NEW FINDINGS IN QUANTUM MECHANICS(PARTIAL ALGEBRAIZATION OF THE SPECTRAL PROBLEM)*)

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We discuss a new class of spectral problems discovered recently which occupies an intermediate position between the exactly-solvable problems (like the famous harmonic oscillator) and all others. The problems belonging to this class are distinguished by the fact that an (arbitrary) part of the eigenvalues and eigenfunctions can be found algebraically, but not the whole spectrum. The reason explaining the existence of the quasi-exactly-solvable problems is a hidden dynamical symmetry present in the Hamiltonian. For one-dimensional motion this hidden symmetry is $SU(2)$. The simplest one-dimensional system admitting algebraization for a part of the spectrum is the anharmonic oscillator with the x^6 anharmonicity and a relation between the coefficients in front of x^2 and x^6 . We review also more complicated cases with the emphasis on pedagogical aspects. The groups $SU(2) \times SU(2)$, $SO(3)$ and $SU(3)$ generate two-dimensional problems with the partial algebraization of the spectrum. Typically we get Schrödinger-type equations in curved space. An intriguing relation between the algebraic structure of the Hamiltonian and the geometry of the space emerges. Another interesting development is the use of the graded algebras which allow one to construct multi-component quasi-exactly-solvable Hamiltonians.

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1. Introduction

More than half of the century passed after creation of quantum mechanics^[1]. The principles discovered in twenties lie in the basis of modern physics which has grown into a huge tree with a lot of important and rather distant branches, from solid state physics to (super)string theory. Every student starts learning modern physics from quantum mechanics, and hundreds of professors during these years have analysed seemingly all possible aspects in this field. So, it seems absolutely impossible now to get radically new result referring to good old quantum mechanics. Surprising though it is, this common belief is wrong. Only 5 years ago the existence of the so called Berry's phase has been understood^[2] (for a review see ref. 3), a breakthrough in the standard theory of the adiabatic approximation in quantum mechanics. The formulation based on the Berry's phase provides us with new insights in such complicated field-theoretic phenomena as, for instance, the chiral anomaly. The present review³ is devoted to another striking finding - the discovery of a new class of quantal problems called *quasi-exactly-solvable*. "Quasi" - because only a part of the eigenvalues and eigenfunctions, and not the whole spectrum, can be found analytically in these problems.

Thus, they occupy an intermediate position between the famous harmonic oscillator (and a few other similar examples) on one

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The second half of this paper is based on original results.

hand, and an infinite set of "normal" spectral problems, on the other hand. The idea of existence of such intermediate hamiltonians gradually crystallized from the results published in the last decade [4-10]. The critical impetus has been given in the works [6,7] and, especially [8-10], done by the Soviet authors. Unfortunately, their publications are not always easily accessible, and this is one of the motivations for writing this review. Another reason is the elegance and simplicity of the general rules ensuring the partial algebraization of the spectrum. Having mastered them one can not stop constructing new and new hamiltonians with striking properties. The game becomes even more versatile and attractive in two dimensions, with the whole wealth of Schrödinger-type quasi-exactly-solvable problems describing motion in curved spaces. The algebraic structure of the hamiltonians determines, as it turns out, the geometric structure of the corresponding spaces [11], a fascinating relation which I would like to explain below.

2. Quasi-exactly-solvable problems or partial algebraization: what does that mean?

We all know that any problem in quantum mechanics ⁴ includes three stages: first, we isolate the relevant degrees of freedom and establish the boundary conditions, then write the hamiltonian and, finally, determine the eigenvalues of the hamiltonian and the

⁴ Although we speak of quantum mechanics one can keep in mind the general formulation of the spectral problem which we encounter, for instance, in computing functional determinants and in many other cases.

corresponding eigenfunctions. Usually the hamiltonian is a linear differential operator of the second order and, consequently, the eigenvalue problem reduces to a solution of the second-order differential equation under certain boundary conditions. Quite obviously, the procedure is not algebraic. In the general case, the spectrum can not be found analytically, i.e. in the form of an explicit function of the parameters entering the hamiltonian. The problems of this type are called exactly-non-solvable.

There are a few famous exceptions, of course, which will be discussed shortly. But before the discussion it is helpful to recall another -- absolutely equivalent -- formulation of the spectral problem. Any hamiltonian H can obviously be represented as an infinite-dimensional hermitean matrix,

$$H \rightarrow \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} & \dots \\ h_{21} & h_{22} & \dots & h_{2n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \dots & h_{nn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (1)$$

To this end choose an arbitrary complete set of orthonormal functions ψ_i satisfying the boundary conditions; then define

$$h_{ij} = \langle \psi_i | H | \psi_j \rangle.$$

Generally speaking, all h_{ij} 's will be non-vanishing.

In this language the solution of the spectral problem reduces simply to a diagonalization of the infinite-dimensional matrix $\{h_{ij}\}$. The eigenfunctions of the hamiltonian are the eigenvectors of the matrix $\{h_{ij}\}$. Unfortunately, unlike the case of the finite matrices, there are no general algebraic rules which would allow one to diagonalize the infinite-dimensional matrix $\{h_{ij}\}$.

Table 1. One-dimensional quantal systems admitting exact solution.

Potential $V^a)$	SU(2) representation $b)$
Harmonic oscillator $V = \frac{\omega^2}{2} x^2$	$-\frac{1}{2} \tau^+ \tau^- + \omega \tau^0$
Morse potential $V = A(e^{-2\alpha x} - 2e^{-\alpha x})$	$-\frac{\alpha^2}{2} \tau^0 \tau^0 + \sqrt{2A} \alpha \tau^- - \alpha(\sqrt{2A} + (j+1/2)\alpha) \tau^0$
Pöschl-Teller potential $V = -\frac{U_0}{\cosh^2 \alpha x}$	$-\frac{\alpha^2}{2} \tau^0 \tau^0 - \frac{\alpha^2}{2} \tau^+ \tau^- + \frac{\alpha^2}{2} [(1+8U_0/\alpha^2)^{1/2} - (2j+1)] \tau^0$
Coulomb potential plus centrifugal barrier $V = -\frac{\alpha}{r} + \frac{l(l+1)}{2r^2}$	$\text{const } \tau^0 \tau^- + \text{const } \tau^- + \text{const } \tau^0 c)$

a) The hamiltonian is defined as follows:

$$H = -\frac{1}{2} \left(\frac{d}{dx} \right)^2 + V(x), \text{ see eq. (9b). Various parameters in the}$$

potentials are introduced in accordance with ref. 12.

b) The meaning of this column is explained in sects. 3, 4. The operators τ^+, τ^-, τ^0 are given in eq. (18).

c) The values of the constants appearing in this entry are given in Appendix B, which contains also other explanations. This Appendix presents an SU(2)-based solution of the Coulomb problem.

The famous exactly-solvable problems are distinguished by the fact that in these problems the matrix $\{h_{ij}\}$ is very specific and is reducible to the diagonal form with the aid of an algebraic procedure. The most well-known example is the harmonic oscillator.

The hamiltonian H can be written as

$$H = \frac{1}{2} (a^+ a + \omega) \quad (2)$$

where a, a^+ are the linear differential operators,

$$a = \omega x + d/dx, \quad a^+ = \omega x - d/dx, \quad (3)$$

satisfying the Heisenberg algebra

$$[a, a^+] = 2\omega. \quad (4)$$

If the wave function ψ_0 is annihilated by the lowering operator a ,

$$a\psi_0 = 0, \quad \psi_0 = (\omega/\pi)^{1/4} \exp(-\omega x^2/2), \quad (5)$$

and the basis ψ_n is chosen as follows

$$\psi_n = (n!(2\omega)^n)^{-1/2} (a^+)^n \psi_0 \quad (6)$$

then in this basis

$$H = \begin{bmatrix} \frac{1}{2}\omega & 0 & 0 & 0 & \dots \\ 0 & \frac{3}{2}\omega & 0 & 0 & \dots \\ 0 & 0 & \frac{5}{2}\omega & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (7)$$

and ψ_n are the eigenfunctions of the hamiltonian. Other known examples admitting the exact solution (exact in this sense) are collected in Table 1. They all describe a one-dimensional motion and allow for algebraization due to the fact that the corresponding hamiltonians can be written as quadratic combinations of the operators forming a closed algebra. We will see later that essentially the same idea, but with the operators obeying a more

complicated algebra than in eq. (4), leads to quasi-exactly solvable systems.

Summarizing, the classical theory of the Schrödinger equation includes two polar cases: the complete diagonalization of the hamiltonian by algebraic methods like in eq. (7), a very rare situation, and the typical case presented in eq. (1) with non-vanishing off-diagonal h_{ij} 's. As a rule, the diagonalization of the matrix $\{h_{ij}\}$ can not be carried out algebraically.

A new direction in the spectral theory, an intermediate link between (7) and (1), has been opened recently [6-10]. Assume that the matrix $\{h_{ij}\}$ has a block structure:

$$H \rightarrow \{h_{ij}\} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1N} & 0 & 0 & 0 & \dots \\ h_{21} & h_{22} & \dots & h_{2N} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{M1} & h_{M2} & \dots & h_{MN} & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & \dots & 0 & \dots & 0 & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (8)$$

Non-vanishing h_{ij} 's

where N is some fixed integer. The block in the upper left corner is an N by N matrix while the second non-vanishing block (in the lower right corner) is an infinite-dimensional matrix. Then, quite obviously, one can immediately diagonalize the finite block without touching the infinite one. The operation is performed just in the same way as for any finite matrix and is purely algebraic. In other words, in the case (8) we analytically determine a part of the spectrum - N eigenvalues and the corresponding eigenfunctions of the hamiltonian H - by algebraic manipulations. Naturally, if we take a random hamiltonian we will not

discover the block structure of eq. (8), rather, we will get a random matrix (1). Miracles do not occur in physics without a deep reason. Actually, there is a regular method ensuring the block structure (8) for specially chosen hamiltonians. A hidden dynamical symmetry is inherent to every quasi-exactly-solvable problem. More exactly, algebraization for a part of the spectrum occurs provided that the hamiltonian can be reduced to a quadratic combination of the generators of a group with finite-dimensional representations. We elucidate the latter assertion in detail in sect. 3 and here just formulate the basic points. Consider some group G with a finite number of the generators T^a , $a = 1, 2, \dots, \nu$ where $\nu = \dim G$. Let us assume that the group G possesses finite-dimensional representations $\{R_j\}$. If we are able to reduce the hamiltonian H to a combination of the generators T^a with the constant coefficients, say,

$$H \rightarrow \sum_{a,b} C_{ab} T^a T^b + \sum_a C_a T^a$$

then this hamiltonian will automatically have the desired form (8) provided that the basis for calculating $\{h_{ij}\}$ is chosen as follows:

$$\{|\psi_j\rangle\} = \{ \text{elements of } R_j + \text{arbitrary set from orthogonal space} \}.$$

The reason is obvious: the action of the generators T^a on the elements of $\{R_j\}$ leaves us inside the same representation. The dimension of the finite block N in eq. (8) coincides with that of $\{R_j\}$. For further details see sect. 3.

As usual, different original papers use quite different languages for description of essentially one and the same phenomenon. The essence of the construction, the reason why it

works, is better seen within the approach proposed in ref. 9, and it will be consistently used throughout the paper. Where necessary I tried to translate the results of other authors in this language. A unified formulation is always useful and can give a stimulating impetus for further development. Actually, this has been the case (sect. 5).

The organization of the paper is as follows. Sect. 3 explains all basic elements of the regular method yielding the partial algebraization of the spectrum. Its aim is basically pedagogical. That is why I deliberately choose here the simplest situation and avoid general but more complicated assertions. The general analysis of one-dimensional quasi-exactly-solvable problems is presented in sect. 4. In this case there exists an exhaustive classification [9,10], a kind of a complete catalogue, and I reproduce a few excerpts. In sect. 5 the relation between the Turbiner and Ushveridze approaches is discussed. An observation of ref. 10 allows one to construct in an elegant way the eigenfunctions belonging to the algebraized part of the spectrum. Sect. 6 opens the part devoted to generalizations. The program leading to multi-dimensional systems with the partial algebraization is sketched here. The results for two-dimensional problems are considered in sect. 7. Although not exhaustive, they still seem to be interesting since they reveal a surprising relation between algebra and geometry in the Schrödinger-type equations. Transition to the multi-component wave functions is discussed in sect. 8. Sect. 9 presents a brief historical survey of the literature. Sect. 10 outlines some issues for further

investigation. A summary of most interesting results is also given here. In Appendix A we remind simple facts concerning differential realizations of the Lie algebras on the space of inhomogeneous polynomials. In Appendix B we show how to expand the class of one-dimensional problems admitting the algebraic solution based on $SU(2)$. The Coulomb system with the centrifugal barrier is discussed here.

3. The simplest example

The most transparent and simple example is the one-dimensional motion in the x^6 potential of a special form. It is ideally suitable for the pedagogical purposes. At first, however, we will dwell on a general element of the method needed not only in the given example, but in any quasi-exactly-solvable problem as well.

3.1. Imaginary phase transformation

Let us write the Schrödinger equation determining the stationary states of the system (the spectral problem) as follows:

$$H \psi(x_1) = E \psi(x_1), \quad (9a)$$

$$H = -\frac{1}{2} (\partial/\partial x_1)^2 + V(x_1). \quad (9b)$$

Here the cartesian coordinates x_1 are used for definiteness⁵, $V(x_1)$ is the potential energy, $\psi(x_1)$ is the wave function. The normalizability condition is implied.

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We reserve the notation x_1 for the coordinates in the flat space; the coordinates in the curved space will be denoted by ξ^1 , see sects. 6, 7 and Appendix A.

$$\int |\psi(x_1)|^2 \prod_1 dx_1 = 1. \quad (10)$$

In the typical situation, if the domain of variation of x_1 's is infinite, the normalizability of the wave function is equivalent to an exponential fall off at large distances.

The overall phase of the wave function is clearly not fixed. Eq. (9a) is invariant under the global transformation $\psi(x_1) \rightarrow \psi(x_1) \exp(i\alpha)$. What will happen if we would like to make a local phase transformation with an x -dependent phase? The reader familiar with electrodynamics will answer immediately. The equation

$$[-\frac{1}{2}(\partial/\partial x_1)^2 + V(x_1)]\psi = E\psi \quad (11)$$

is non-invariant under the local transformation

$$\psi(x_1) = \tilde{\psi}(x_1) e^{i\alpha(x_1)} \quad (12)$$

It is equivalent, however, to a transformed equation

$$[-\frac{1}{2}(\partial/\partial x_1 + iA_1(x))^2 + V(x)]\tilde{\psi} = E\tilde{\psi}, \quad (13)$$

$$A_1 = \partial\alpha/\partial x_1$$

describing the new "wave function" $\tilde{\psi}$. The ordinary derivative ∂_1 is substituted by the covariant derivative $\partial_1 + iA_1$ where A_1 is the vector potential.

It is not difficult to recognize the standard gauge transformation in the given construction. The derivatives $\partial_1 \alpha$ play the role of an external electromagnetic potential. Instead of the Schrödinger wave function $\psi(x_1)$ two functions $\tilde{\psi}(x_1)$ and $\alpha(x_1)$ are introduced in eq. (12) but to compensate for this the gauge freedom is present now. Eq. (13) substituting the original eq. (9) is obviously invariant under further gauge transformations,

$$\tilde{\psi}(x_1) \rightarrow \tilde{\psi}(x_1) e^{i\beta(x_1)}, \quad A_1 \rightarrow A_1 + i\partial\beta/\partial x_1. \quad (14)$$

Any gauge is suitable; having solved eq. (13) for the arbitrary gauge one can always return back to the Schrödinger equation which corresponds to the gauge $A_1 = 0$.

The gauge transformation (12) is unitary. It does not change the probability density. Actually, we will need below a formal analytic continuation, an "imaginary phase", which makes the transformation non-unitary. This is perfectly acceptable since the equivalence of eqs. (9) and (13) takes place for any α .

Let us assume that the function α is purely imaginary, $\alpha =$

$$i a, \quad \psi(x) = \tilde{\psi}(x) e^{-a(x)}. \quad (15)$$

For the imaginary phase transformation it is convenient to rewrite eq. (13) as follows

$$H_{nc} \tilde{\psi} = E \tilde{\psi}, \quad H_{nc} = -\frac{1}{2}(\partial/\partial x_1 - A_1(x))^2 + V(x), \quad (16)$$

$$A_1 = \partial a/\partial x_1.$$

Thus, we are dealing here with a peculiar "gauge" transformation changing, in particular, the asymptotic behaviour of the wave function $\tilde{\psi}$. (The quotation marks will remind us that this is not a real gauge transformation). The information on the probability amplitude is now coded in two functions, $\psi(x)$ and $a(x)$, and can be arbitrarily redistributed between them. The "gauge" freedom allows one to realize the most convenient choice of $a(x)$. The standard possibility is $a(x) = 0$. Then $\tilde{\psi}(x) = \psi(x)$ is the Schrödinger wave function. On the contrary, one can trivialize $\tilde{\psi}(x) = 1$ referring all dynamical information to the "imaginary phase" $a(x)$. The well-known Riccati equation emerges in this way

for one degree of freedom,

$$\frac{1}{2} (A^2 - A'^2) + (V - E) = 0. \quad (17)$$

Intermediate choices are also possible and useful. For instance, refs. 9, 11 suggest using the gauge freedom in order to try to find the phase a such that H_{ac} would be representable as a quadratic combination $H_{ac} = C_{ab} T^a T^b + C_a T^a$ (let us parenthetically note that in this case $a(x)$ describes the leading asymptotic behaviour while the pre-exponential factor $\tilde{\psi}(x)$ is responsible for necessary modulations distinguishing the ground state from the higher excitations). In any case, it is important to understand that the solution of eq. (16) in any "gauge" entails the knowledge of the spectrum and eigenfunctions of the original Schrödinger equation. It is always possible to make further "gauge" transformations.

$$\tilde{\psi}(x_1) \rightarrow \tilde{\psi}(x_1) e^{-b(x)}, \quad A_1 \rightarrow A_1 + \partial b / \partial x_1,$$

and, in particular, return back to the original wave functions and the original Schrödinger equation.

Since the transformation (15) is not a real gauge transformation due to its non-unitary nature the boundary conditions and the normalization condition for the original wave function ψ and for $\tilde{\psi}$ differ. This trivial fact should always be taken into account.

3.2. The simplest quasi-exactly-solvable problem: one-dimensional anharmonic oscillator in an x^6 potential

In this subsection we will give a pedagogical sketch of the method for constructing the potentials ensuring partial algebraization of the spectral problem. As was already mentioned,

of crucial importance is the fact that the hamiltonian can be built from the generators of a finite-dimensional group. The simplest non-abelian group is $SU(2)$, and it is quite natural that this group leads us to the simplest example.

The starting observation is as follows. Three $SU(2)$ generators can be realized in the differential form; more exactly, as linear functions of the differential operator $d/d\xi$. The generators act on polynomials of ξ . Specifically,

$$\begin{aligned} T^+ &= +2j\xi - \xi^2(d/d\xi), \\ T^0 &= -j + \xi(d/d\xi), \\ T^- &= + (d/d\xi) \end{aligned} \quad (18)$$

where j is a number (see Appendix A)⁶. It is a trivial exercise to check that the commutation relations for T^+ , T^0 are indeed those inherent to the $SU(2)$ group:

$$[T^+, T^-] = 2T^0, \quad [T^+, T^0] = -T^+, \quad [T^-, T^0] = +T^-, \quad (19)$$

Moreover, all finite-dimensional representations of $SU(2)$ can be catalogued with the aid of a single (semi-integer) number, spin. The dimension of the representation of spin j is $(2j+1)$. If the generators are chosen in the form (18) the corresponding finite-dimensional representation is

$$R_j = \{ \xi^0, \xi^1, \dots, \xi^{2j} \} \quad (20)$$

provided that the parameter j is semi-integer. Thus, j in eq. (18) stands for the spin. Eqs. (20), (18) explicitly demonstrate that

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Warning: the sign convention for the generators T^a accepted here does not always coincide with that used in refs. 9, 11. The relative signs in eq. (18) are in accord with the standard matrix representation (A.1).

$\bar{T}^+$ plays the role of the raising operator while $\bar{T}^-$ of the lowering one.

Armed with this knowledge let us proceed to construction of the hamiltonian. In the problem under consideration the potential $V(x)$ will depend only on x^2 and, correspondingly, all states will possess definite parity under the reflection $x \rightarrow -x$. In other words, we are dealing here with an additional symmetry commuting with the hamiltonian. All states can be characterized by the corresponding additional quantum number. In such a case one should analyse separately the states with the definite value of this quantum number. We will consider here the wave functions even under the reflection $x \rightarrow -x$ keeping in mind that treatment of the odd wave functions is a straightforward generalization.

Since everything depends on x^2 it is convenient to introduce a new variable, $\xi = x^2$. This specific relation between ξ and x is not the only possibility, of course. As we will see in sect. 4 there exists a rich spectrum of options. We have chosen $\xi = x^2$ to ensure the simplest form of the potential $V(x)$.

Now, let us assume that we are able to find the "gauge" in such a way that the hamiltonian $H_{\text{e.g.}}$ (see eq. (16)) reduces to a quadratic combination of the $SU(2)$ generators $\bar{T}^+, \bar{T}^0$,

$$H_{\text{e.g.}} = \sum_{a,b=2,0} C_{ab} \bar{T}^a \bar{T}^b + \sum_{a=2,0} C_a \bar{T}^a + \text{const} \quad (21)$$

where C 's are numerical parameters. Choose the basis as follows:

$$\{ \tilde{\psi} \} = \{ \xi^0, \xi^1, \dots, \xi^{2j}, \tilde{\psi}_{2j+2}, \tilde{\psi}_{2j+3}, \dots \} \quad (22)$$

with $\{ \tilde{\psi}_{2j+2}, \tilde{\psi}_{2j+3}, \dots \}$ being the arbitrary set of functions

orthogonal to $\xi^0, \xi^1, \dots, \xi^{2j}$ with the weight $\exp(-2a)$:

$$\int x^{2n} \tilde{\psi}_{2j+k} e^{-2a(x)} dx = 0, \quad n = 0, 1, \dots, 2j; \quad k = 2, 3, \dots \quad (23)$$

The only requirement is that the set (22) must be complete. It is quite obvious that in the given basis the matrix representation for the hamiltonian will have the desired block form (8). The finite matrix splitted from the infinite one will have the dimension $(2j+1) \times (2j+1)$. The most crucial point ensuring the validity of this assertion is that the action of the generators $\bar{T}^+, \bar{T}^0$ on the elements of the representation $\{ R_j \}$ leaves us inside the same representation, $\bar{T}^{\pm,0} | \text{element of } R_j \rangle = | \text{element of } R_j \rangle$. If so,

$$h_{n, 2j+k} = \int \tilde{\psi}_{2j+k} e^{-a} H x^{2n} e^{-a} dx = \quad (24)$$

$$= \int e^{-2a} \tilde{\psi}_{2j+k} H_{\text{e.g.}} x^{2n} dx = \int e^{-2a} \tilde{\psi}_{2j+k} \sum_{l=0}^{2j} c_l x^{2l} dx = 0,$$

q.e.d. Here $n = 0, 1, \dots, 2j$, and $k = 2, 3, \dots$; c_l are numerical coefficients.

The standard algorithm of diagonalization of the finite $(2j+1) \times (2j+1)$ matrix -- it may be cumbersome but is purely algebraic -- will give us $(2j+1)$ eigenvalues and the corresponding eigenfunctions of the hamiltonian.

After these general remarks determination of the simplest quasi-exactly-solvable potential should not seem difficult.

We observe that

$$\frac{d}{dx} = 2x \frac{d}{d\xi}, \quad \frac{d^2}{dx^2} = 4\xi \frac{d^2}{d\xi^2} + 2 \frac{d}{d\xi}, \quad (25)$$

and, hence, the kinetic term

$$-\frac{1}{2} \frac{d^2}{dx^2} = -2 \Gamma^0 \Gamma^- - (2j+1) \Gamma^- \quad (26)$$

has the desired form. In order to generate the potential term let us add, say, $-\nu \Gamma^+ + \mu \Gamma^0$ where μ, ν are numerical parameters,

$$H_{uc} = -2 \Gamma^0 \Gamma^- - (2j+1) \Gamma^- - \nu \Gamma^+ + \mu \Gamma^0. \quad (27)$$

Then, using eq. (18) we immediately get

$$H_{uc} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} (\nu x^3 + \mu x) \frac{d}{dx} - 2\nu j x^2 + \text{const}, \quad (28)$$

to be compared with the general expression

$$H_{uc} = -\frac{1}{2} \frac{d^2}{dx^2} + A(x) \frac{d}{dx} + \Delta V, \quad (29a)$$

$$\Delta V = V(x) + \frac{1}{2} A' - \frac{1}{2} A^2. \quad (29b)$$

As we see, there exists a one-to-one correspondence, which allows us, first of all, to identify the "gauge" potential,

$$A(x) = \frac{1}{2} (\nu x^3 + \mu x)$$

and the corresponding "phase",

$$a(x) = \int A(x) dx = \frac{\nu x^4}{8} + \frac{\mu x^2}{4}.$$

(In one dimension any "gauge" potential $A(x)$ is, obviously, pure "gauge", $A(x) = \partial a / \partial x$; this is not so in higher dimensions, see sects. 6, 7). It is important that for $\nu > 0$ the "phase" grows at $x \rightarrow \infty$, so that the normalizability of the original wave function is guaranteed.

Having fixed the "gauge" we can now make the next step and determine the potential $V(x)$. Comparing again eqs. (28) and (29) it is not difficult to establish that

$$V(x) = \frac{\nu^2}{8} x^6 + \frac{\mu\nu}{4} x^4 + \left[\frac{\mu^2}{8} - 2\nu \left(j + \frac{3}{8} \right) \right] x^2. \quad (30)$$

(j semi-integer)

The potential is a polynomial of the sixth order, or, more exactly, a two-parameter family of polynomials with a relation between the numerical coefficients. If we did not know the true reason we would of course be struck by the fact that the anharmonic oscillator with the x^4 and x^6 anharmonicity can be solved, at least partly, by an algebraic procedure.

We pause here to make a few remarks. A question which immediately comes to one's mind is: What happens if the parameter j becomes very large? In the large j limit the number of the algebraically determined levels grows tending to infinity and, as one may expect, in the limit $j \rightarrow \infty$ the exactly solvable quantal problem is recovered. This is indeed the case provided that a smooth extrapolation is ensured. What does it mean, the smooth extrapolation? If we will simply allow the parameter j in eq. (30) to grow with μ and ν fixed, then even the ground-state energy will become infinite in the limit $j \rightarrow \infty$ (a simple exercise for the reader). Such an extrapolation is not smooth, of course. The smooth extrapolation assumes that the lowest level can not "run away". To achieve this purpose one should rescale ν simultaneously with increasing j . The appropriate regime is as follows:

$$\nu = \text{const}/j, \quad \mu = \text{const}.$$

Then, in the large j limit, the x^4 and x^6 anharmonicity in the potential (30) becomes inessential and we are left with the harmonic oscillator, the classical example of the spectral problem admitting the algebraic solution. Such a situation is quite general. Every exactly-solvable quantal problem has an associated tower of quasi-exactly-solvable problems built on it, and the

tower degenerates back into the "parent" exactly-solvable problem in the limit $j \rightarrow \infty$ provided the extrapolation is smooth⁷.

The second question, which is also obvious, is connected with the ordering of the levels. Are the $(2j+1)$ levels belonging to the algebraic sector the lowest levels of the system? In one dimension it is very easy to give the answer -- actually it is positive -- due to the existence of the oscillation theorem. Indeed, all $(2j+1)$ eigenfunctions have the generic form

$$\psi(x) = \mathcal{P}_{2j}(x^2) \exp\left(-\frac{\nu}{8}x^4 - \frac{\mu}{4}x^2\right) \quad (31)$$

where $\mathcal{P}_{2j}$ is a polynomial in $\xi = x^2$ of the degree $2j$. Thus, it has not more than $2j$ zeroes on the ξ axis and, as a result, the eigenstates we deal with are the lowest $(2j+1)$ levels of the given parity. Using the oscillation theorem in more complicated one-dimensional problems one can always find the ordering of the levels. In two dimensions (and the more so in higher dimensions) where such a powerful classification tool as the oscillation

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Perhaps, it is worth noting that the notion of smoothness in the extrapolation procedure is rather vague and does not exclude certain exceptional regimes. What will happen, for instance, if we adjust the parameters in eq. (30) in such a way that $\mu^2 = 2\nu(8j+3)$ and, moreover, $\mu = 2(8j+3)^{1/3}$? The x^2 term will vanish and we will arrive at the potential [13] $V = 2^{-1}(8j+3)^{-2/3}x^6 + x^4$. The limit $j \rightarrow \infty$ is obviously smooth according to the above definition since the ground state does not run away. On the other hand, if j is very large the potential tends to x^4 , the notorious anharmonic oscillator, which certainly can not be solved exactly. In this case the pre-exponential polynomial factor in the wave function, $\mathcal{P}_{4j}(x)$, is so important that it actually changes the exponential asymptotic behaviour of $\psi(x)$ in the intermediate domain $1 \ll x^2 \ll 2(8j+3)^{2/3}$.

theorem is absent it is much more difficult to establish the relative position of the algebraic sector. The question still remains open. It is not ruled out that in certain problems with $d > 1$ the exactly-known levels can be mixed with those which do not belong to the algebraized part of the spectrum.

Finally, it is instructive to discuss the question: "what is to be changed in order to pass from the states even under the parity transformation $x \rightarrow -x$ to those odd under this reflection?"

First of all let us supplement the hamiltonian (27) by the extra term $(-2T^-)$. As a result, the expression for the "gauge" potential will become

$$A(x)_{\text{odd}} = \frac{\nu}{2}x^3 + \frac{\mu}{2}x - \frac{1}{x},$$

where the subscript "odd" indicates hereafter that we are constructing the odd states. The corresponding new "phase" differs from the old expression,

$$a(x)_{\text{odd}} = a(x) - \ln x.$$

Hence,

$$\exp(-a_{\text{odd}}) = x \exp(-a),$$

and the pre-exponential factor which will emerge from diagonalization of H_{nc} will now effectively contain odd powers of x .

It is important to realize that the potential $V(x)$ will also change and, instead of (30), we will get a system of the odd states for the hamiltonian with

$$V_{\text{odd}} = \frac{\nu^2}{8}x^6 + \frac{\mu\nu}{4}x^4 + \left(\frac{\mu^2}{8} - 2\nu(j + \frac{5}{8})\right)x^2.$$

Since j is semi-integer it is impossible to exactly reproduce eq. (30) by adjusting the parameters. Some of the coefficients will be

always slightly different.

Concluding this pedagogical part I propose to do an exercise for training.

Exercise: put $\mu = 0$, $\nu = 2$ in eq. (30) and determine the eigenvalues and eigenfunctions of $(2j+1)$ lowest states (even under reflection $x \rightarrow -x$) for the hamiltonian

$$H = \frac{1}{2} [p^2 + (x^6 - (8j+3)x^2)] .$$

The corresponding double-well potential is depicted on fig. 1. Recently this potential was used by Prof. Martin for illustrating his exact theorems on the level ordering in quantum-mechanical problems [13].

Solution.

According to eq. (27)

$$H_{"c"} = -2T^0 T^+ - (2j+1)T^+ - 2T^+ .$$

The reduced wave function $\tilde{\psi}$ should be chosen in the form

$$\tilde{\psi} = 1 + \alpha\xi + \beta\xi^2 + \dots + \delta\xi^{2j}$$

where $\alpha, \beta, \dots, \delta$ are numerical coefficients, to be determined from the eigenvalue equation $H_{"c"}\tilde{\psi} = E\tilde{\psi}$. The rules according to which T 's act on ξ^k follow from eq. (18).

(1) $j = 1$, three levels:

$$E = -4, \quad \psi = (1 + 4\xi + 2\xi^2)\exp(-x^4/4), \quad \xi = x^2.$$

This wave function has no zeroes in the physical domain $\xi > 0$. Two zeroes of the pre-exponential polynomial lie in the unphysical domain $\xi < 0$.

$$E = 0, \quad \psi = (1 - \frac{2}{3}\xi^2)\exp(-x^4/4).$$

One of the zeroes of the pre-exponential polynomial is in the physical domain $\xi = x^2 > 0$, and the second zero is in the unphysical domain. (We mean the zeroes in ξ . Naturally, each

Fig. 1. The double-well potential in the simplest quasi-exactly-solvable problem.

positive zero on the ξ -axis corresponds to two zeroes in x , $x_0 = \pm \xi_0^{1/2}$.

$$E = 4, \quad \psi = (1 - 4\xi + 2\xi^2)\exp(-x^4/4).$$

Both zeroes are in the physical domain.

(ii) $j = 3/2$, four levels:

$$E = -(30+12\sqrt{5})^{1/2}, \quad \alpha = (30+12\sqrt{5})^{1/2}, \quad \beta = 4+2\sqrt{5}, \quad \gamma = \frac{2\beta}{\alpha},$$

no zeroes in the physical domain;

$$E = -(30-12\sqrt{5})^{1/2}, \quad \alpha = (30-12\sqrt{5})^{1/2}, \quad \beta = 4-2\sqrt{5}, \quad \gamma = \frac{2\beta}{\alpha},$$

one zero at $\xi > 0$;

$$E = (30-12\sqrt{5})^{1/2}, \quad \alpha = -(30-12\sqrt{5})^{1/2}, \quad \beta = 4-2\sqrt{5}, \quad \gamma = \frac{2\beta}{\alpha},$$

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three zeroes at $\xi > 0$.

(iii) $j = 2$, five levels:

$$E = -(80+24\sqrt{6})^{1/2}, \quad \alpha = (80+24\sqrt{6})^{1/2}, \quad \beta = 12+4\sqrt{6},$$

$$\gamma = \frac{\alpha(\beta-6)}{15}, \quad \delta = \frac{2\gamma}{\alpha},$$

no zeroes at $\xi > 0$;

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one zero at $\xi > 0$;

$$E = 0, \quad \alpha = 0, \quad \beta = -4/3, \quad \gamma = 0, \quad \delta = 4/21,$$

two zeroes at $\xi > 0$;

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four zeroes at $\xi > 0$.

Notice that in all three cases considered the position of the

energy levels is symmetric with respect to zero. Moreover, moving from the ground state to the excited ones we first observe "condensation" of the levels (the energy splitting decrease) and then, above the zero energy mark, the regime changes and the energy splittings begin to grow. Actually, the both properties can be derived from general arguments [13].

4. Other quasi-exactly-solvable problems associated with the $SU(2)$ group

One-dimensional hamiltonians with the hidden dynamical $SU(2)$ symmetry admit an exhaustive classification (see refs. 9, 10). In this section we will sketch the general procedure and different particular cases which seem to be most interesting.

An important remark for connoisseurs: it is more accurate to speak about $SL(2)$, not $SU(2)$. The both groups have the same generators $T^{1,2,3}$, but $SL(2)$ allows for complex coefficients in front of $T^{1,2,3}$, as actually takes the place in eq. (27).

Let us return to the general expression for $H_{"q"}$ ensuring algebraizations, eq. (21). Without loss of generality we can take the coefficients C_{ab} symmetric. The antisymmetric part is equivalent to a redefinition of the coefficients C_a . Then eq. (18) implies that

$$H_{"q"} = -\frac{1}{2} P_4(\xi) \frac{d^2}{d\xi^2} + P_3(\xi) \frac{d}{d\xi} + P_2(\xi) \quad (32)$$

where $P_n(\xi)$ are polynomials of the n -th order,

$$\begin{aligned} P_4(\xi) &= -2[C_{++}\xi^4 - 2C_{+0}\xi^3 + (C_{--} - 2C_{+-})\xi^2 + 2C_{-0}\xi + C_{--}], \\ P_3(\xi) &= -2kC_{++}\xi^3 + (3kC_{+0} - C_{+-})\xi^2 + \\ &\quad + (-k(C_{00} - 2C_{+-}) + C_{-0})\xi + (-kC_{-0} + C_{-}), \end{aligned} \quad (33)$$

positive zero on the ξ -axis corresponds to two zeroes in x , $x_0 = \pm \xi_0^{1/2}$.

$$E = 4, \quad \psi = (1 - 4\xi + 2\xi^2)\exp(-x^4/4).$$

Both zeroes are in the physical domain.

(ii) $j = 3/2$, four levels:

$$E = -(30+12\sqrt{5})^{1/2}, \quad \alpha = (30+12\sqrt{5})^{1/2}, \quad \beta = 4+2\sqrt{5}, \quad \gamma = \frac{2\beta}{\alpha},$$

no zeroes in the physical domain;

$$E = -(30-12\sqrt{5})^{1/2}, \quad \alpha = (30-12\sqrt{5})^{1/2}, \quad \beta = 4-2\sqrt{5}, \quad \gamma = \frac{2\beta}{\alpha},$$

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no zeroes at $\xi > 0$;

$$E = -(80-24\sqrt{6})^{1/2}, \quad \alpha = (80-24\sqrt{6})^{1/2}, \quad \beta = 12-4\sqrt{6},$$

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one zero at $\xi > 0$;

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$$E = (80+24\sqrt{6})^{1/2}, \quad \alpha = -(80+24\sqrt{6})^{1/2}, \quad \beta = 12+4\sqrt{6},$$

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An important remark for connoisseurs: it is more accurate to speak about $SL(2)$, not $SU(2)$. The both groups have the same generators $T^{1,2,3}$, but $SL(2)$ allows for complex coefficients in front of $T^{1,2,3}$, as actually takes the place in eq. (27).

Let us return to the general expression for H_{qc} ensuring algebraizations, eq. (21). Without loss of generality we can take the coefficients C_{ab} symmetric. The antisymmetric part is equivalent to a redefinition of the coefficients C_a . Then eq. (18) implies that

$$H_{qc} = -\frac{1}{2} P_4(\xi) \frac{d^2}{d\xi^2} + P_3(\xi) \frac{d}{d\xi} + P_2(\xi) \quad (32)$$

where $P_n(\xi)$ are polynomials of the n -th order,

$$\begin{aligned} P_4(\xi) &= -2[C_{++}\xi^4 - 2C_{+0}\xi^3 + (C_{00} - 2C_{+-})\xi^2 + 2C_{-0}\xi + C_{--}], \\ P_3(\xi) &= -2kC_{++}\xi^3 + (3kC_{+0} - C_{+})\xi^2 + \\ &\quad + (-k(C_{00} - 2C_{+-}) + C_{-0})\xi + (-kC_{-0} + C_{-}), \end{aligned} \quad (33)$$

$$P_2 = 2jK C_{++} \xi^2 + (-2jK C_{+0} + 2jC_+) \xi$$

Here

$$k = 2j-1 \quad (34)$$

and in $P_2(\xi)$ I have omitted an inessential constant term which simply shifts the overall energy scale. Notice that the coefficients C_{∞} and C_{+-} enter only in the combination $(C_{\infty} - 2C_{+-})$. This is not by chance, of course. The point is that the quadratic Casimir operator in the given differential realization reduces to a constant,

$$\sum_{i=1,2,3} (\vec{r}^i)^2 = \frac{1}{2} (\vec{r}^+ \vec{r}^- + \vec{r}^- \vec{r}^+) + \vec{r}^0 \vec{r}^0 = j(j+1).$$

Since the constant term in the hamiltonian is inessential there is no need to keep the both combinations, $\{\vec{r}^+ \vec{r}^-\}_+$ and $\vec{r}^0 \vec{r}^0$, in eq. (21); it does not give anything new. It is sufficient to keep only one of them, say, $\vec{r}^0 \vec{r}^0$. Thus, the coefficient C_{+-} will be consistently neglected, $C_{+-} = 0$.

In order to transform $H_{"C"}$ to the Schrödinger form we must get rid of P_4 , see eq. (29). The following change of variables, obviously, does the job:

$$\xi \rightarrow x, \quad x(\xi) = \int d\xi (P_4(\xi))^{-1/2} \quad (35)$$

so that

$$\frac{d}{d\xi} = \frac{1}{P_4^{1/2}} \frac{d}{dx}, \quad \frac{d^2}{d\xi^2} = \frac{1}{P_4} \frac{d^2}{dx^2} - \frac{1}{2} \frac{P_4'}{P_4^{3/2}} \frac{d}{dx}, \quad (36)$$

where $P_4' = dP_4(\xi)/d\xi$.

The next steps conceptually are very simple. Instead of ξ we introduce a new variable x according to eq. (35). In terms of this new variable the operator $H_{"C"}$ takes the form

$$H_{"C"} = -\frac{1}{2} \left(\frac{d}{dx} \right)^2 + \frac{(P_4'/4) + P_4}{P_4^{1/2}} \frac{d}{dx} + P_4 \quad (37)$$

where the coefficient in front of d/dx and the function $P_2(\xi)$ are to be considered as functions of x . In other words, eq. (35) which gives x as a function of ξ must be reversed. Technically, for arbitrary polynomials $P_4(\xi)$ it is rather difficult to explicitly perform the reversion because the integral in the right-hand side of eq. (35) reduces in the general case to an elliptic function.

Comparing eqs. (29a) and (37) we determine the "gauge" potential and ΔV :

$$A(x) = \frac{(P_4'/4) + P_4}{P_4^{1/2}} \bigg|_{\xi=\xi(x)}, \quad a(x) = \int A(x) dx = \int \frac{(P_4'/4) + P_4}{P_4} d\xi \bigg|_{\xi=\xi(x)},$$

$$\Delta V = P_2(\xi) \big|_{\xi=\xi(x)} \quad (38)$$

We can now construct $V(x)$ according to the formula (29b), and the corresponding quantal system will be quasi-exactly-solvable; $(2j+1)$ eigenvalues and eigenfunctions will be determined algebraically.

As has been already noted the general situation corresponds to potentials constructed from the elliptic functions. It is important to realize that any set of the coefficients C in eq. (33) leads in the one-dimensional case to a sensible quantal system. One can choose C 's at will and then get a quasi-exactly-solvable problem. The only delicate point is verifying the normalizability of the corresponding wave functions. (For further discussion see sect. 6). The general expressions are very cumbersome, however, and we will not dwell on them here. Instead, we will consider in brief some particular cases resulting in rather elegant expressions for the potential.

(1). Exactly-solvable systems

It is quite obvious that if the parameter j , the spin of the representation, does not appear in the final expression (37) for H_{nc} , then we actually have exactly-solvable hamiltonian. Indeed, we can take any value of j , find algebraically $(2j+1)$ levels, then proceed to a larger value of j , etc. Since the potential is j -independent we will algebraically reconstruct larger and larger parts of the spectrum of one and the same system, and in the limit $j \rightarrow \infty$ will find the whole spectrum.

Under what conditions j drops out? From eq. (33) it stems that the necessary and sufficient conditions are

$$C_{++} = 0, \quad C_{+0} = 0, \quad C_+ = 0;$$

C_{00} , C_{-0} , C_{--} are j -independent constants;

$(C_0 - k C_{00})$, $(C_- - k C_{-0})$ are j -independent combinations.

(Recall that C_{+-} is put equal to zero, once and forever).

Thus, for a five parameter family of hamiltonians we ensure complete -- not partial -- algebraization. In this case H_{nc} reduces to

$$H_{nc}(\text{exactly-solv.}) = -\frac{1}{2} Q_2(\xi) \frac{d^2}{d\xi^2} + Q_1(\xi) \frac{d}{d\xi} \quad (39)$$

where Q_1 is a polynomial of the i -th degree,

$$Q_2(\xi) = -2C_{00}\xi^2 - 4C_{-0}\xi - 2C_{--}, \quad Q_1(\xi) = (C_0 - kC_{00})\xi + (C_- - kC_{-0}).$$

The general formulae (35), (38) now take the form

$$H_{nc} = -\frac{1}{2} \left(\frac{d}{dx} \right)^2 + A(x) \frac{d}{dx}, \quad (40a)$$

$$x(\xi) = \int d\xi (Q_2(\xi))^{-1/2}, \quad A(x) = \frac{(Q_2'/4) + Q_1}{Q_2^{1/2}} \xi - \xi(x) \quad (40b)$$

Varying the five parameters⁸, three coefficients in Q_2 and two coefficients in Q_1 , one smoothly interpolates between the known exactly-solvable problems. For instance, if $Q_2 = 1$, $Q_1 = \omega\xi$, eqs. (40) and (29b) imply that $V(x) = \frac{1}{2} \omega^2 x^2$ (harmonic oscillator). A more general situation, D -dimensional harmonic oscillator with a generalized centrifugal barrier,

$$H = -\frac{1}{2} \frac{d^2}{dr^2} - \frac{D-1}{2r} \frac{d}{dr} + \frac{C}{r^2} + \frac{\omega^2 r^2}{2},$$

can also be described. This problem is not shown in Table 1, although the complete algebraic solution, certainly, exists. Actually, it is given in point (e) below.

Furthermore, choosing $Q_2 = \alpha^2 \xi^2$ and $Q_1 = \sqrt{2A} \alpha - (\sqrt{2A} + \alpha)\alpha\xi$ one arrives at the Morse potential $V(x) = A [\exp(-2\alpha x) - 2\exp(-\alpha x)]$. Finally, if $Q_2 = \alpha^2(\xi^2+1)$ and $Q_1 = [-\alpha^2 + \frac{1}{2}\alpha^2(1 + \frac{8U_0}{\alpha^2})^{1/2}]\xi$ we get the Pöschl-Teller potential $V(x) = -U_0/\cosh^2 \alpha x$. All three cases are summarized in Table 1.

It is worth noting that the Morse and Pöschl-Teller potentials have been obtained from a rather close algebraic consideration long ago [5]. As for the Coulomb problem also indicated in Table 1, the method described here does not work literally in this case. The $SU(2)$ -based algebraic solution of the Coulomb problem requires an extension of the method. The extension is described in great detail in Appendix B.

⁸ Out of these five parameters in essence only two are non-trivial because two parameters can be absorbed in the remaining three, and one of those three parameters simply fixes the units on the x -axis.

(11). Quasi-exactly-solvable problems, brief catalog

(a) Polynomial potential.

The simplest example -- the anharmonic oscillator with x^4 and x^6 terms in $V(x)$ -- has been analysed in subsect 3.2. The expression for $H_{"C"}$ is given in eq. (27) and for the potential in eq. (30). The polynomials $P_4(\xi)$ and $P_3(\xi)$ degenerate to linear functions of ξ ($P_4 = 4\xi$, $P_3 = -2\nu j\xi$) while P_3 is quadratic in ξ , $P_3 = \nu\xi^2 + \mu\xi - 1$.

(b) Exponential potential

$$V(x) = \frac{1}{2} \{a e^x + (b-2j)\}^2 + \frac{1}{2} \{ce^{-x} + (b+1)\}^2.$$

The corresponding expression for $H_{"C"}$ is

$$\begin{aligned} H_{"C"} &= -\frac{1}{2} \Gamma \Gamma^0 - a \Gamma^+ + c \Gamma^- + (b-j + \frac{1}{2}) \Gamma^0 = \\ &= -\frac{1}{2} \xi^2 d_\xi^2 + (a\xi^2 + b\xi + c) d_\xi - 2aj\xi. \end{aligned}$$

Here a , b , c are numerical coefficients.

(c) Periodic potential

$$V(x) = (a^2/2) \cos^2 x + a(2j + \frac{1}{2}) \sin x,$$

stemming from

$$\begin{aligned} H_{"C"} &= \frac{1}{2} (\Gamma \Gamma^0 - \Gamma^- \Gamma^-) + j\Gamma^0 + a(\Gamma^- + \Gamma^+) = \\ &= -\frac{1}{2} (1 - \xi^2) d_\xi^2 + (a - a\xi^2 + \xi/2) d_\xi + 2ja\xi. \end{aligned}$$

It is instructive to note that the eigenfunctions in this case reduce to a polynomial of $\sin x$ times $\exp(-a \sin x)$ and, thus, are non-normalizable. The requirement of normalizability is absent, however, for periodic potentials. The wave functions we get here are strictly periodic, with no Bloch phase. This means that they

correspond to the boundaries of the Bloch zones.

(d) Potential built from the hyperbolic functions

$$V(x) = (a^2/2) \sinh^2 x - a(2j + \frac{1}{2}) \cosh x$$

is obtained analogously to the previous case.

$$\begin{aligned} H_{"C"} &= -\frac{1}{2} (\Gamma \Gamma^0 - \Gamma^- \Gamma^-) - j\Gamma^0 - a(\Gamma^- + \Gamma^+) = \\ &= -\frac{1}{2} (\xi^2 - 1) d_\xi^2 + (a\xi^2 - a - \xi/2) d_\xi - 2ja\xi; \\ \psi &\sim [\text{polynomial}(\cosh x)] \exp(-a \cosh x). \end{aligned}$$

(e) To conclude our brief catalog of selected variants let us consider a situation slightly differing from the point (a). The situation is relevant to the motion in the radially symmetric D-dimensional potential:

$$\begin{aligned} H &= -\frac{1}{2} d_r^2 - \frac{D-1}{2r} d_r + V(r), \\ V(r) &= \frac{C}{r^2} + a r^2 + b r^4 + \alpha r^6, \end{aligned}$$

where

$$a = \sqrt{2\alpha} \left[\frac{b^2}{4\alpha\sqrt{2\alpha}} - 4j - 2 - \sqrt{\frac{(D-2)^2}{4} + 2C} \right]$$

and C , b , α are numerical parameters. The novel feature here is the appearance in the potential of the singular term Cr^{-2} characteristic to a centrifugal barrier. Technically its origin is due to a modified (in comparison with the point (a)) coefficient in front of Γ^- ,

$$\begin{aligned} H_{"C"} &= -\frac{1}{2} \Gamma \Gamma^0 - [2j + 2 + \{(D-2)^2 + 8C\}^{1/2}] \Gamma^- - 2\sqrt{2\alpha} \Gamma^+ + \\ &\quad + \frac{2b}{\sqrt{2\alpha}} \Gamma^0. \end{aligned}$$

If we put $C = 0$ and then analytically continue to the point $D=1$

all formulae given here match with those quoted in eqs. (27), (30) under appropriate redefinitions of the parameters. On the other hand, in the limit $\alpha \rightarrow 0$, $b \rightarrow 0$, $(b^2/2\alpha) = \omega^2$, fixed, the D-dimensional harmonic oscillator with the generalized centrifugal barrier is obtained. In this limit H_{nc} is actually j-independent (see eq. (18)) and, hence, the problem becomes exactly-solvable.

5. Turbinder and Ushveridze approaches

Turbinder and Ushveridze started together [8] but later they departed and each of them has proposed his own method or, better to say, procedure for generating quasi-exactly-solvable systems. Thus far, I put the main emphasis on algebraic aspects [9], only marginally mentioning the results of ref. 10. Now it is time to discuss them in more detail.

The procedure described in sects. 3, 4 is algorithmic. It consists of several elementary steps, or elementary prescriptions, with the absolutely clear physical content:

- step 1 -- fix the coefficients in eq. (21) and find H_{nc} ;
- step 2 -- determine the finite block of the matrix $\{h_{ij}\}$;
- step 3 -- diagonalize the finite block, get the eigenvalues and eigenfunctions $\tilde{\psi}$;
- step 4 -- determine the "gauge" phase a from H_{nc} and pass to the Schrödinger-type equation and to the wave functions $\psi = \tilde{\psi} \times \exp(-a)$.

Actually, practically next to nothing is said in the present paper about the third step. It is implied that diagonalization of any finite matrix is an algebraic operation which can always be performed, and I was completely satisfied by this possibility

existing in principle.

The Ushveridze program [10] fuses all four steps into one. The starting (and the only) point of the program is an ansatz, a set of analytic functions depending on a few numerical parameters, to be fitted to make the ansatz compatible with the equation under consideration. From this ansatz one gets simultaneously a quasi-exactly-solvable hamiltonian and the corresponding eigenfunctions ψ in a "ready-for-consumption" form. The only job to do is to fit the numerical parameters. Once they are fitted, diagonalization of the finite block in eq. (8) is achieved.

Since the result of ref. 10 is presented in such a condensed form it is not easy to understand why the ansatz is reasonable, what is the origin of the partial algebraization, etc. So, I tried to translate the assertions of ref. 10 in the language accepted here. (I even compiled a small dictionary, see Table 2). It turned out that the recipe formulated in [10] is absolutely equivalent to the algebraic treatment based on $SU(2)$ (sects. 3, 4). Moreover, it became clear that one of the aspects of the Ushveridze approach can be successfully borrowed and applied for diagonalization of H_{nc} . Another aspect -- consideration of three distinct ansätze, the so called rational, trigonometric and elliptic cases in [10] -- I believe, is superfluous since all three cases are reducible to each other. And, finally, the third aspect, the existence of finite series, at the moment seems to be wrong (see below).

Let us start from the first point, an interesting technical device, perhaps implying a deep connection between partial algebraization in quantal systems and those integrable

Table 2. Correspondence between the Ushveridze ansatz [10] and the algebraic approach [9] accepted in the present paper ^{a)}

Reference 10	The present paper
$\exp (\int y(\lambda) d \lambda)$	$\tilde{\psi}(\xi)$
$A(\lambda) = A_0 \exp \left[-\frac{1}{\alpha} \int v(\lambda - a_\alpha) \right]$ ^{b)}	$-\frac{1}{2} P_4(\xi)$
$2 A(\lambda) B(\lambda)$	$P_3(\xi)$
$A(\lambda) C(\lambda)$	$P_2(\xi)$
a_α	the zeroes of $P_4(\xi)$
λ_1	the zeroes of $\tilde{\psi}(\xi)$

^{a)} The translation presented in the Table is not absolutely exact because of different sign convention and normalization of the kinetic term (according to our eq. (9b) it includes $1/2$ while in ref. 10 the coefficient is 1).

^{b)} The minus sign in the exponent is omitted in ref. 10 evidently due to a misprint.

two-dimensional field theories which are treated within the Bethe ansatz. To explain the idea it is convenient to use the example discussed at the end of sect. 3 (see the Exercise). Later we will arrive at a more general formulation.

Thus, take the double-well potential of fig. 1. The corresponding quasi-exactly-solvable hamiltonian,

$$H = \frac{1}{2} [p^2 + x^6 - (8J+3)x^2],$$

as we already know, after a "gauge" transformation converts into

$$H_{"G"} = -2\xi d_\xi^2 + (2\xi^2 - 1) d_\xi - 4J\xi, \quad (\xi = x^2)$$

The eigenfunctions of $H_{"G"}$ belonging to the algebraic sector have the following generic form:

$$\tilde{\psi}(\xi) = \mathcal{P}_{2j}(\xi) = (\xi - \xi_1)(\xi - \xi_2) \dots (\xi - \xi_{2j})$$

Here $\mathcal{P}_{2j}$ denotes a polynomial of the degree $2j$ which can be always represented as a product of $2j$ elementary factors figuring in the right-hand side; $\xi_1, \xi_2, \dots, \xi_{2j}$ are numbers determining the position of the zeroes of $\mathcal{P}_{2j}(\xi)$. In other words, ξ_i 's (or, $x_i = \pm \xi_i^{1/2}$) are the nodes of the wave function. Shortly we will give to these parameters one more interpretation.

Now diagonalization is equivalent to fitting ξ_i 's. As a byproduct of the fitting procedure we get the corresponding energy eigenvalues.

After these remarks the eigenvalue equation $H_{"G"} \tilde{\psi} = E \tilde{\psi}$ can be immediately rewritten as

$$-2\xi \sum_{\substack{n=1 \\ n \neq 2j}}^{2j} (\xi - \xi_1) \dots (\xi - \xi_n) \dots (\xi - \xi_{2j}) + \\ + (2\xi^2 - 1) \sum_{n=1}^{2j} (\xi - \xi_1) \dots (\xi - \xi_n) \dots (\xi - \xi_{2j}) -$$

$$- (4j\xi + E)(\xi - \xi_1) \dots (\xi - \xi_{2j}) = 0,$$

where the slash means that the corresponding factor is absent.

Dividing the left- and right-hand sides by

$$4\xi \prod_{a=1}^{2j} (\xi - \xi_a)$$

we get

$$- \frac{1}{2} \sum_{\substack{n=1 \\ n \neq n}}^{2j} \frac{1}{\xi - \xi_n} \frac{1}{\xi - \xi_n} + \left(\frac{\xi}{2} - \frac{1}{4\xi} \right) \sum_{n=1}^{2j} \frac{1}{\xi - \xi_n} - j - \frac{E}{4\xi} = 0.$$

The parameters ξ_n and E must (and can) be chosen in such a way as to ensure the vanishing of the left-hand side. It is convenient, although not necessary, to decompose it in the sum of elementary pole terms,

$$- \sum_{\substack{n=1 \\ n \neq n}}^{2j} \frac{1}{\xi - \xi_n} \left[\frac{1}{\xi - \xi_n} - \frac{1}{\xi - \xi_n} \right] + \sum_{n=1}^{2j} \left[\frac{1}{2} + \frac{\xi_n}{2(\xi - \xi_n)} + \frac{1}{4\xi \xi_n} - \frac{1}{4\xi_n(\xi - \xi_n)} \right] - j - \frac{E}{4\xi} = 0.$$

Notice that the constant term $-j + \Sigma(1/2)$ cancels, the manifestation of the fact that the original hamiltonian admits algebraization for semi-integer values of j !

The conditions on ξ_n 's are obvious: the residue of every pole must vanish. In this way we arrive at the following system of equations:

$$- \sum_{\substack{n=1 \\ n \neq k}}^{2j} \frac{1}{\xi_k - \xi_n} + \frac{\xi_k}{2} - \frac{1}{4\xi_k} = 0, \quad k = 1, 2, \dots, 2j. \quad (41)$$

After fixing ξ_n the energy of the level E is entirely determined.

$$E = \sum_{n=1}^{2j} \frac{1}{\xi_n}$$

Let us pause here to notice that we are just approaching the point where an elegant observation of ref. 10 can be used. Instead of solving $2j$ equations (41) observe that they all, taken together, are nothing else than the stability conditions of a classical analog system^[10]. The system includes $2j$ interacting "particles" which can "move" along the line (fig. 2), plus a fixed force centre at the origin. The "coordinates" of the "particles" are $\xi_1, \xi_2, \dots, \xi_{2j}$ while their interaction "potential" is

$$W = - \sum_{\substack{n, n=1 \\ n < n}}^{2j} \ln |\xi_n - \xi_n| - \frac{1}{4} \sum_{n=1}^{2j} \ln |\xi_n| + \sum_{n=1}^{2j} \frac{\xi_n^2}{4}.$$

The stationary equilibrium of the analog system is achieved in the position with the minimal potential energy. The minimal energy means that $\partial W / \partial \xi_n = 0$ ($n = 1, 2, \dots, 2j$), and $2j$ equations $\partial W / \partial \xi_n = 0$ are precisely the conditions (41).

What did we gain by introducing the analog system of "particles"? The answer is obvious: the rich intuition everybody has in classical mechanics. In particular, looking at the function $W(\xi_n)$ everybody will immediately say that the equilibrium position does exist!

Indeed, the "potential" $W(\xi_n)$ describes $2j$ "particles" each with a "charge" one. The interaction between the particles is Coulomb (recall that the Coulomb potential is logarithmic in one dimension); they repel each other. Furthermore, they interact also with the force centre at the origin which has the "Coulomb charge" $1/4$. At short distances the centre also repels the particles. They

Fig. 2. The classical analog system for the simplest quasi-exactly-solvable problem with the potential depicted on fig. 1. The closed circle denotes the position of the fixed force centre while the crosses denote moving particles. Between all the particles there is a repulsion. The force centre at short distances also repels, but at large distances it attracts the particles with the linearly growing force.

can not run away at infinity, however, because of the attraction term, $\xi^2/4$, which becomes important at large distances. Hence, the equilibrium is necessarily established at some finite distances. There are several equilibrium positions, depending on the number of the "particles" situated to the right from the origin. More exactly, there are $(2j+1)$ possibilities:

(1). No "particles" on the right semi-axis (no zeroes of the wave function in the physical domain $\xi = x^2 > 0$);

(2). One "particle" on the right semi-axis (one zero of $\tilde{\psi}$ in the physical domain);

.....;

$(2j+1)$. $2j$ "particles" on the right semi-axis ($2j$ zeroes of $\tilde{\psi}$ in the physical domain).

Each equilibrium position corresponds to a distinct eigenstate. Thus, the situation depicted on fig. 2 describes the third excited (even) level of the system with five levels in the algebraized sector (point (111) of the Exercise in sect. 3).

Thus far we considered a specific example. What changes in the general case? To answer this question let us return to sect. 4 and analyse the expression for H_{nc} , quoted in eq. (32). In the most general situation the coefficient in front of $-d_\xi^2/2$ can be represented as

$$P_4(\xi) = -2C_{++}(\xi-a_1)(\xi-a_2)(\xi-a_3)(\xi-a_4)$$

where a_α are numerical constants, $\alpha = 1, 2, 3, 4$, and C_{++} is assumed to be non-vanishing⁹. Some of the parameters a_α can be

⁹ If $C_{++} = 0$ the polynomials P_4 and P_3 degenerate into some polynomials of the 3-d and 2-nd degrees respectively (see next page)

complex; then we actually deal with the pairs of the complex-conjugated parameters. This fact is inessential for us here since our purpose is limited; we are trying to translate the results of ref. 10 in the algebraic language which seems to be more transparent. Therefore, we assume for simplicity a_1, a_2, a_3 , and a_4 to be real.

Moreover, polynomials $P_3(\xi)$ and $P_2(\xi)$ are obviously representable in the following form

$$P_3(\xi) = P_4(\xi) \sum_{\alpha=1}^4 \frac{b_\alpha}{\xi - a_\alpha}, \quad (P_2(\xi) + E) = P_4(\xi) \cdot \sum_{\alpha=1}^4 \frac{c_\alpha}{\xi - a_\alpha},$$

where b_α and c_α are some numerical coefficients. From eq. (33) it is easy to see that a_1, a_2, a_3 , and a_4 can be chosen arbitrarily, while the parameters b_α and c_α are constrained. There exists one relation for b_α 's,

$$\sum_{\alpha=1}^4 b_\alpha = 2J - 1.$$

In our approach it follows from the requirement (21) (plus (18)) while Ushveridze obtains it empirically, demanding self-consistency of his ansatz. Once a_α and b_α are chosen the parameters c_α are (almost) fixed. They can be calculated from eq. (33). (In [10] they are again fitted to make the ansatz compatible with the equation under consideration). Three conditions on c_α are

This situation corresponds to the choice $N=3$ in ref. 10 where N is the number of the force centres. If $C_{++} = C_{+0} = 0$ the polynomial is actually of the second degree and we arrive at exactly solvable problems (the case $N=2$ in ref. 10, see point (i) of sect 4). In the both degenerate cases the analysis is similar to that presented here but technically simpler.

rather obvious from eqs. (33),

$$\sum_{\alpha=1}^4 c_\alpha = 0, \quad \sum_{\alpha=1}^4 c_\alpha a_\alpha = -j(2j-1), \quad \sum_{\alpha=1}^4 c_\alpha a_\alpha^2 = -2j \sum_{\alpha=1}^4 b_\alpha a_\alpha,$$

where the constraint on b_α is taken into account. The remaining freedom in c_α 's reflects the possibility of having different energy eigenvalues.

Now we proceed in the same vein as in the example discussed above. The eigenvalue equation $H_{C_n} \tilde{\psi} = E \tilde{\psi}$ is equivalent to

$$-\frac{1}{2} \sum_{m,n=1}^{2j} \frac{1}{\xi - \xi_m} \frac{1}{\xi - \xi_n} + \sum_{\alpha=1}^4 \frac{b_\alpha}{\xi - a_\alpha} \sum_{m=1}^{2j} \frac{1}{\xi - \xi_m} + \sum_{\alpha=1}^4 \frac{c_\alpha}{\xi - a_\alpha} = 0$$

provided that the eigenfunction $\tilde{\psi}(\xi)$ is chosen in the form

$$\tilde{\psi}(\xi) = (\xi - \xi_1)(\xi - \xi_2) \dots (\xi - \xi_{2j}).$$

(From the algebraic arguments we do know that $\tilde{\psi}(\xi) = \mathcal{P}_{2j}(\xi)$). Then reducing the left-hand side to the sum of simple poles and requiring all residues to vanish we get the equations on ξ_k :

$$-\sum_{m=1}^{2j} \frac{1}{\xi_k - \xi_m} - \sum_{\alpha=1}^4 \frac{(-b_\alpha)}{\xi_k - a_\alpha} = 0, \quad k = 1, 2, \dots, 2j. \quad (43a)$$

Again it is very convenient to interpret these equations as the equilibrium condition of an analog mechanical system of $2j$ "particles" moving along the ξ axis and interacting according to the Coulomb law,

$$W(\xi_k) = - \sum_{\substack{m,n=1 \\ m < n}}^{2j} \ln |\xi_m - \xi_n| - \sum_{\alpha=1}^{2j} (-b_\alpha) \ln |\xi_m - a_\alpha|. \quad (43b)$$

The charges of all moving particles are 1 and they repel each other. The force centres located at a_1, a_2, a_3 , and a_4 have the "charges" $(-b_1), \dots, (-b_4)$, respectively, and can either repel or

attract the "particles". Notice that we can not have all the force centres repelling because of the condition $\sum b_\alpha = 2j-1$. Still, the equilibrium is possible, for instance, in the situation depicted on fig. 3.

Having determined the parameters ξ_α we fix c_α , also from vanishing of the pole residues,

$$c_\alpha = \sum_{m=1}^{2j} \frac{b_\alpha}{\xi_\alpha - a_\alpha}, \quad \alpha = 1, 2, 3, 4.$$

The latter expression yields, in turn, the energy eigenvalues,

$$E = -2C_{++1234} \sum_{\alpha} \frac{a_\alpha}{c_\alpha} \left(\frac{c_\alpha}{a_\alpha} \right).$$

Different equilibrium positions in the physical interval (e.g. the interval (a_2, a_3) on fig. 3) correspond to different eigenstates and energy eigenvalues of the original Schrödinger equation.

Summarizing, we see that the proposal of ref. 10 is, in essence, a method of diagonalization of the finite block in H_{nc} , especially useful for large N where N is the matrix dimension (eq. (8)). The analogy noted in [10] can go further, however, because the equilibrium conditions for the analog mechanical system, quoted above, exactly coincide [10] with the Bethe equations for completely integrable systems in the quasi-classical limit [14]. Typical hamiltonians which can be treated within the Bethe ansatz are those of interacting spins on one-dimensional lattice. In each lattice vertex there is a spin operator belonging to $O(3)$ and coupled to its neighbours according to certain rules.

It seems remarkable that the systems with a large number of interacting spin operators situated in different lattice vertices are in one-to-one correspondence with quasi-exactly-solvable

Fig. 3. The classical analog system corresponding to the hamiltonian (32). The force centres a_1, a_2, a_3, a_4 are situated at the zeroes of $P_\alpha(\xi)$ and their number coincides with the degree of this polynomial. The interaction of the moving particles and the force centres is determined by eq. (43b). The Coulomb charges of the force centres are $-b_\alpha$.

quantal systems whose hamiltonian is quadratic in the $SU(2)$ generators (18) but these generators carry no indication on any lattice. This parallel obviously calls for further investigation; it may well turn out that a formal coincidence of equations referring to different spheres is more than pure coincidence, and one may expect from it new insights or, even, a breakthrough both in the Bethe ansatz and in the partial algebraization in the Schrödinger-type problems.

The reader is perhaps surprised by the fact that the force centres of the analog mechanical system (43b) possess only the Coulomb interaction while in the simplest example (42) there was also the oscillator-like attraction. The point is that the most interesting particular cases listed in sect. 4 are characterized by degenerate polynomials $P_4(\xi)$ and $P_3(\xi)$. In order to obtain them from the general expression (43b) one must send some of the force centres to infinity simultaneously rescaling their "charges". The limiting procedure is rather delicate and yields, as a result, non-Coulomb terms in the potential of the analog mechanical system. Practically it is very inconvenient to use the general formula (43b) in the concrete quasi-exactly-solvable problems discussed in sect. 4. The shortest way in each particular instance is to start from the very beginning and get a dedicated analog system especially fitted for the given degenerate situation.

A few words about other assertions of ref. 10. One of them can be reformulated as follows. It is claimed that for certain hamiltonians of the type

$$H_{\text{nc}} = -\frac{1}{2} P_\ell(\xi) d_\xi^2 + P_{\ell-1}(\xi) d_\xi + \tilde{P}_{\ell-1}(\xi), \quad \ell \geq 5 \quad (44)$$

there exist exceptional finite series -- algebraized sectors including up to seven eigenstates. (P_ℓ , $P_{\ell-1}$ and $\tilde{P}_{\ell-1}$ in eq. (44) are the polynomials of the degree ℓ and $\ell-1$, respectively; ℓ is the number of the force centres). From what we already know such sectors seem to be ruled out if $\ell > 4$.

The last remark concerns the fact that actually not one, but three different ansätze have been proposed in [10]. Above we discussed the rational ansatz.

$$\tilde{\psi}(\xi) = \prod_i (\xi - \xi_i)$$

Now I would like to explain that two other ansätze of ref. 10, trigonometric and elliptic, present nothing new in the conceptual aspect although, perhaps, technically they still further simplify the task of diagonalization.

Consider for definiteness the trigonometric ansatz which can be written as

$$\tilde{\psi}(\lambda) = \prod_{i=1}^{2J} \sin(\lambda - \lambda_i) \quad (45)$$

The Ushveridze variable λ is preserved here intentionally although it plays the same role as ξ ; the reason will become clear shortly.

It is not difficult to understand that the wave function (45) is a linear combination (with specially chosen coefficients) of the elements belonging to the following set:

$$R_j = \{ e^{-2Ji\lambda}, e^{-(2J-2)i\lambda}, \dots, e^{(2J-2)i\lambda}, e^{2Ji\lambda} \} \quad (46)$$

where $i = \sqrt{-1}$ and J is a semi-integer number. Comparing eqs. (20) and (46) we immediately guess that we are actually dealing with the same $SU(2)$ algebra as previously but in a slightly different realization. In this realization the $SU(2)$ generators are

The curvature corresponding to this metric is

$$\sum_{a,b} C_{ab} \Gamma^a \Gamma^b + \sum_a C_a \Gamma^a \quad (49)$$

does not always contain the second derivatives

$$\partial_{ij} \partial_j = \partial^2 / \partial \xi^i \partial \xi^j$$

with the unit coefficient. In the general case, for arbitrary choice of C's, the derivatives ∂_{ij} enter being multiplied by functions of ξ . Thus, the expression of the type $g^{ij} \partial_{ij}$ naturally appears where g^{ij} plays the role of a metric. In other words, the straightforward generalization of the one-dimensional procedure leads us (in two, three, etc. dimensions) to consideration of motion in a gravitational background.

$$-\frac{1}{2} \left[\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ik} \partial_k \psi) \right] + V \psi = E \psi, \quad (50)$$

where the term in the square bracket is the covariant laplacian substituting the ordinary laplacian of the flat space. As usually,

$$g = \det \{g_{ik}\} = 1/\det \{g^{ik}\}.$$

Let us parenthetically note that even in one dimension we do not get the kinetic term $(d/dx)^2$ directly from the combination (21). Instead, we obtain $P_4(\xi)(d/d\xi)^2$ and then are forced to perform a change of variables. The change of variables (35) can be considered as a transition to the flat metric which is always possible in $D = 1$ (see below). This is not the case for $D > 1$. Already in two dimensions in the general case the best one can do by changing the variables is to pass to a conformally flat metric.

Thus, in multi-dimensional spaces the quasi-exactly-solvable systems will be characterized not only by the potential $V(\xi, \eta, \dots)$ but also by invariant geometrical properties of the manifold on which the covariant laplacian acts, i.e. by the curvature tensor.

$$\Gamma^+ = \frac{1}{2I} e^{2I\lambda} (2Ij - d_\lambda), \quad \Gamma^0 = \frac{1}{2I} d_\lambda, \quad \Gamma^- = \frac{1}{2I} e^{-2I\lambda} (2Ij + d_\lambda) \quad (47)$$

The transition from the standard form, eqs. (18), (20) to eqs. (46), (47) is rather obvious. At first, we make the "gauge" transformation with the phase $a(\xi) = -j \ln \xi$. Then

$$\begin{aligned} d_\xi &\rightarrow d_\xi - A(\xi) = \\ &= d_\xi - \frac{da}{d\xi} = d_\xi + j \xi^{-1}. \end{aligned}$$

The second and the last step is the change of variables,

$$\xi = e^{2I\lambda}. \quad (48)$$

Thus, we see that the general analysis of sect. 4 completely covers this ansatz. In principle, there is no need in a separate consideration although, of course, it can be carried out for practical purposes.

6. Multidimensional space. Generalities

Partial algebraization of the spectrum is not specific for one-dimensional problems. The idea is universal. If there are appropriate differential realizations of finite-dimensional groups involving more than one variables one can construct a multi-dimensional differential operator of the second order, quadratic in the generators of the group. As explained in Appendix A such differential realizations do exist. Unlike the one-dimensional case, however, this quadratic operator is not necessarily of the form relevant to the Schrödinger-type equation.

Let us explain the point in more detail. From the explicit expressions for the generators Γ^a , eqs. (59), (64), (80) it is clear that the combination

The next step is just the same as for $D = 1$. We introduce the "gauge" freedom. Moreover, after the phase transformation $\psi = \tilde{\psi} \exp(-a)$ eq. (50), evidently, converts into

$$\left[-\frac{1}{2} g^{ik} \partial_1 \partial_k - \frac{1}{2} (\partial_1 g^{ik}) \partial_k - \frac{1}{2} (\partial_1 \ln \sqrt{g}) g^{ik} \partial_k + g^{ik} \mathcal{A}_1 \partial_k + \Delta V \right] \tilde{\psi} = E \tilde{\psi} \quad (51)$$

where

$$\begin{aligned} \mathcal{A}_1 &= \partial_1 a = \partial a / \partial \xi^1, \\ V &= \frac{1}{2} g^{ik} \mathcal{A}_1 \partial_k - \frac{1}{2} (\partial_1 \ln \sqrt{g}) g^{ik} \mathcal{A}_k - \frac{1}{2} g^{ik} (\partial_1 \mathcal{A}_k) - \frac{1}{2} (\partial_1 g^{ik}) \mathcal{A}_k + \Delta V. \end{aligned} \quad (52)$$

Along with the "vector potential" $\mathcal{A}_1$ it is convenient to introduce a new auxiliary "potential"

$$\tilde{\mathcal{A}}_1 = \partial_1 (\ln \sqrt{g}) - 2\mathcal{A}_1 = \partial_1 (\ln \sqrt{g} - 2a). \quad (53)$$

Then the spectral equation (50) takes the form

$$-2 H_{\alpha\alpha} \tilde{\psi} = \left[g^{ik} \partial_1 \partial_k + (\partial_1 g^{ik}) \partial_k + \tilde{\mathcal{A}}_1 g^{ik} \partial_k - 2\Delta V \right] \tilde{\psi} = -2E \tilde{\psi}. \quad (54)$$

Now, let us compare the left-hand side of eq. (54) with the already familiar bilinear combination (49). Assume that the coefficients C_{ab} are chosen in a certain way. The set of the coefficients C_{ab} first of all uniquely fixes the metric g^{ik} . The bilinear piece $C_{ab} \Gamma^{ab}$ produces some terms linear in ∂_k as well. Next, we add the terms $C_a \Gamma^a$ and demand the linear in ∂_k part of $C_{ab} \Gamma^{ab} + C_a \Gamma^a$ to be equal to

$$[(\partial_1 + \tilde{\mathcal{A}}_1) g^{ik}] \partial_k.$$

Since g^{ik} is already fixed this requirement gives us D equations on $\tilde{\mathcal{A}}_1$.

$$(\partial_1 + \tilde{\mathcal{A}}_1) g^{ik} = \mathcal{H}^k, \quad k = 1, 2, \dots, D \quad (55)$$

where $\mathcal{H}^k$ are the coefficients in front of ∂_k in $C_{ab} \Gamma^{ab} + C_a \Gamma^a$ (they are built from polynomials of ξ^1 and depend on C_{ab} and C_a). D is the number of variables ξ^1 .

For non-degenerate metrics eq. (55) can always be solved,

$$\tilde{\mathcal{A}}_1 = g_{ik} (\mathcal{H}^k - \partial_n g^{nk}), \quad i = 1, 2, \dots, D. \quad (56)$$

Alas! This solution is not the end of the story. Certainly, eq. (56) ensures the desired form of $H_{\alpha\alpha}$, eq. (49), but we must remember that actually at our disposal is only one gauge freedom, one phase a . The functions $\tilde{\mathcal{A}}_1$ must be pure gauge (eq. (53)). This requirement imposes the following constraints on $\tilde{\mathcal{A}}_1$:

$$\partial_k \tilde{\mathcal{A}}_1 - \partial_1 \tilde{\mathcal{A}}_k = 0, \quad (57)$$

to be considered as self-consistency conditions. If and only if the functions $\tilde{\mathcal{A}}_1$ given in eq. (56) are pure gauge can we speak of the quasi-exactly-solvable problem of the Schrödinger type, eq. (50).

At present I am aware of no general solution of the constraints (57). For some specific choices of the coefficients C the result for $\tilde{\mathcal{A}}_1$ turns out to be pure gauge automatically. Several such examples will be discussed below.

In this respect the situations in one dimension and in two (or three, etc.) dimensions are drastically different. The one dimensional case could be described in the same terms as here, but in one-dimensional problems we deal with only one potential $\mathcal{A}$, and the metric $\{g^{ik}\}$ degenerates to a single function g^{11} . Hence:

-- There is no consistency condition like (57), and $\mathcal{A}$ is always pure gauge (modulo the issue of normalizability of the resulting wave functions, which should be analysed separately).

-- The metric g^{11} is always reducible to the flat space by a change of variables.

These remarks explain why in sect. 4 devoted to one-dimensional motion we were able to give the complete classification of D=1 quasi-exactly-solvable systems emerging for arbitrary set of the coefficients C. Already in two dimensions, first of all, not every choice of C's is admissible from the point of view of the constraints (57). Second, even if C's are such that $\tilde{A}_1$ satisfy the consistency conditions (pure gauge) the metric (g^{1k}) is not necessarily flat. Generally speaking, the corresponding curvature will be non-vanishing, although there may exist some variants which correspond to the flat space, see below.

Passing to higher dimensions we increase the number of constraints (57) and the general analysis becomes more and more difficult.

Summarizing the multi-dimensional situation it is worth emphasizing again that at present there is no complete analysis analogous to those given in sect. 4 for D = 1. Definitely not every set of the coefficients C leads to a quasi-exactly-solvable hamiltonian. What are the general conditions on the coefficients C guaranteeing that $C_a r^a r^b + C_b r^a r^b$ will be equal to the "normal" H_{nc} , eq. (54) ? The answer seems to be hidden somewhere in the mathematical literature but at the moment is unknown to me.

First attempts to construct multi-dimensional quasi-exactly-solvable systems were undertaken in ref. 15. This is a large series of papers discussing different properties of the ansätze proposed earlier [10]. One of the directions is the extension to a

larger number of variables. The publication [11] started a systematic analysis based on the algebraic approach, and below I will closely follow this work.

7. Two-dimensional motion.

As discussed in Appendix A only three groups provide us with the differential realizations built on two variables, ξ and η , suitable for our program. Namely,

$$SU(2) \times SU(2), \quad SO(3), \quad \text{and} \quad SU(3). \quad (58)$$

There are no other possibilities because higher groups require more than two variables. We will start with the most obvious case, $SU(2) \times SU(2)$, then discuss in brief $SO(3)$ -based hamiltonians and, finally, consider some quasi-exactly-solvable systems with the hidden $SU(3)$. The latter group generates the widest range of problems since it has the largest number of the generators and their form is more versatile. The $SO(3)$ group realized on two variables, ξ and η , presents a subgroup of $SU(3)$. In other words, the $SO(3)$ generators (64) are linear combinations of some of the $SU(3)$ generators (80).

For each group from the family (58) a few representative examples will be discussed. No complete classification is available at present.

Before submerging into constructing D=2 hamiltonians with the partial algebraization two remarks are in order.

The variables ξ and η will be treated as the coordinates with the upper indices, $\xi^1 = \xi$, $\xi^2 = \eta$; then, according to the general rules the corresponding derivatives $\partial_\xi = \partial/\partial\xi$, $\partial_\eta = \partial/\partial\eta$ are to be interpreted as ∂_1 and ∂_2 with the lower indices.

In two dimensions the curvature tensor has only one non-trivial component, R_{1212} . Therefore, the 2-dim manifolds are completely characterized by $\sqrt{g}$ and the scalar curvature R .

(1) $SU(2) \times SU(2)$

This group is isomorphic to $O(4)$; six generators of this group can be written down in the following way:

$$\tilde{T}^+ = 2j\xi - \xi^2\partial_\xi, \quad \tilde{T}^+ = 2\tilde{j}\eta - \eta^2\partial_\eta,$$

$$\tilde{T}^0 = -j + \xi\partial_\xi, \quad \tilde{T}^0 = -\tilde{j} + \eta\partial_\eta,$$

$$\tilde{T}^- = \partial_\xi, \quad \tilde{T}^- = \partial_\eta, \quad (59)$$

where j and $\tilde{j}$ are semi-integer numbers, ξ and η are independent coordinates. The dimension of the representation $R_{j\tilde{j}}$ is $N = (2j+1)(2\tilde{j}+1)$. This is also the number of levels in the algebraized sector.

Now, it is desirable to include in the combination (49) a coupling of the tilted and untilted generators because otherwise we get separated variables from the very beginning, a trivial situation which we would like to avoid. The desired "mixing" can take place only in the bilinear part $C_{ab} \tilde{T}^a \tilde{T}^b$.

I must admit that we failed [11] to invent anything attractive in this case. One of the simplest hamiltonians associated with $SU(2) \times SU(2)$ describes the motion in the domain $\xi > 0, \eta > 0$ (the potential is singular at $\xi=0$ and/or $\eta=0$),

$$-2H_{"G"} = \tilde{T}^0 \tilde{T}^0 + \frac{1}{2} \{ \tilde{T}^0 \tilde{T}^- \}_+ + \tilde{T}^0 \tilde{T}^0 + \frac{1}{2} \{ \tilde{T}^0 \tilde{T}^- \}_+ - 2\tilde{T}^0 \tilde{T}^0 + c\tilde{T}^+ - (c+2\tilde{j}-2j)\tilde{T}^0 + (j+1/2)\tilde{T}^- + c\tilde{T}^+ - (c+2j-2\tilde{j})\tilde{T}^0 + (\tilde{j}+1/2)\tilde{T}^- =$$

$$= \xi(1+\xi)\partial_\xi^2 + \eta(1+\eta)\partial_\eta^2 - 2\xi\eta\partial_\xi\partial_\eta + (\xi+1)(1-c\xi)\partial_\xi + (\eta+1)(1-c\eta)\partial_\eta + 2c(j\xi+\tilde{j}\eta) \quad (60)$$

where $\{\dots\}_+$ denotes hereafter the anticommutator and c is an arbitrary parameter.

Then,

$$(g^{1k}) = \begin{bmatrix} \xi(1+\xi) & -\xi\eta \\ -\xi\eta & \eta(1+\eta) \end{bmatrix} \quad (61)$$

and the curvature

$$R = -\frac{\xi^2 + \eta^2 - 10\xi\eta - 1}{2(1+\xi+\eta)^2} \quad (62)$$

For large ξ and η the curvature tends to a constant which depends, however, on the direction in the (ξ, η) plane. I quote also the results for the phase a and the potential V without further comments:

$$a = \frac{c}{2}(\xi\eta + \xi + \eta) - \frac{1}{4} \ln(\xi\eta(1+\xi+\eta)),$$

$$V = c(\xi+\eta)[2c(\xi+\eta+\xi\eta+1) - \frac{1}{2}] + \frac{20\xi\eta - (\xi+\eta)^2}{2(\xi+\eta+1)^2} - \frac{\xi(\xi+1)^2 + \eta(\eta+1)^2}{4\xi\eta(\xi+\eta+1)^2}$$

$$- c(j\xi + \tilde{j}\eta). \quad (63)$$

Another example with still more sophisticated potential can be found in ref. 11.

(11) $SO(3)$

The algebra of $SO(3)$ is locally isomorphic to that of $SU(2)$. The lowest-dimension representation in $SO(3)$, however, is triplet. Correspondingly, the number of variables needed for the differential realization of $SO(3)$ is not less than two. Three generators (in the differential form) are given Appendix A, eqs.

(A.15), (A.16). These expressions are general, i.e. valid for any SO(3) representation. For convenience we reproduce them here, substituting the integer parameter n by j , just to keep the tradition.

$$\begin{aligned} T^1 &= \eta \partial_\xi - \xi \partial_\eta, \quad T^2 = (1+\eta^2) \partial_\eta + \xi \eta \partial_\xi - j \eta, \\ T^3 &= -(1+\xi^2) \partial_\xi - \xi \eta \partial_\eta + j \xi. \end{aligned} \quad (64)$$

Observe, however, that unlike the SU(2) group, in eq. (64) j is integer if we want to deal with the finite-dimensional representations. The dimension of the representation R_j and, hence, the number of the levels in the algebraized sector is $N = (1+j)(1+\frac{j}{2})$.

Eqs. (64) imply that the generic form of the SO(3) generators is

$$T^a = h^{a1} \partial_1 + p_1^a \xi^1 \quad (65)$$

where h^{a1} are polynomial functions of ξ, η , and p_1^a are constants,

$$\begin{aligned} h^{11} &= \eta, \quad h^{12} = -\xi, \quad p_1^1 = 0, \\ h^{21} &= \xi \eta, \quad h^{22} = 1 + \eta^2, \quad p_1^2 = 0, \quad p_2^2 = -j, \\ h^{31} &= -1 - \xi^2, \quad h^{32} = -\xi \eta, \quad p_1^3 = j, \quad p_2^3 = 0. \end{aligned} \quad (66)$$

The class of the quasi-exactly-solvable problems which will be discussed below is a partial solution of the consistency condition (57). The solution is valid for arbitrary set of C_{ab} provided that $C_a = 0$. More general situations with $C_a \neq 0$ are not found yet.

For arbitrary C_{ab} 's and $C_a = 0$

$$-2H_{\text{nc}} = C_{ab} T^a T^b = C_{ab} h^{a1} h^{b1} \partial_1 \partial_1 + C_{ab} h^{a1} (\partial_1 h^{bk}) \partial_k + 2C_{ab} h^{a1} p_1^b \xi^m \partial_m$$

$$+ C_{ab} p_1^a p_1^b \xi^{1k} + C_{ab} h^{a1} p_1^b \quad (67)$$

where the symmetry of C_{ab} has been taken into account. Notice that H_{nc} , along with the $\partial_1 \partial_k$ term contains also terms linear in ∂_k . This fact is important because otherwise we would automatically get separable variables.

From eq. (67) it stems that

$$g^{1k} = C_{ab} h^{a1} h^{bk}. \quad (68)$$

Moreover, comparing eqs. (67) and (54) and using the latter expression for g^{1k} we arrive at the following relations for $\tilde{A}_1$:

$$C_{ab} (\tilde{A}_1 h^{a1}) h^{bk} = C_{ab} (2p_1^a \xi^{11} - \partial_1 h^{a1}) h^{bk}. \quad (69)$$

It is easy to see that there exists a solution satisfying

$$\tilde{A}_1 h^{a1} = 2p_1^a \xi^{11} - \partial_1 h^{a1}, \quad (70)$$

and, hence, valid for any set of C_{ab} irrespectively of the concrete values of these coefficients. Specifically,

$$\tilde{A}_1 = -\frac{2j+3}{2} \partial_1 \ln(1 + \xi^2 + \eta^2). \quad (71)$$

The corresponding expression for the phase a is logarithmic,

$$a = \frac{j+3/2}{2} \ln(1 + \xi^2 + \eta^2) + \frac{1}{4} \ln g, \quad (72)$$

which results in a power-like fall off of the eigenfunctions from the algebraized part of the spectrum. Therefore, it is necessary to check the normalizability of the wave functions.

Up to this point the coefficients C_{ab} are unspecified. To have an idea of the metric and the potential V let us consider several simple choices.

$$(a). \quad C_{11} = 0, \quad C_{ab} = 0 \text{ if } a \neq b, \quad C_{22} = C_{33} = 1.$$

Then

$$\{g^{ik}\} = \begin{bmatrix} \xi^2 \eta^2 + (1+\xi^2)^2 & \xi \eta (2+\xi^2+\eta^2) \\ \xi \eta (2+\xi^2+\eta^2) & \xi^2 \eta^2 + (1+\eta^2)^2 \end{bmatrix} \quad (73)$$

With this metric, after some algebra one gets

$$\sqrt{g} = (1+\xi^2+\eta^2)^{-1},$$

$$R = -4(\xi^2 + \eta^2 + 1). \quad (74)$$

Moreover, the potential V turns out to be bilinear in ξ^1 ,

$$V = -\frac{3}{8}(\xi^2 + \eta^2). \quad (75)$$

It is interesting that the parameter j is absent in the final answer for V and, hence, we actually deal with the exactly-solvable problem. Moreover, the variables can be separated in the polar coordinates, $\xi = r \cos \theta$, $\eta = r \sin \theta$. In these coordinates the metric takes the form $\{g^{ik}\} = \text{diag}((r^2+1)^2, r^{-2})$. In spite of the fact that the potential V in eq. (75) is negative the eigenfunctions are normalizable. The reason is that the curvature R is non-vanishing and, moreover, tends to infinity at large r .

(b). $C_{11} = C_{22} = C_{33} = 1$, $C_{ab} = 0$ if $a \neq b$.

$$\{g^{ik}\} = \begin{bmatrix} (\xi^2+1)^2 + \xi^2 \eta^2 + \eta^2 & \xi \eta (\xi^2 + \eta^2 + 1) \\ \xi \eta (\xi^2 + \eta^2 + 1) & (\eta^2+1)^2 + \xi^2 \eta^2 + \xi^2 \end{bmatrix}. \quad (76)$$

It is very interesting that this example corresponds to a constant curvature, $R = 2$. The potential V vanishes,

$$V = 0.$$

(c). $C_{22} = C_{33} = 2C_{23} = 1$, other C 's zero.

$$\{g^{ik}\} = \begin{bmatrix} (\xi^2+1-\xi\eta)^2 + \xi\eta(1+\xi^2) & (\xi^2+\eta^2-\xi\eta)(2\xi\eta-1)+3\xi\eta-1)/2 \\ ((\xi^2+\eta^2-\xi\eta)(2\xi\eta-1)+3\xi\eta-1)/2 & (\eta^2+1-\xi\eta)+\xi\eta(1+\eta^2) \end{bmatrix}$$

The curvature corresponding to this metric is

$$R = -4(\xi^2 + \eta^2 + 1 - \xi\eta),$$

while the potential

$$V = \frac{1}{8} \xi \eta (2j-1)(2j-3) - \frac{3}{8} (\xi^2 + \eta^2).$$

Comparing the results for R and V we see that there is no obvious substitution leading to separation of the variables.

(d) $2C_{12} = 1$, other C 's zero.

$$\{g^{ik}\} = \begin{bmatrix} \xi \eta^2 & \eta(\eta^2+1-\xi^2)/2 \\ \eta(\eta^2+1-\xi^2)/2 & -\xi(\eta^2+1) \end{bmatrix} \quad (77)$$

This example is interesting by the fact that the potential is singular,

$$V = -(\xi/8)(2j-3/\eta^2). \quad (78)$$

We arrive at the quasi-exactly-solvable spectral problem for the Laplace operator in the space with the curvature which is also singular,

$$R = 4\xi/\eta^2. \quad (79)$$

The reader can convince himself/herself that in all the problems (a) to (d) the wave functions from the algebraized sector are normalizable,

$$\int e^{-2a\tilde{\psi}^2} \sqrt{g} d\xi d\eta < \infty.$$

Further examples are given in ref. 11. Varying the set of the coefficients C we come across many other non-trivial and interesting situations. Their number is infinite, of course, and what one badly needs is a kind of a guide-book or a catalogue. Let us hope that it will appear soon.

(111) $SU(3)$

It is worth recalling that the arbitrary SU(3) representation can be obtained as a product of several triplets and antitriplets. Thus, it is characterized by two integers, n and m, the numbers of triplets and antitriplets, respectively. The differential realization of SU(3) valid in the general case requires three, not two variables. (This fact follows from the mathematical literature but is also readily verifiable empirically, see Appendix A). Therefore, the realization of the general form is relevant to 3-dim motion. If we limit ourselves, however, to a narrower class of SU(3) representations built exclusively from triplets (n=0, n≠0) we can find a differential realization on two variables. Then eight SU(3) generators are

$$\begin{aligned} T_2^3 &= -\eta^2 \partial_\eta - \xi \eta \partial_\xi + j \eta, & T_2^1 &= -\xi^2 \partial_\xi - \xi \eta \partial_\eta + j \xi, \\ T_1^3 &= \eta \partial_\xi, & T_1^2 &= \partial_\xi, & T_3^2 &= \partial_\eta, & T_3^1 &= \xi \partial_\eta, \end{aligned} \quad (80)$$

$$T_d = \eta \partial_\eta + 2\xi \partial_\xi - j, \quad \tilde{T}_d = -2\eta \partial_\eta - \xi \partial_\xi + j.$$

where j should be considered as an integer number. The multiplicity is the same as in the SO(3) case, $N = (1+j)(1+\frac{j}{2})$.

It is interesting to notice that in the case at hand one can construct quasi-exactly-solvable problems referring to the flat space. Indeed, let us choose the hamiltonian as follows:

$$\begin{aligned} -2H_{\text{nc}} &= \{T_1^2 (\frac{2}{3} \tilde{T}_d + \frac{4}{3} T_d) + \{T_3^2 (\frac{2}{3} T_d + \frac{4}{3} \tilde{T}_d) + \frac{4j}{3} (T_1^2 + T_3^2) + 16\alpha (T_1^2 + T_3^2) - 8\beta (\frac{1}{3} \tilde{T}_d + \frac{2}{3} T_d) + 8\gamma (\frac{1}{3} T_d + \frac{2}{3} \tilde{T}_d)\} = \\ &= 4\xi \partial_\xi^2 + 4\eta \partial_\eta^2 + 2\partial_\xi + 2\partial_\eta - 16\alpha (\xi^2 \partial_\xi + \xi \eta \partial_\eta - j\xi + \eta^2 \partial_\eta + \xi \eta \partial_\xi - \\ &\quad - j\eta) - 8\beta (\xi \partial_\xi - j/3) - 8\gamma (\eta \partial_\eta - j/3). \end{aligned} \quad (81)$$

The corresponding metric $\{g^{lk}\} = \text{diag}(4\xi, 4\eta)$ is characterized by the zero curvature. The following obvious change of variables ensures the transition to cartesian coordinates (x,y):

$$\xi = x^2, \quad \eta = y^2.$$

Moreover, the solution of eqs. (53), (55) is

$$a = \alpha(\xi + \eta)^2 + \beta\xi + \gamma\eta = \alpha r^4 + \beta x^2 + \gamma y^2 \quad (82)$$

where $r^2 = x^2 + y^2$. For the potential $V(x,y)$ one can easily get with the aid of eq. (52):

$$V = 8\alpha^2 r^6 + 8\alpha r^2 (\beta x^2 + \gamma y^2) + 2(\beta^2 x^2 + \gamma^2 y^2) - 8\alpha(j+1) r^2. \quad (83)$$

Thus, we see that the hamiltonian (81) is a direct 2-dim generalization of the problem discussed in sect. 3. The analogy goes even further because in the Schrödinger equation (9) with the potential (82) one can actually separate the variables by passing to the so called generalized elliptic coordinates.

The potential (83) has been obtained independently in [11, 15].

More instructive are, perhaps, typical situations [11] of curved, not flat spaces. An example which immediately comes to one's mind is:

$$\begin{aligned} -2H_{\text{nc}} &= T_{11}^2 + T_{11}^3 + T_{33}^2 + T_{33}^1 - \{T_{31}^1, T_{13}^3\} - \alpha(T_d - \tilde{T}_d) = \\ &= (1 + \eta^2) \partial_\xi^2 + (1 + \xi^2) \partial_\eta^2 - 2\xi \eta \partial_\xi \partial_\eta - (1+3\alpha) \eta \partial_\eta - (1+3\alpha) \xi \partial_\xi. \end{aligned} \quad (84)$$

The coefficient of $\partial_1 \partial_k$ in the latter expression determines the metric.

$$\{g^{lk}\} = \begin{bmatrix} 1 + \eta^2 & -\xi\eta \\ -\xi\eta & 1 + \xi^2 \end{bmatrix}, \quad (85)$$

which is obviously α -independent. As far as the phase a is

concerned it is equal to

$$a = \frac{1}{2} \ln \sqrt{g} + \frac{3\alpha}{4} (\xi^2 + \eta^2). \quad (86)$$

Moreover, the curvature has the form

$$R = 6/(\xi^2 + \eta^2 + 1)^2 \quad (87)$$

and tends to zero asymptotically. It is worth noting that although the corresponding space is non-compact, the Euler characteristic $p = \frac{1}{2\pi} \int R(g)^{1/2} d\xi d\eta = 2$. The potential is j -independent,

$$V = \frac{9}{32} \alpha^2 r^2 - \frac{1}{8(r^2 + 1)} + \frac{5}{8(r^2 + 1)^2} \quad (88)$$

where $r = (\xi^2 + \eta^2)^{1/2}$. After simple manipulations the problem reduces to the standard 2-dim symmetric harmonic oscillator.

Finally, the last $SU(3)$ -associated quasi-exactly-solvable problem discussed below describes the motion in the half-plane $\xi + \eta > \alpha$ where α is a numerical parameter,

$$\begin{aligned} -2H_{\text{osc}} = & \left\{ T_1^2, \left(\frac{1}{6} \tilde{T}_d + \frac{1}{3} T_1 \right) \right\}_+ + T_1^2 T_d^2 - \left\{ T_3^2, \left(\frac{1}{6} \tilde{T}_d + \frac{1}{3} T_1 \right) \right\}_+ + T_1^2 T_3^2 + \\ & + 2\alpha T_3^2 T_1^2 + \left(\frac{1}{2} + \frac{1}{3} \right) (T_3^2 T_1^2 + T_1^2 T_3^2) + \beta (T_2^3 + T_1^3) + \beta \alpha (T_3^1 + T_1^3) = \\ = & (\xi + \eta) \partial_\xi^2 + (\xi + \eta) \partial_\eta^2 + 2\alpha \partial_\xi \partial_\eta - \beta (j\eta - \xi(\xi + \eta) \partial_\xi - \eta(\xi + \eta) \partial_\eta + j\xi) + \\ & + \partial_\xi + \partial_\eta + \beta \alpha (\eta \partial_\xi + \xi \partial_\eta). \end{aligned} \quad (89)$$

Reading the metric off eq. (89) we find

$$\{g\}^{ik} = \begin{bmatrix} \xi + \eta & \alpha \\ \alpha & \xi + \eta \end{bmatrix} \quad (90)$$

Now the standard recipe for calculating the curvature yields [11]

$$R = - \frac{12 \alpha^2}{(\xi + \eta + \alpha)(\xi + \eta - \alpha)^2} - 2 \frac{\xi + \eta + 4\alpha}{(\xi + \eta + \alpha)(\xi + \eta - \alpha)}. \quad (91)$$

We quote also the results of ref. 11 for the phase α and the potential V :

$$a = \frac{1}{2} \ln \sqrt{g} - \frac{\beta}{4} (\xi^2 + \eta^2), \quad (92)$$

$$\begin{aligned} V = & \frac{\beta^2}{8} (\xi + \eta)^3 - \frac{\beta^2}{4} \xi \eta (\xi + \eta) + \frac{\beta^2}{4} (\xi + \eta)^2 + \frac{\beta^2}{4} \alpha \xi \eta + \frac{\beta}{4} (\alpha \beta + \beta + 1 + 2j) \times \\ & \times (\xi + \eta) - \frac{\xi + \eta + 3\alpha}{4(\xi + \eta + \alpha)(\xi + \eta - \alpha)} - \frac{5}{4} \frac{\alpha^2}{(\xi + \eta + \alpha)(\xi + \eta - \alpha)}. \end{aligned} \quad (93)$$

4. Graded algebras

The simple idea which forms the basis of the approach presented here allows for a whole spectrum of generalizations. One of the promising directions is a class of problems with the multi-component wave functions. Up to now we have considered the Schrödinger-type equations on a single scalar function ψ . Quite often, however, one has to deal with the wave function with more than one component. For instance, such is the situation in the coupled-channel problems, in the problems describing the motion of spin-1/2 particles in magnetic fields, etc. In this section we will demonstrate that in this case there exist quasi-exactly solvable hamiltonians as well¹¹. The discussion below gives an idea on the possibilities of the method in constructing multicomponent Schrödinger-type equations with the algebraized part of the spectrum.

In order to discuss the multi-component equations in the present context it is necessary to invoke extensions of the ordinary Lie algebras, namely the graded algebras (in the physical literature they are called also supersymmetric algebras). Here we limit ourselves to the simplest example, $SU(2/1)$ in the Kac notation.

In this group in addition to the standard three SU(2) generators T^a and one U(1) generator J there are four odd elements of the algebra, Q_α and $\bar{Q}_\alpha$ ($\alpha = 1, 2$) transforming as SU(2) doublets. The even elements T^a , J are characterized by the usual commutation relations

$$[T^+, T^-] = 2T^0, \quad [T^\pm, T^0] = \mp T^\pm, \quad (94)$$

$$[J, T^a] = 0 \quad (95)$$

while the odd elements Q_α and $\bar{Q}_\alpha$ anticommute according to the group law

$$\{\bar{Q}_1, Q_2\}_+ = -T^+, \quad \{\bar{Q}_2, Q_1\}_+ = -T^-, \quad \frac{1}{2}(\bar{Q}_1, Q_1 - \bar{Q}_2, Q_2)_+ = T^0,$$

$$\frac{1}{2}(\bar{Q}_1, Q_1 + \bar{Q}_2, Q_2)_+ = J, \quad (96)$$

expressing the fact that the product of doublet and antidoublet yields triplet plus singlet. Finally, the commutation relations between the odd and even elements of the algebra are also quite transparent,

$$[T^+, Q_1] = Q_2, \quad [T^+, Q_2] = 0, \quad [T^-, Q_2] = Q_1, \quad [T^-, Q_1] = 0,$$

$$[T^-, \bar{Q}_1] = \bar{Q}_2, \quad \text{etc.} \quad (97)$$

As far as the U(1) charge of Q and $\bar{Q}$ is concerned, it is opposite for Q and $\bar{Q}$

$$[Q_\alpha, J] = -\frac{1}{2}Q_\alpha, \quad [\bar{Q}_\alpha, J] = +\frac{1}{2}\bar{Q}_\alpha, \quad (98)$$

so that T^a , being the bilinear product of Q and $\bar{Q}$ (eq. (96)), is neutral, see eq. (95).

The group structure presented in eqs. (94) - (98) is certainly not the minimal graded algebra associated with SU(2). From the point of view of the group theory we could easily get rid of the U(1) factor and manage with only two "supercharges", not

four as written down in eq. (96). The minimal version, however, does not seem suitable for quasi-exactly-solvable problems, as will be explained shortly.

Now, can one construct a differential realization of the given graded algebra? The answer is positive, but such a realization will necessarily involve the Grassmann variable θ along with the ordinary variable ξ . Specifically,

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \partial_\theta \\ \xi \partial_\theta \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix} = \begin{bmatrix} \xi \partial_\xi - 2j\theta \\ -\partial_\xi \end{bmatrix} \quad (99)$$

where $\partial_\xi = \partial/\partial\xi$ and $\partial_\theta = \partial/\partial\theta$. We remind that θ and ∂_θ anticommute, so that $\theta^2 = \partial_\theta^2 = 0$, $\partial_\theta(\theta f) = f - \theta\partial_\theta f$, $\{\partial_\theta, \theta\} = 1$.

It is not difficult to check that all commutation relations (94)-(98) are indeed satisfied provided that

$$T^+ = 2j\xi - \xi^2\partial_\xi - \xi\partial_\theta, \quad T^0 = -j + \xi\partial_\xi + \frac{1}{2}\theta\partial_\theta, \quad T^- = \partial_\xi, \quad (100)$$

$$J = -j - \frac{1}{2}\theta\partial_\theta.$$

Thus, the graded algebra generalizing SU(2) is at our disposal, and the next issue to consider is that of finite dimensional representations. Inspection of the generators (66), (100) shows that the finite-dimensional representations exist provided j is seminteger, in perfect analogy with the situation considered in sects. 3, 4:

$$R_{j, \text{graded}} = \{\xi^0, \xi^1, \dots, \xi^j; \xi^0\theta, \xi^1\theta, \dots, \xi^{2j-1}\theta\}; \quad (101)$$

the dimension of $R_{j, \text{graded}}$ is $(4j+1)$.

Before proceeding to construction of the quasi-exactly-solvable systems associated with the given graded algebra a remark is in order concerning the matrix representation

of the Grassmann variables. The fact that the anticommuting variables can be represented in the matrix form is well-known. We will not go into generalities here, and simply formulate the answer expecting that the reader can easily convince himself/herself in the validity of all necessary properties. The recipe is as follows: substitute θ and ∂_θ in the generators (99), (100) by the matrices σ^+ and σ^- , respectively, ($\sigma^\pm = (\sigma^1 \pm i\sigma^2)/2$) acting on two-component spinors; moreover, in the representation (101)

$$\theta \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \partial_\theta \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (102)$$

In the explicit form

$$Q = \begin{bmatrix} \sigma^- \\ \xi \sigma^- \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} \xi \sigma^+ \partial_\xi - 2j\sigma^+ \\ -\sigma^+ \partial_\xi \end{bmatrix},$$

$$R_{J, \text{graded}} = \{ \xi^0 | \downarrow \rangle, \dots, \xi^{2j} | \downarrow \rangle; \xi^0 | \uparrow \rangle, \dots, \xi^{2j-1} | \uparrow \rangle \}. \quad (103)$$

After this remark it should be clear that the quantal hamiltonian we will finally arrive at will act on a two-component wave-function.

The next steps are conceptually the stages of the program outlined in sects. 3, 4 with a single but important exception. Namely, we start from the Schrödinger equation

$$\left[-\frac{1}{2} \frac{d^2}{dx^2} + \hat{V}(x) \right] \psi(x) = E \psi(x) \quad (104)$$

where $\hat{V}(x)$ now is a two-by-two hermitean matrix (the caret over V is to remind us about this fact) while ψ is a two-component wave function,

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

After making the "gauge" transformation we obtain the transformed hamiltonian H_{nc} and try to find the gauge in such a way that H_{nc} would reduce to a quadratic combination of the generators (99), (100). The only peculiarity is that the class of allowed "gauge" transformations becomes much wider: instead of a purely abelian substitution $\psi = \exp(-a)\tilde{\psi}$ we can invoke non-abelian transformations

$$\psi = U \tilde{\psi} \quad (105)$$

where U is, generally speaking, an arbitrary two-by-two matrix.

Then

$$H_{nc} = -\frac{1}{2} (d_x + \hat{A})^2 + U^{-1} \hat{V} U = -\frac{1}{2} d_x^2 - \hat{A} d_x + \Delta V \quad (106)$$

where

$$\hat{A} = U^{-1} (d_x U), \quad \Delta V = U^{-1} \hat{V} U - \frac{1}{2} \hat{A}' - \frac{1}{2} \hat{A}^2. \quad (107)$$

and the prime denotes differentiation over x .

Now, we compare eq. (73) with $C_a T_{ab}^a + C_a T^a$ where the set of the generators T^a includes now all generators written down in eqs. (66), (67). Moreover, identifying the coefficients in front of d_x we can always find $\hat{A}$ and the corresponding matrix U , constructing in this way a quasi-exactly-solvable hamiltonian which acts on a two-component wave function.

Unfortunately, the situation is less favourable than that described in sect. 4. For $SU(2)$ group any combination $C_a T_{ab}^a + C_a T^a$ leads to a "reasonable" hamiltonian (modulo the issue of normalizability of the wave functions, of course!). In the case at hand, however, we obtain something for H_{nc} , and this "something" after the reverse gauge transformation produces the potential $\hat{V}$

$$\hat{V} = U(\Delta V)U^{-1} + \frac{1}{2}(\partial^2 U)U^{-1} \quad (108)$$

which often turns out to be non-hermitean. If we insist on the hermitean hamiltonian H the requirement of hermiticity of $\hat{V}$ imposes an additional condition on $\hat{A}$. The general solution of this constraint is unknown. Thus, we encounter the obstacle perfectly similar to that discussed in sect. 6.

In the absence of the complete solution it seems instructive to consider particular examples. One of them is presented below. (It is worth noticing in passing that with one supercharge no examples with the hermitean potentials have been found thus far. That is the reason why we discuss a non-minimal supersymmetric extension of $SU(2)$).

The starting hamiltonian is

$$H_{"0"} = -\{\tau^0 \tau^1\}_+ - 2j\tau^1 - \bar{Q}_1 Q_1 + \alpha \tau^0 - i\beta(Q_2 \tau^1 + \bar{Q}_1 + 2jQ_1) + \frac{i}{2}\alpha\beta Q_2 - \frac{i}{2}\beta Q_1 \quad (109)$$

where α and β are numerical parameters. Using eqs. (100) and (103) we get after simple substitutions

$$\begin{aligned} H_{"0"} &= -2\xi\partial_\xi^2 - \partial_\xi + \alpha(\xi\partial_\xi + \frac{1}{4}\sigma_3) - i\beta(\xi\partial_\xi\sigma_1 - 2j\sigma_2) + \frac{i}{2}\alpha\beta\xi\sigma^- - \frac{i}{2}\beta\sigma^- \\ &= -\frac{1}{2}\frac{d^2}{dx^2} + (\frac{1}{2}\alpha x - \frac{i}{2}\beta x\sigma^-)\frac{d}{dx} + \frac{\alpha}{4}\sigma_3 - 2\beta j\sigma_2 + \frac{i}{2}\alpha\beta x^2\sigma^- - \frac{i}{2}\beta\sigma^- \quad (110) \end{aligned}$$

(a constant term has been omitted here).

Comparison with eq. (106) yields

$$\hat{A} = -\frac{\alpha x}{2} + \frac{i\beta x}{2}\sigma_1, \quad \Delta V = \frac{\alpha}{4}\sigma_3 - 2\beta j\sigma_2 + \frac{i\alpha\beta}{2}x^2\sigma^- - \frac{i\beta}{2}x\sigma^- \quad (111)$$

which implies, in turn, the following expression for U :

$$U = \exp(-\frac{\alpha x^2}{4} + \frac{i\beta x^2}{4}\sigma_1) \quad (112)$$

Now we are finally able to present the potential $\hat{V}$, a two-by-two matrix, determining a quantal system possessing an algebraized part of the spectrum,

$$\begin{aligned} \hat{V} &= \frac{1}{8}(\alpha^2 - \beta^2)x^2 + \sigma_2(-2j\beta - \frac{\beta}{4} + \frac{\alpha\beta x^2}{4} - \frac{\alpha}{4}\ln\frac{\beta x^2}{2})\cos\frac{\beta x^2}{2} + \\ &+ \sigma_3(-2j\beta - \frac{\beta}{4} + \frac{\alpha\beta x^2}{4} - \frac{\alpha}{4}\operatorname{ctn}\frac{\beta x^2}{2})\sin\frac{\beta x^2}{2}. \quad (113) \end{aligned}$$

It is seen that the potential $\hat{V}$ is hermitean, as we tended to get, and stable; (4j+1) eigenvalues and eigenfunctions (obviously normalizable provided that $\alpha > \beta$) can be found algebraically.

9. Comments on the literature

As was noted in sect. 1 the history of discovery of the spectral problems with the partial algebraization is short. The first publication presenting a one-dimensional potential with two and three exactly known levels seems to have appeared in 1980 [4]. Actually, the author of ref. 4 empirically found the potential built from the hyperbolic functions (see point (d) of sect. 4). A polynomial potential with the similar properties was written first in ref. 16. A systematic investigation of the polynomial potential was undertaken in [8a]. The authors succeeded in constructing a whole family of hamiltonians depending on an integer parameter $2j$; for any given value of this integer parameter $(2j+1)$ levels were shown to be calculable algebraically. The term "quasi-exactly-solvable" has been introduced in this paper.

Different types of one-dimensional hamiltonians ensuring the partial algebraization (about a dozen) are proposed in later works [8-10]. In [8b], in particular, it is proven that eq. (30) is the only possible choice among the polynomial potentials. What happens in the limit $j \rightarrow \infty$ is another question addressed in [8b]. Ref. 10 formulates a procedure opening the way to industrial methods of constructing $\tilde{\psi}$ in the algebraic sector.

Approximately by this time the understanding that the source of the algebraization is a hidden dynamical symmetry matured. The fact that some quantal hamiltonians can be rewritten in terms of the $SU(2)$ generators seems to have been noted for the first time in [5]. This paper is devoted mainly to study of the scattering states; it is called "Group approach to scattering". It contains, however, the assertion that the exact solutions in the Morse and

Pöschl-Teller potentials can be reproduced algebraically if one uses a representation of the corresponding hamiltonians bilinear in the $SU(2)$ generators. (The differential realization of the $SU(2)$ generators accepted in [5] is more complicated than that quoted in eq. (18)).

The first application of the $SU(2)$ algebra in quasi-exactly-solvable systems refers to 1984. The hamiltonian discovered in [4] was rewritten [6] in terms of the $SU(2)$ generators thus explaining the fact that several eigenvalues and eigenfunctions are exactly calculable. The next step is done in an essentially unnoticed work [7] where many examples of the $SU(2)$ -generated hamiltonians with the partial algebraization are considered. Finally, the general understanding that any quadratic combination of the generators (18) leads to a certain 1-dim quasi-exactly-solvable problem was achieved in [9]. This work presents the exhaustive classification of all 1-dim systems and outlines a regular method for obtaining algebraization. An interesting technical device, supplementing the ideas of ref. 9, can be extracted from [10] (see sect. 5). The latter work is interesting by itself because it reveals a connection between the partial algebraization in the spectral problems and the Bethe ansatz in 2-dim field theory. Further investigations in this direction are reported in ref. 15. Mechanical analog systems presented in sect. 5 are actually introduced in these papers which also discuss the properties of the analog systems from different points of view. In particular, the reader can find in [15] the answer to the question: "what happens if the zeroes a_α and/or ξ_α are in the complex plane, not on the ξ axis".

Perhaps, it is worth adding that the work [9] is distinguished both, historically and psychologically: it completes the first stage and opens the modern stage of development. At present extremely promising and interesting generalizations are under investigation. The most obvious direction is the multi-dimensional spectral problems with the partial algebraization. Actually, their analysis (based on the non-algebraic approach [10]) started [15] independently of the publication [9] and in parallel with the work [11]. In ref. 11 the idea of the hidden symmetry $(SU(2) \times SU(2), SO(3), SU(3))$ is used in full for constructing two-dimensional quasi-exactly-solvable systems on a regular basis. Moreover, it is shown [11] that graded algebras lead to multi-component Schrödinger-type equations with the same properties.

Some related issues are discussed in [17].

It is worth noting that only in retrospective the development described above seems logical and continuous. As a matter of fact, many of the authors mentioned here did not know about the existence of the predecessors and/or parallel work. The results already existing in the literature were independently rediscovered more than once.

10. Summary and conclusions. Prospects for future

The partial algebraization of the spectrum is a new phenomenon discovered in the very old problem. Although serious practical applications are not seen at the moment, from the theoretical point of view the idea and its realization are simple and beautiful; work on this subject gives pleasure. At least, I had a lot of fun.

A dynamical symmetry present in the hamiltonian but invisible

for those who are not prepared is a desirable finding for every theorist. The approaches reducing the hamiltonian to simpler operators obeying certain algebra are usually especially fruitful. Suffice it to mention supersymmetric field theories where an enormous number of results have been obtained due to the fact that $H \sim Q\bar{Q}$ (Q stands for the supercharge). Perhaps, even more close in spirit to the ideas exploited here are the works [18] well-known to experts in 2-dim field theory. D=2 field theories considered in ref. 18 have a remarkable property: the energy-momentum tensor $\Theta_{\mu\nu}(x)$ turns out to be proportional to the product of certain conserved currents $j_\mu(x)$ existing in these models. This fact alone allows one to solve them exactly [18]. At present similar methods are successfully developed and used in conformal field theory.

The novel feature is the possibility of representing the Schrödinger-type hamiltonians in the form (21) or (49). It is worth emphasizing that the sphere of applications is by no means limited to quantum mechanics. We encounter similar spectral problems when computing all types of functional determinants, a usual business of contemporary theorists.

Although the algebraic origin of the phenomenon has been understood only recently, the results are already rather numerous. In one-dimensional case there exists an exhaustive analysis of quasi-exactly-solvable problems, and many especially attractive examples are discussed in great detail. For $D > 1$ investigations started. The most interesting finding, to my mind, is an intriguing connection between the algebraic structure (49) and the geometric structure of the emerging manifolds. I would like to understand this connection more deeply and in both directions. What set of the

coefficients C must be chosen in order to ensure the desired geometry of the manifold? What manifolds can be obtained in this way at all? Can one give their general classification? These are the questions to be answered in future. The use of the graded algebras in the multi-component systems also seems very promising.

Another direction of future investigations which seems to be extremely important stems from the fact that the quasi-exactly-solvable hamiltonians are in one-to-one correspondence with two-dimensional field theories integrable within the Bethe ansatz. The nodes of the wave functions from the algebraic sector obey precisely the same equations as the rapidities of particles in the scattering matrix in certain completely integrable 2-dim models! Actually, equations like (43a) in the field theory play the role of the famous conditions of the S-matrix factorizability. In the field theory we have a spacial distribution of, say, spins while in the quasi-exactly-solvable problems there is only one "spin". And still, there exists a mysterious kinship between these two systems which, being understood, could lead to a real breakthrough. I keep in mind something like the Eguchi-Kawai solution of the multi-color QCD.

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Appendix A

Lie groups: differential realizations in the space of inhomogeneous polynomials

The basic idea of the approach presented above is the substitution of the second order differential operator (hamiltonian) by a quadratic combination of the generators of some group. Such a transformation is possible only because the group generators can be represented as linear functions of derivatives acting on polynomials of a certain number of variables. This representation is known in the mathematical literature. Physicists are more familiar, however, with the matrix representation and, therefore, it seems useful to explain the relation between them, sketch the basic points of the derivation and present explicit formulae for most important cases.

The simplest differential realization of the Lie groups is built on the homogeneous polynomials. Let us consider the example most convenient for the pedagogical purposes, $SU(2)$.

The fundamental representation is two-dimensional, and the standard matrix form of the generators is

$$T^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad T^3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (A.1)$$

In this form two elements of the doublet representation are

$$R_2 = \{ | \uparrow \rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } | \downarrow \rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}. \quad (A.2)$$

Since we would like to construct a realization on polynomials we introduce two independent variables, x and y , and identify them with the elements of the doublet representation,

$$|\alpha\rangle \leftrightarrow y, \quad |\beta\rangle \leftrightarrow x. \quad (\text{A.3})$$

Notice that we can not formulate a non-trivial orthogonality condition for these elements. It can be imposed only after the operation of the hermitean conjugation is defined, so that we have both bra- and ket-states. We will consistently work only with the ket-states avoiding introduction of the hermitean conjugation and bra-states. (Under this condition the notion of orthogonality for $|\alpha\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\beta\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not exist either).

Now, eqs. (A.1), (A.3) imply

$$\begin{aligned} T^+ x &= y, & T^+ y &= 0; \\ T^- x &= 0, & T^- y &= x; \\ T^0 x &= -\frac{1}{2} x, & T^0 y &= \frac{1}{2} y. \end{aligned} \quad (\text{A.4})$$

Eqs. (A.4) immediately allow us to write the generators in the differential form

$$T^+ = yd/dx, \quad T^- = xd/dy, \quad T^3 = \frac{1}{2} (yd/dy - xd/dx). \quad (\text{A.5})$$

This realization is obviously valid not only for the doublet realization (A.2) but for all higher representations of the type $(R_2 \times R_2 \times \dots \times R_2)_{\text{symmetrized}}$. This is a very important general feature of all differential realizations with the group generators linear in derivatives. If a realization is found for some representation it can be applied to the product. Indeed, for any representation R_j constructed as a product of lower representations,

$$R_j = (R_1 \times R_2 \times \dots)$$

the action of the generators T^a , by definition, must reduce to

$$T^a R_j = (T^a R_1) \times R_2 \times \dots + R_1 \times (T^a R_2) \times \dots + \dots$$

This law exactly coincides with the standard rules of differentiation.

The symmetrization property mentioned above is obtained automatically because all doublets are identical. If we would like to avoid symmetrization it would be necessary to introduce more states, and more variables. Notice that in SU(2) the symmetrized product (A.6) covers all possible representations. This is not the case for higher groups, see below.

The elements of (A.6) are the homogeneous expressions $x^k y^{n-k}$ of the n-th order ($k = 0, 1, \dots, n$ where n is the number of factors in (A.6)). Applying the generators (A.5) to $x^k y^{n-k}$ we get an expression from the same multiplet, as it should be for any finite-dimensional representation.

The homogeneous realization (A.5) does not suit for our purposes, however, since the bilinear combination $C_{ab} T^a T^b + C_a T^a$ does not lead in this case to physically interesting hamiltonians. We need a differential realization built on inhomogeneous polynomials. It is rather obvious that such a realization exists. Indeed, the generators (A.5) transform the elements of

$$\begin{aligned} (R_2 \times R_2 \times \dots) \equiv R_2^n &= \{ x^n y^0, x^{n-1} y^1, x^{n-2} y^2, \dots, x^0 y^n \} \equiv \\ &\equiv x^n \{ 1, \xi, \xi^2, \dots, \xi^n \}, \end{aligned}$$

$$\xi = \frac{y}{x},$$

into each other. If one is not interested in the overall factor x^n one can trivially rewrite the SU(2) generators so that they will act on $\{ 1, \xi, \xi^2, \dots, \xi^n \}$ thus mixing different powers of ξ . To this end let us notice that

$$T^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad T^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad T^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (A.9)$$

Here I changed the normalization as compared to (A.1) so that

$$[T^a T^b] = \varepsilon^{abc} T^c$$

(the i factor in the right-hand side is absent). The corresponding triplet representation consists of three vectors

$$R_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad (A.10)$$

Our aim is to pass from (A.9), (A.10) to the differential realization with

$$R_3 = \{x, y, z\} \quad (\text{the homogeneous space}), \quad (A.11)$$

or

$$R_3 = \{1, -\xi, \eta\}, \quad \xi = -\frac{y}{x}, \quad \eta = -\frac{z}{x} \quad (\text{the inhomogeneous space}). \quad (A.12)$$

Comparing (A.9) - (A.11) it is not difficult to get

$$T^1 x = 0, \quad T^1 y = -z, \quad T^1 z = y;$$

$$T^2 x = -z, \quad T^2 y = 0, \quad T^2 z = x; \quad (A.13)$$

$$T^3 x = -y, \quad T^3 y = x, \quad T^3 z = 0.$$

Eqs. (A.13) imply in turn that, say,

$$T^2 \{ \underbrace{R_3 \times R_3 \times \dots \times R_3}_n \} = (T^2 x^n) \{ (1, -\xi, \eta) \times \dots \times (1, -\xi, \eta) \} +$$

$$x^n \sum \{ (1, -\xi, \eta) \times \dots \times (T^2 (1, -\xi, \eta)) \times \dots \times (1, -\xi, \eta) \} =$$

$$= x^n \{ (-n\eta) \{ (1, -\xi, \eta) \times \dots \times (1, -\xi, \eta) \} + \sum \{ (1, -\xi, \eta) \times \dots \times$$

$$(0, -\xi\eta, 1+\eta^2) \times \dots \times (1, -\xi, \eta) \} \quad (A.14)$$

Now we can read off the expression for T^2 :

$$T^2 = -n\eta + \eta\xi\partial_\xi + \partial_\eta + \eta^2\partial_\eta \quad (A.15)$$

$$T^+ \{ \underbrace{(x, y) \times (x, y) \times \dots}_n \} = T^+ x^n \{ \underbrace{(1, \xi) \times (1, \xi) \times \dots}_n \} =$$

$$(T^+ x^n) \{ \underbrace{(1, \xi) \times (1, \xi) \times \dots}_n \} + x^n \sum_k \{ \underbrace{(1, \xi) \times \dots \times (T^+ (1, \xi)) \times \dots \times (1, \xi)}_{n-k-1} \} =$$

$$n\xi \xi \{ \underbrace{(1, \xi) \times (1, \xi) \times \dots}_n \} + x^n \sum_k \{ \underbrace{(1, \xi) \times \dots \times (0, -\xi^2) \times \dots \times (1, \xi)}_{n-k-1} \}, \quad (A.7)$$

plus similar relations for T^- and T^3 . Omitting the overall factor x^n we see from eq. (A.7) that

$$T^+ = n\xi - \xi^2 \frac{d}{d\xi}, \quad T^0 = -\frac{1}{2}n\xi + \xi \frac{d}{d\xi}, \quad T^- = \frac{d}{d\xi}. \quad (A.8)$$

(Compare with eq. (18) where $j = n/2$).

A few remarks are in order here concerning the derivation given above. We started from the doublet representation of SU(2).

This is not necessary, of course. If the matrix form of the generators is known for some representation we can immediately translate this knowledge into differential expressions like (A.8).

The corresponding differential realization will be built on D variables where $D = \dim R - 1$, and will be applicable not only to the original representation R but also to all symmetrized products $R \times R \times \dots$. The differential realization obtained in this way is not necessarily the most economic one. Thus, in SU(2) the realization

(A.8) which includes only one variable covers all possible representations. If we started from the triplet in SU(2) we would need two variables and we would get expressions (linear in ∂_ξ and ∂_η) for the generators acting on the elements of $((n+1)(n+2)/2)$ -dimensional representations. Actually, the latter construction refers to SO(3) and it is worthwhile discussing it separately.

In the matrix form the O(3) generators are

Absolutely analogously

$$T_1^1 = \eta \partial_\xi - \xi \partial_\eta, \quad (A.16)$$

$$T_2^3 = n\xi - \partial_\xi - \xi^2 \partial_\xi - \xi \eta \partial_\eta \quad (A.16)$$

where n is an arbitrary integer number. For $SU(3)$ one triplet is sufficient for constructing all other representations. Therefore, the realization (A.15), (A.16) exhausts the problem, in the same way as eq. (A.8) describes all representations in $SU(2)$.

The situation in $SU(3)$ is more complicated. The fundamental representation in $SU(3)$ is triplet. This means that acting in the same vein as above (cf. eqs. (A.1) - (A.4)) one can easily construct a differential realization on two variables, ξ and η .

Namely, if

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow y, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow x, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow z$$

$$R = (1, \xi, \eta) \times \dots \times (1, \xi, \eta), \quad \xi = \frac{y}{x}, \quad \eta = \frac{z}{x},$$

then, for instance,

$$T_2^1 y = 0, \quad T_2^1 x = y, \quad T_2^1 z = 0,$$

and, hence,

$$T_2^1 = n\xi - \xi^2 \partial_\xi - \xi \eta \partial_\eta. \quad (A.17)$$

Here

$$T_2^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (A.18)$$

For other generators we have

$$T_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow \partial_\xi, \quad T_3^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow \xi \partial_\eta.$$

$$T_1^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \leftrightarrow \eta \partial_\xi, \quad T_3^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow \partial_\eta, \quad (A.19)$$

$$T_2^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \leftrightarrow n\eta - \eta \xi \partial_\xi - \eta^2 \partial_\eta,$$

$$T_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow -n + 2\xi \partial_\xi + \eta \partial_\eta,$$

$$T_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \leftrightarrow n - \xi \partial_\xi - 2\eta \partial_\eta.$$

If the parameter n is integer the realization (A.17), (A.19) possesses finite-dimensional representations,

$$R = \{\xi^n, \xi^{n-1}\eta, \dots, \eta^n; \xi^{n-1}, \xi^{n-2}\eta, \dots, \eta^{n-1}; \dots; \xi, \eta, 1\} \quad (A.20)$$

with

$$\dim R = \frac{1}{2}(n+1)(n+2). \quad (A.21)$$

Already from eq. (21) we see that the simple construction presented here does not include all possible representations of $SU(3)$. Actually, as was explained above, it covers only those representations which are the (symmetrized) product of one triplet. Thus, they are marked by a single integer, n , the number of triplets (the letter j is used in sect. 7 instead of n). It is well-known that the arbitrary $SU(3)$ representation is built from n triplets and m antitriplets and is marked by two integers, n and m .

Therefore, the realization (A.17), (A.19) based on two variables is applicable only to the degenerate class of the $SU(3)$ representations. The differential realization of $SU(3)$ of the general form requires minimally three variables, ξ, η, ζ . Corresponding formulae exist in the mathematical literature, but, unfortunately, at the moment I am aware of no simple derivation. Therefore, I just quote one of the possible answers without further

explanations:

$$T_1^2 = \partial_\xi, \quad T_1^3 = \partial_\eta, \quad T_2^3 = -\xi \partial_\eta + \partial_\xi;$$

$$T_2^1 = -\xi^2 \partial_\xi - \xi \eta \partial_\eta + \xi \zeta \partial_\zeta + \eta \partial_\zeta + n \xi,$$

$$T_3^1 = -\eta^2 \partial_\eta - \xi \eta \partial_\xi - \eta \zeta \partial_\zeta - \xi \zeta^2 \partial_\zeta + m \xi \zeta + (m+n)\eta,$$

$$T_3^2 = -\zeta^2 \partial_\zeta - \eta \partial_\xi + m \zeta,$$

$$T_d = 2\xi \partial_\xi + \eta \partial_\eta - \zeta \partial_\zeta - n,$$

$$\tilde{T}_d = 2\zeta \partial_\zeta - \xi \partial_\xi + \eta \partial_\eta - m. \quad (A.22)$$

Here n and m are integers playing the role of the number of triplets and antitriplets, respectively. The reader can check himself that all commutation relations for the generators (A.22) are precisely the ones that are obtained for the standard matrix form, see eqs. (A.18), (A.19). The simplest non-trivial representations, triplet and antitriplet, are

$$R_3 = \{-\eta, -\xi, 1\}, \quad (m=0, n=1);$$

$$R_{\bar{3}} = \{1, \zeta, \xi+\eta\}, \quad (m=1, n=0).$$

Notice that for R_3 and $R_{\bar{3}}$ the notion of the scalar product $(R_3 R_{\bar{3}})$ is well-defined, and the construction presented here incorporates the condition $(R_3 R_{\bar{3}}) = 0$.

It is easy to understand the fact that for $SU(3)$ it is sufficient to have three independent variables. Indeed, any representation of $SU(3)$ can be built from one triplet and one antitriplet. If we work in the space of the inhomogeneous polynomials each of them requires two variables (four altogether) but we impose the additional condition, namely, assume that the singlet built from the triplet and antitriplet vanishes.

The general formula for the minimal number of variables D sufficient for all representations of an arbitrary group G is

$$D = \frac{1}{2} (\dim G - \text{rank } G),$$

where $\dim G$ is the dimension of the adjoint representation and $\text{rank } G$ is the rank of the group.

If the minimal differential realizations of the general form require certain work, the simplified procedure described in the beginning of the Appendix is readily generalizable. For instance, for $O(N)$ and $SU(N)$ ($N = 4, 5, \dots$) the corresponding expressions only trivially differ from eqs. (A.15) - (A.17), (A.19).

Appendix B

Algebraic solution of the Coulomb problem

In sect. 4 I mentioned that the spectrum in the Coulomb potential can be found algebraically, essentially using the same ideas that were described in sects. 3, 4. The crucial point is the possibility of representing the corresponding hamiltonian H_{Coul} as a quadratic combination of the generators (18). Conceptually this case belongs to the class of the exactly-solvable problems discussed in great detail in the beginning of sect. 4. (point (1)). There is a technical nuance, however, which distinguishes the Coulomb motion from other exactly-solvable problems admitting a standardized treatment (the five-parameter family of the generalized Pöschl-Teller potentials). The Coulomb potential is not included directly in this five-parameter family.

Below we will explain the technical difficulty which does not allow one to obtain the Coulomb potential directly from eqs. (39).

(29b) and indicate a roundabout way. But first of all, the starting definitions.

The Coulomb system will be considered in three-dimensional space, although this assumption is inessential. Then the radial part of the hamiltonian is

$$H_{\text{Coul}} = -\frac{1}{2} d_r^2 - \frac{1}{r} d_r + \frac{C}{r^2} - \frac{\alpha}{r} \quad (\text{B.1})$$

where $d_r = (d/dr)$ and, usually, $C = \ell(\ell+1)/2$, ℓ stands for the angular momentum. The concrete value of C is not important, and we will treat it as a free parameter till the very end.

A short reflection shows that, unfortunately, the "gauge" transformation reducing the hamiltonian (B.1) to the general form (40a),

$$A(r) = C_1 + \frac{C_2}{r} \quad (C_{1,2} \text{ are constants})$$

is not compatible with the representation (40b). Thus, we can not treat the Coulomb problem (B.1) directly within the standard SU(2) program.

Instead, let us introduce an auxiliary hamiltonian

$$H = \gamma \left(-\frac{1}{2} d_r^2 - \frac{1}{r} d_r + \frac{C}{r^2} \right) - \frac{\alpha}{r} + \frac{\gamma}{2} \delta^2 \quad (\text{B.2})$$

where γ and δ are constant coefficients,

$$\delta = \frac{\alpha}{\gamma \left(\frac{1}{2} + \left(\frac{1}{4} + 2C \right)^{1/2} \right)} \quad (\text{B.3})$$

The expression (B.2) differs from (B.1) by a renormalization of the kinetic term and an additive constant.

Now, starting from (B.2) we calculate the "gauge"-transformed hamiltonian choosing the following gauge potential

$$A(r) = \delta + \frac{\nu}{r}, \quad (a = \delta r + \nu \ln r) \quad (\text{B.4})$$

where

$$\nu = \frac{1}{2} - \left(\frac{1}{4} + 2C \right)^{1/2} \quad (\text{B.5})$$

Then

$$H_{\text{nc}} = \gamma \left(-\frac{1}{2} d_r^2 - \frac{1}{r} d_r + \frac{1}{r} d_r \right) \quad (\text{B.6})$$

It is important that the derivative-free term in H_{nc} vanishes exactly, and not up to an additive constant, as was the case in the examples given in sect. 3, 4 (the reason will become clear shortly).

As has been already noticed, the hamiltonian (B.6) does not fit the general expression (39), (40). But rH_{nc} does! Actually, rH_{nc} is precisely of the form (39) provided we identify r^4 and ξ . So, we can readily find all eigenvalues and eigenfunctions of rH_{nc} with the aid of the simple algebra:

$$rH_{\text{nc}} \tilde{\psi}_n = E_n(\gamma) \tilde{\psi}_n \quad (\text{B.7})$$

In eq. (B.7) I have explicitly indicated that all eigenvalues E_n depend on γ (α and C are assumed to be fixed numbers).

But eq. (B.7) is equivalent to

$$H_{\text{nc}} \tilde{\psi}_n = (r^{-1} E_n(\gamma)) \tilde{\psi}_n \quad (\text{B.8})$$

which implies, in turn,

$$\left(H - \frac{E_n}{r} \right) \tilde{\psi}_n = 0 \quad (\text{B.9})$$

where

$$\tilde{\psi}_n = \psi_n \exp(-\delta r - \nu \ln r). \quad (\text{B.10})$$

Using now the explicit expression (B.2) for the hamiltonian H we arrive at

$$\left[\gamma \left(-\frac{1}{2} d_r^2 - \frac{1}{r} d_r + \frac{C}{r^2} \right) - \frac{\alpha + E_n(\gamma)}{r} + \frac{\gamma}{2} \delta^2 \right] \psi_n = 0 \quad (B.11)$$

Dividing by γ and regrouping the terms

$$\left[\left(-\frac{1}{2} d_r^2 - \frac{1}{r} d_r + \frac{C}{r^2} \right) - \frac{\alpha + E_n(\gamma)}{\gamma} \frac{1}{r} \right] \psi_n = -\frac{1}{2} \delta^2 \psi_n \quad (B.12)$$

We observe that we have actually solved the original Coulomb problem (B.1) provided that the parameter γ is chosen as follows

$$\frac{\alpha + E_n(\gamma)}{\gamma} = \alpha \quad (B.13)$$

For each n the algebraic procedure at first fixes $E_n(\gamma)$ and then γ from eq. (B.13). In this way we determine γ_n , the value of the effective parameter in (B.2), relevant to the n -th level. Moreover, $(-\delta^2/2)$ in eq. (B.12) plays the role of the eigenvalue, and δ also depends on n via γ (cf. eq. (B.3)).

Now it is hopefully clear why I introduced the auxiliary parameter γ in eq. (B.2) and why it was necessary to keep the constant term $(-\gamma\delta^2/2)$ which normally can be safely omitted. The point is that unlike the standard situation the SU(2)-based treatment of the Coulomb problem requires adjusting the numerical parameters in the auxiliary hamiltonian (B.2) so that they become n -dependent.

To complete the analysis return to eq. (B.7), which can be rewritten as

$$\gamma \left[-\frac{1}{2} r d_r^2 + (\delta r + (\nu-1)) d_r \right] \tilde{\psi}_n = E_n \tilde{\psi}_n \quad (B.14)$$

It is quite obvious that

$$\tilde{\psi}_n = r^n + C_{n-1} r^{n-1} + \dots + C_0, \quad n = 0, 1, 2, \dots \quad (B.15)$$

where C_j are constant coefficients, to be determined from a trivial recurrence relation. We leave this task as an exercise, and will

find in this Appendix only E_n . This is even a more trivial exercise because the only relevant term then is $\gamma \delta r$. Hence,

$$E_n(\gamma) = \gamma \delta n \quad (B.16)$$

Substituting eqs. (B.16), (B.3) in eq. (B.13) yields

$$\gamma_n = 1 + \frac{n}{\frac{1}{2} + (\frac{1}{4} + 2C)^{1/2}} \quad \delta_n = \frac{\alpha}{\frac{1}{2} + (\frac{1}{4} + 2C)^{1/2} + n} \quad (B.17)$$

Now we are able to say that the eigenvalues in the Coulomb problem (B.1) are

$$-\frac{1}{2} \delta_n^2 = -\frac{\alpha^2}{2(\frac{1}{2} + (\frac{1}{4} + 2C)^{1/2} + n)^2}, \quad n = 0, 1, 2, \dots \quad (B.18)$$

(cf. eq. (B.12)). If $C = \ell(\ell+1)/2$ this expression reduces to the standard form

$$-\frac{\alpha^2}{2n^2}, \quad n = \ell+1, \ell+2, \dots \quad (B.19)$$

To obtain the eigenfunctions one combines (B.10) and (B.15).

Fig. 1. The double-well potential in the simplest quasi-exactly-solvable problem.

Fig. 2. The classical analog system for the simplest quasi-exactly-solvable problem with the potential depicted on fig. 1. The closed circle denotes the position of the fixed force centre while the crosses denote moving particles. Between all the particles there is a repulsion. The force centre at short distances also repels, but at large distances it attracts the particles with the linearly growing force.

Fig. 3. The classical analog system corresponding to the hamiltonian (32). The force centres a_1, a_2, a_3, a_4 are situated at the zeroes of $P_4(\xi)$ and their number coincides with the degree of this polynomial. The interaction of the moving particles and the force centres is determined by eq. (43b). The Coulomb charges of the force centres are $-b_a$.

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