ suggest that a finite volume V decreases μ_D by an amount proportional to $1/V^{1/3}$.

Comparing the full QCD value with the pure gauge case both obtained with $n = 1$, we see that the former is generally larger than the latter. This is in accord with the expectation that quark-antiquark pair creation and annihilation enhance the shielding effect of the plasma. We also note that the Debye mass is fairly large already slightly above the transition ($\mu_D/T \sim 3 - 4$ for full QCD and ~ 2 for pure gauge at $T/T_c \sim 1.1$). Closer to the transition, the pure gauge value decreases sharply^{21,46,59} while the full QCD values stay large. The range reached in full QCD is still fairly small ($\tau T \leq 1.5$ as compared to $\sim 3 - 4$ for the pure gauge case), however.

Let us turn to the high temperature behavior and comparison with the perturbative prediction. To the lowest non-trivial order, the weak-coupling expansion in the continuum gives $\mu_D = 2m_E$ and $n = 2$ with

$$m_E^2 = \Pi_{44}(k \rightarrow 0, \omega = 0) = (1 + \frac{N_f}{6})g^2 T^2 + O(g^3), \quad (9)$$

the electric mass for the time component of the gauge potential. In the pure gauge theory for $T/T_c > 2 - 3$, the power $n = 2$ appears to fit the data better than $n = 1$, and the resulting Debye mass is consistent with (9)^{45,59} (see fig. 7 for the $N_t = 4$ case). The agreement is lost close to the critical temperature. In fact the power n seems to change toward $n \sim 1$ close to

the critical temperature^{57-59,45}. The amount of increase for full QCD seen in fig. 7 is larger than the weak-coupling prediction (9). It is too early to take the difference as significant: the full QCD estimates were obtained assuming $n = 1$ which tends to give a larger value than $n = 2$, as discussed above. The finite volume corrections might be large and positive for the quark contribution to the electric mass⁶¹.

Perturbation theory at finite temperatures has the well-known problem of infrared divergence⁶². There are some discussions in the literature⁶³ as to whether the lowest order calculation can be trusted because of this effect. The agreement in the pure gauge case may be taken as a sign that higher order corrections are not very large.

Hadronic screening length. The screening length is generally dependent on the quantum numbers. The Debye mass measures the shielding of the color charge in the fundamental representation of $SU(3)$. Another interesting choice is the quantum number of hadrons. The corresponding screening length is extracted from hadron Green's function at a spatial separation,

$$\sum_{t,x,y} \langle \mathcal{O}(t,x,y,z) \mathcal{O}(0)^\dagger \rangle_H \sim e^{-z/\xi_H}, \quad z \rightarrow \infty, \quad (10)$$

with $\mathcal{O}_H$ a hadronic operator. Calculation of ξ_H or the corresponding "mass" $\mu_H \equiv 1/\xi_H$ were first^{9,64} made with the Kogut-Susskind form of the quark action and have recently been extended¹⁶ to the Wilson form. The results show two characteristic features: i) above the chiral transition, the pion mass ceases to decrease with the quark mass and remains finite at $m_q = 0$. The parity partners ($\pi - \sigma, \rho - A_1$ for mesons and $N_+ - N_-$ for nucleon) become degenerate above T_c , while they are well separated below; ii) the masses of degenerate partners are rather large ($\mu_{\pi-\sigma}/T \sim 3 - 4$ and $\mu_{N_+-N_-}/T \sim 10$).

The first feature shows quite nicely that the chiral symmetry is restored in the high temperature phase, consistent with the behavior of the chiral order parameter $\langle \bar{\chi}\chi \rangle > 0$ found in the phase diagram analyses. The interpretation of ii) is controversial. It has originally been argued⁹ that it might imply the existence of heavy real-time excitations with hadronic quantum numbers. The reasoning used analytic continuation of the hadronic states in the low temperature phase through the gap of singularity in the phase diagram at intermediate quark mass. This may be criticized on two grounds: It gives no information on the lifetime of such modes⁹. The Green's function (10) is measured along the spatial direction and basically explores the dynamics of the spatial components of the gluon fields. This is considered to be confining independent of the temperature, which by itself will give the exponential decay (10) independent of the presence or absence of hadron-like modes⁶⁵.

The second criticism probably applies at very high temperatures where the temporal size of the space-time becomes small. Close to the critical temperature, the problem is open at the moment. We point out in this connection that numerical simulations are ill-equipped to study the real-time dynamics. Being closely tied up with the imaginary time formalism, analytic continuation back to real time has to be made numerically, and this is not an easy task⁶⁶.

4. Finite baryon number density.

There has been little progress in understanding the phase structure with increasing baryon number density. We briefly recall why this problem is so difficult, and then describe some recent attempts which managed to produce some results, albeit in an unphysical limit of strong coupling $\beta = 0$ ($g^2 = \infty$).

The baryon number density is controlled by the quark chemical potential μ_q . For lattice implementation, one multiplies the kinetic term of the quark action by $e^{\pm i\mu_q a}$ for positive and negative time directions⁶⁷. This makes the determinant of the lattice Dirac operator $\det D_q$ complex. In principle one can still make numerical simulations using the real part of the effective action and incorporating the imaginary part into the observables. One can also try to apply the complex extension⁶⁸ of the Langevin algorithm. These do not work for QCD. The basic reason is a global oscillation of the phase of the determinant over the phase space⁶⁹. Theoretical⁶⁹ as well as numerical⁷⁰ analyses show that large phase fluctuations occur when the chemical potential approaches the critical value. In such a circumstance, one cannot hope to obtain reliable estimates of observable averages unless one properly takes into account the ensuing cancellations. The known simulation methods, however, are all local in phase space in that they determine the direction of motion by examining the immediate neighborhood of the current position. Thus these methods are not aware of the possibility of global cancellations, and therefore not capable of producing meaningful results in the critical region. This argument also shows that the quenched approximation that ignores the determinant is incorrect^{71,69}.

There are a few recent attempts which have tried to overcome the difficulty. One of them⁷² integrates out the phase explicitly by working at the strong-coupling limit $g^2 = \infty$. This method has been extended to $SU(3)$ recently⁷³. The result shows a clear first-order phase transition restoring chiral symmetry. Another approach⁷⁰ counts the number of states with a given value of the phase. Unfortunately both methods are difficult to extend to realistic simulations. The former seems restricted to the strong coupling limit and the latter is too time consuming to make except on a small lattice such as 2^4 or 4^4 . Radically new ideas are needed to make a headway on the problem of large baryon number density.

5. Development in the pure gauge theory.

Universality arguments¹⁸ initially suggested that the deconfinement transition in the pure gluon theory with the $SU(3)$ gauge group is first order. A large number of pure gauge simulations including some finite size studies^{74,39} strongly supported the conclusion. This has recently been questioned by the Ape collaboration²¹ based on the data produced on their custom-made computer.

Two points were stressed by the Ape group: i) the first-order criteria used so far such as a flip-flop behavior and a discontinuity of observables have to be coupled with a detailed analysis of the spatial volume dependence for establishing the order of the transition; ii)

the correlation length should give a good guide since it will not diverge at the critical point for a first-order transition, in contrast to second order transitions. The Ape group made measurements of the correlation length using a cold wall and the decay of the Polyakov line from it on three lattices $8^2 \times 32$, $12^2 \times 48$ and $16^2 \times 64$, all with $N_t = 4$. On each lattice, they found that the correlation length close to the transition is comparable to the shorter side of the spatial lattice. They interpret this as a signal that the correlation length at the critical point diverges in the infinite spatial volume limit. They conclude that there are no clear signs of a first-order phase transition, and suggest that it may be second order.

At about the same time, the Columbia group⁴⁶ carried out an independent calculation, also using their own custom-made computer, on 16^3 and 24^3 lattices with $N_t = 4$ and 6. The standard procedures without cold wall were used. On the $24^3 \times 4$ lattice, they found a finite value of the correlation length (~ 5 in lattice units) close to the transition and a clear flip-flop signal with, however, a latent heat of $\Delta\epsilon_g/T_c^4 = 2.54(12)$ which is substantially smaller than the previous estimates on a smaller lattice (for example, $5.4(7)$ on a $10^3 \times 4$ lattice⁷⁴). They conclude that the transition is weak but first order, and emphasize the importance of a large spatial volume to obtain accurate results close to the critical point.

The controversy has not been settled yet. We shall restrict ourselves to a few comments. i) The question concerns the pure gauge system without dynamical quarks. It is not directly related to the chiral transition that we expect in nature. It does, however, remind us of the necessity of a large volume and finite size scaling analyses for establishing the nature of the transition. ii) Correlation length is a delicate quantity. As has recently been emphasized⁷⁵, careful considerations are required in its extraction from the correlation function on a finite volume, especially for first-order transitions.

6. Conclusions.

In this review we have placed much greater emphasis on full QCD with dynamical quarks than in the reviews at the past Quark Matter conferences. This reflects a definite shift in recent lattice QCD studies and an increasingly large amount of full QCD data that have become available. This is not to say that the pure gauge studies are at a satisfactory stage; the question of the order of the deconfinement transition has to be settled since it forms one of the bases of our understanding of the phase diagram.

The full QCD simulations carried out so far support the existence in reality of a first-order phase transition restoring chiral symmetry, driven essentially by the up and down quarks. The high temperature behavior of the plasma phase do not show an apparent breakdown of the weak-coupling perturbation theory once the temperature exceeds $T/T_c \sim 2 - 3$. The dynamics in the vicinity of the chiral transition is much more complex, however, as envisaged by the behavior of pressure.

Current full QCD simulations are $O(50) - O(100)$ more time-consuming than the pure gauge counterparts, in spite of the considerable progress in the dynamical fermion

algorithms⁷⁶. For this reason, the lattice size employed has been typically $8^3 \times 4$ and the lattice spacing $a^{-1} \sim 1$ GeV, which are roughly half of those of the most advanced pure gauge simulations. The studies over the last year have shown that these parameters are not large enough to obtain quantitatively reliable results on the phase transition characteristics and the behavior close to the transition. The spatial size relative to the temporal size has to be increased to have a better control over the finite volume dependence. The lattice spacing (equivalently the temporal lattice size) should be decreased (increased) to diminish artifacts of a finite lattice spacing. Roughly speaking at least doubling the lattice size both in time and space directions appears needed. This means an increase of two orders of magnitude in the amount of computations.

Clearly future progress will depend very much on the improvement of the computing power. This may turn out to be rather fast. Concerning the general purpose supercomputers, the models with a peak speed of a few Gflops or more (1 Gflops = 10^9 floating point operations/second) are becoming available for QCD simulations replacing the existing ones such as CRAY-XMP with the range of 300 Mflops to 1 Gflops. More important perhaps is the appearance of custom-made computers dedicated for lattice QCD. Two machines already exist (Columbia parallel processor and Ape computer, both having a peak speed of 1 Gflops). In the pure gauge simulations, they have started producing results^{21,46} larger in scale than those carried out on general purpose computers. There are also several ongoing projects of dedicated computers with a designed peak speed of 5 to 10 Gflops, aiming at completion within a year or so⁷⁷. A ten-fold increase in speed is therefore quite imminent. Looking further ahead we note an ambitious proposal of the Ape collaboration to reach 100 Gflops in 3 to 4 years from now⁷⁸. With this range of speed, full QCD simulations will reach and perhaps surpass the lattice size and the lattice spacing suggested above for a quantitative understanding of the chiral phase transition.

We should not forget that there are also theoretical questions awaiting progress. The problem of a large baryon number density is the most urgent one at the moment. The finite temperature behavior with the Wilson quark action should also be understood better^{15,16}.

It is our hope that the theoretical as well as the technological efforts will produce much deeper insight into the thermodynamics of QCD by the time of the next Quark Matter conference.

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References.

1. F. Karsch, in *Proc. VIIth Intern. Conf. on Ultra-relativistic Heavy Ion Collisions-Quark Matter 1987*, Z. Phys. C38 (1988) 147.
2. M. Fukugita and A. Ukawa, Phys. Rev. Lett. 57 (1986) 503.
3. R. Gupta, G. Guralnik, G. W. Kilcup, A. Patel and S. R. Sharpe, Phys. Rev. Lett. 57 (1986) 2621.
4. E. V. E. Kovacs, D. K. Sinclair, J. B. Kogut, Phys. Rev. Lett. 58 (1987) 751; Nucl. Phys. B290[FS20] (1987) 431.
5. F. Karsch, J. B. Kogut, D. K. Sinclair and H. W. Wyld, Phys. Lett. 188B (1987) 353.
6. M. Fukugita, S. Ohta, Y. Oyanagi and A. Ukawa, Phys. Rev. Lett. 58 (1987) 2515.
7. S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, Phys. Rev. D35 (1987) 3972.
8. R. V. Gavai, J. Potvin and S. Sanielevici, Phys. Rev. Lett. 58 (1987) 2519.
9. C. DeTar and J. B. Kogut, Phys. Rev. Lett. 59 (1987) 399; Phys. Rev. D36 (1987) 2828.
10. *Proceedings of the 1988 Symposium on Lattice Field Theory*, FNAL, September 22-25, 1988, ed. by W. A. Bardeen *et al.* (to appear in Nucl. Phys. B (Proc. Suppl.)). See in particular the reviews by M. Fukugita and F. Karsch (finite temperature QCD), E. Marinari (hadron mass spectrum), D. Weingarten (dynamical fermion algorithm) and N. Christ (dedicated computers).
11. The previous meetings are recorded in:
Field Theories on the Lattice, Seillac, France, September 28-October 2, 1987, ed. by A. Billoire *et al.*, Nucl. Phys. B (Proc. Suppl.) 4 (1988);
Lattice Gauge Theory '86, Brookhaven, September 15-19, 1986, ed. by H. Satz *et al.* (Plenum, New York, 1987);
NATO Workshop on Lattice Gauge Theory, Wuppertal, November 5-7, 1985, ed. by B. Bunk *et al.* (Plenum, New York, 1986).
12. A. Ukawa, CERN-TH.5245/88 (1988).
13. M. Fukugita, in *Lattice Gauge Theory using Parallel Processors*, ed. X. Y. Li, Z. M. Qiu and H. C. Ren, (Gordon and Breach, London, 1987).
14. L. Susskind, Phys. Rev. D16 (1977) 3031.
15. M. Fukugita, S. Ohta and A. Ukawa, Phys. Rev. Lett. 57 (1986) 1974; Ph. de Forcrand, M. Haraguchi, H. C. Hege, V. Linke, A. Nakamura and I. O. Stamatescu, Phys. Rev. Lett. 58 (1987) 2011; Ph. de Forcrand and I. O. Stamatescu, Nucl. Phys. B304 (1988) 628; Ph. de Forcrand, V. Linke and I. O. Stamatescu, Nucl. Phys. B304 (1988) 645; Y. Iwasaki and Y. Tsuboi, preprint UTHEP-185 (1988); A. Patel, C. Bailie, R. Gupta, G. Guralnik, G. W. Kilcup and S. R. Sharpe, preprint LA-UR-88-998 (1988).
16. A. Ukawa, CERN-TH-5244/88 (1988).

17. K. G. Wilson, in *New Phenomena in Subnuclear Physics*, ed. A. Zichichi (Plenum, New York, 1977).
18. L. G. Yaffe and B. Svetitsky, *Phys. Rev. D* **26** (1982) 963; B. Svetitsky and L. G. Yaffe, *Nucl. Phys. B* **210**[FS6] (1982) 423.
19. T. Banks and A. Ukawa, *Nucl. Phys. B* **225**[FS9] (1983) 145; T. A. DeGrand and C. E. DeTar, *Nucl. Phys. B* **225**[FS9] (1983) 590.
20. M. Fukugita and A. Ukawa, *Phys. Rev. D* **38** (1988) 1971.
21. Ape collaboration, P. Baileiri, E. Remiddi, G. M. Todesco, M. Bernaschi, S. Cabasino, N. Cabibbo, L. A. Fernández, E. Marinari, P. Paolucci, G. Parisi, G. Salina, A. Tarancón, F. Coppola, M. P. Lombardo, E. Simeone, R. Tripiccone, G. Fiorentini, A. Lai, P. A. Marchesini, F. Marzano, F. Rapuano, W. Tross and R. Russack, *Phys. Rev. Lett.* **61** (1988) 1545; ROM2F-88-022(1988).
22. R. V. Gavai, J. Potvin and S. Sanielevici, *Phys. Lett. B* **200** (1988) 137.
23. R. Gupta, G. Guralnik, G. W. Kilcup, A. Patel and S. R. Sharpe, *Phys. Lett. B* **201B** (1988) 503.
24. J. B. Kogut and D. K. Sinclair, *Nucl. Phys. B* **295**[FS21] (1988) 480.
25. R. D. Pisarsky and F. Wilczek, *Phys. Rev. D* **29** (1984) 338; H. Goldberg, *Phys. Lett. B* **131B** (1983) 133.
26. A. M. Polyakov, *Phys. Lett. B* **72** (1978) 477; L. Susskind, *Phys. Rev. D* **20** (1979) 2610.
27. S. Duane, *Nucl. Phys. B* **257**[FS14] (1985) 652; S. Duane and J. B. Kogut, *Nucl. Phys. B* **275**[FS17] (1986) 398; S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, *Phys. Rev. D* **35** (1987) 2531.
28. A. Iråäck, F. Karsch, B. Petersson and H. W. Wyld, CERN-TH-5130/88 (1988), (*Phys. Lett. B*, in press).
29. A. Ukawa and M. Fukugita, *Phys. Rev. Lett.* **55** (1985) 1854; G. G. Batrouni, G. R. Katz, A. S. Kronfeld, P. Lepage, B. Svetitsky and K. G. Wilson, *Phys. Rev. D* **32** (1985) 2736.
30. S. Duane, A. D. Kennedy, B. J. Pendleton and D. Roweth, *Phys. Lett. B* **195B** (1987) 216.
31. R. Gupta, G. W. Kilcup and S. R. Sharpe, *Phys. Rev. D* **38** (1988) 1288.
32. B. Nienhuis and M. Nauenberg, *Phys. Rev. Lett.* **35** (1975) 477; Y. Imry, *Phys. Rev. B* **21** (1980) 2042; M. E. Fisher and A. N. Berker, *Phys. Rev. B* **26** (1982) 2052.
33. K. M. Bitar, A. D. Kennedy, R. Horsley, S. Meyer and P. Rossi, SCRI preprint, FSU-SCRI-88-54 (1988).
34. R. V. Gavai, J. Potvin and S. Sanielevici, *Phys. Rev. D* **37** (1988) 1343.
35. J. B. Kogut and D. K. Sinclair, *Phys. Rev. Lett.* **60** (1988) 1250.
36. R. Gavai, J. Potvin and S. Sanielevici, TIFR/TH/88-29//BUHEP-88-21 (1988).

37. S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, *Phys. Rev. Lett.* **59** (1987) 1513; FERMILAB-PUB-88/62-T (1988).
38. A. Hasenfratz and P. Hasenfratz, *Phys. Lett.* **93B** (1980) 165; *Nucl. Phys. B* **193** (1981) 210; H. S. Sharatchandra, H. J. Thun and P. Weisz, *Nucl. Phys. B* **192** (1981) 205.
39. A. D. Kennedy, J. Kuti, S. Meyer and B. Pendleton, *Phys. Rev. Lett.* **54** (1985) 87; S. Gottlieb, J. Kuti, D. Toussaint, A. D. Kennedy, S. Meyer, D. J. Pendleton and R. L. Sugar, *Phys. Rev. Lett.* **55** (1985) 1958; D. Toussaint, in *Proc. Symposium on Lattice Gauge Theory 1986* (Plenum, New York, 1987).
40. N. H. Christ and A. E. Terrano, *Phys. Rev. Lett.* **56** (1986) 111; N. H. Christ, in *Non-perturbative methods in Field Theory*, *Nucl. Phys. B* (Proc. Suppl.) **1A** (1987) 95; in *Field Theories on the Lattice*, *Nucl. Phys. B* (Proc. Suppl.) **4** (1988) 241; H. Q. Ding, in *Lattice Gauge Theory using Parallel Processors*, ed. X. Y. Li, Z. M. Qiu and H. C. Ren, (Gordon and Breach, London, 1987).
41. M. Fukugita, Y. Oyanagi and A. Ukawa, *Phys. Rev. Lett.* **57** (1986) 953; *Phys. Rev. D* **36** (1987) 824; *Phys. Lett.* **203B** (1988) 145; A. Patel, CERN-TH-5230/88 (1988); M. Fukugita, S. Ohta, Y. Oyanagi and A. Ukawa, *Phys. Lett.* **191B** (1987) 164.
42. F. Karsch and H. W. Wyld, *Phys. Lett.* **213B** (1988) 505.
43. U. M. Heller and F. Karsch, *Nucl. Phys. B* **251**[FS13] (1985) 254; *Nucl. Phys. B* **258** (1985) 29.
44. F. Karsch, review in ref. 10.
45. N. Attig, F. Karsch, B. Petersson, H. Satz and M. Wolff, in preparation, quoted in ref. 42.
46. F. Brown, N. H. Christ, Y. Deng, M. Gao and T. J. Woch, *Phys. Rev. Lett.* **61** (1988) 2058.
47. F. Karsch, unpublished (1987).
48. J. I. Kapusta, *Nucl. Phys. B* **148** (1979) 461; O. K. Kalashnikov and V. V. Klimov, *Phys. Lett. B* **88** (1979) 328.
49. T. A. DeGrand and C. E. DeTar, *Phys. Rev. D* **35** (1987) 742.
50. Y. Deng, Columbia preprint (1988).
51. J. Engels, talk in ref. 10.
52. S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, *Phys. Rev. Lett.* **59** (1987) 2247; preprint IUHET-143 (1988).
53. T. Matsui and H. Satz, *Phys. Lett.* **178B** (1986) 416.
54. M. Fukugita, T. Kaneko and A. Ukawa, *Phys. Lett.* **154B** (1985) 185.
55. T. DeGrand and C. E. DeTar, *Phys. Rev. D* **34** (1986) 2469.
56. K. Kanaya and H. Satz, *Phys. Rev. D* **34** (1986) 3193.
57. N. Attig, F. Karsch, B. Petersson, H. Satz and M. Wolff, *Phys. Lett.* **209B** (1988) 65.
58. J. Engels, F. Karsch and H. Satz, CERN-TH-5024/88 (1988), (*Nucl. Phys. B*, in press).

59. M. Gao, Columbia preprint (1988); talk in ref. 10.
60. R. V. Gavai, M. Lev, B. Petersson and H. Satz, Phys. Lett. **203B** (1988) 295.
61. H.-Th. Elze, K. Kajantie and J. Kapusta, Nucl. Phys. **B304** (1988) 832.
62. A. D. Linde, Rep. Prog. Phys., **42** (1979) 25; D. J. Gross, R. D. Pisarsky and L. G. Yaffe, Rev. Mod. Phys., **53** (1981) 43.
63. Z. Wanyun, Nuovo Cimento, **76** (1983) 525;
- S. Nadkarni, Phys. Rev. D **33** (1986) 3738; Phys. Rev. D **34** (1986) 3904.
64. S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, Phys. Rev. Lett. **59** (1987) 1881.
65. See, for example, A. Gocksch, P. Rossi and U. M. Heller, Phys. Lett. **205B** (1988) 334; see also, A. Mueller, this volume.
66. For a related attempt, see F. Karsch and H. W. Wyld, Phys. Rev. D **35** (1987) 2518.
67. P. Hasenfratz and F. Karsch, Phys. Lett. **125B** (1983) 308;
- R. V. Gavai, Phys. Rev. D **32** (1985) 519.
68. G. Parisi, Phys. Lett. **131B** (1983) 393; J. R. Klauder, Acta Phys. Austriaca, Suppl. **25** (1983) 251.
69. P. E. Gibbs, Phys. Lett. **182B** (1986) 369.
70. A. Gocksch, preprint UCSD-PTH-88/02 (1988).
71. I. Barbour, N. E. Bihlil, E. Dagotto, F. Karsch, A. Moreo, M. Stone and H. W. Wyld, Nucl. Phys. **B275**[FS17] (1986) 296.
72. E. Dagotto, A. Moreo and U. Wolff, Phys. Rev. Lett. **57** (1986) 1292; Phys. Lett. **186B** (1987) 395.
73. F. Karsch and K.-H. Mütter, CERN-TH-5063/88 (1988), (Nucl. Phys. **B**, in press).
74. T. Çelik, J. Engels and H. Satz, Phys. Lett. **129B** (1983) 323; Z. Phys. **C22** (1984) 301.
75. F. Karsch, R. V. Gavai and B. Petersson, CERN-TH-5221/88 (1988).
76. Reviewed by D. Weingarten in ref. 10.
77. Reviewed by N. Christ in ref. 10.
78. E. Remiddi, talk in ref. 10.