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LOOP-CORRECTED STRING FIELD EQUATIONS FROM WORLD-SHEET WEYL INVARIANCE

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Abstract: We investigate short distance singularities inside the Polyakov path integral in an arbitrary closed bosonic string background due to vertex operators and small handles or holes. The latter are represented in terms of handle or hole operators, which we construct explicitly. The requirement of global Weyl invariance generates loop-corrected equations of motion for the string modes including tadpole terms and self-energy corrections. The relation to the S -matrix and superstrings are discussed briefly.

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1. Introduction

The derivation of string field equations of motion via the requirement of conformal invariance of a two-dimensional field theory has become a useful tool for investigations of string theory. In the case of massless string fields one has to investigate a two-dimensional sigma model involving a finite number of coupling constant functions which are associated with the corresponding vertex operators, and the vanishing of the β -functions carries indeed the same amount of information as the equations of motion derived from string scattering amplitudes in all examples studied so far [1 – 10].

In practice one has to confine oneself, however, to a perturbative expansion in the inverse string tension α' corresponding to a loop-wise expansion in the two-dimensional field theory. This can be overcome, and more insight can be gained, if one treats the massless and the massive string modes on the same footing. Then one has to investigate a two-dimensional field theory including operators of arbitrarily high (classical) dimension. It has been proposed by Banks and Martinec [11] to apply the exact renormalization group for such a system [12] to string theory. Not only can one derive string field equations including the massive modes this way [13 – 19], it is also possible to learn something about the string gauge symmetries [11, 19]. There exists a choice of the cutoff (dimensional regularization [18] or modifying the propagator on the world sheet [11, 17, 19]) such that the β -functions are at most bilinear in the coupling constants. This suggests a close connection to such string field theories which involve as interaction term just a three-string vertex and, correspondingly, just bilinear terms in the equation of motion.

At first sight the consideration of string loop effects in this context seems to be problematic: Contributions to the β -functions arise from short-distance singularities on the world sheet and hence are independent from its topology. On the other hand string loop contributions to the β -functions, and thus to the string equations of motion, require the existence of topology-dependent conformal anomalies.

In order to reconcile these statements let us first identify the sources of divergences in, say, closed bosonic one-loop amplitudes [20, 21]. These are

- a) The emission of an on-mass-shell dilaton (and zero momentum tachyon) from the closed string tree which is then attached to the torus (the dilaton/tachyon tadpole) (or: all vertex operators on the world sheet are simultaneously very close to each other);
- b) External-leg or self-energy corrections, where a handle is attached to the tube corresponding to an incoming or outgoing state (or: all vertex operators but one are simultaneously very close to each other).

Fischler and Susskind [22] have observed that the divergence a) can in fact be viewed as a kind of short-distance singularity on the world sheet: The torus may be parametrized in terms of the complex plane with two holes, the boundaries of which are identified. Two integrals over the location of vertex operators on the complex plane can be rewritten in terms of integrals over the location of the holes. A divergence arises in the limit where the distance between the holes (and their size) becomes very small and is proportional to the insertion of a dilaton (and tachyon) vertex operator. After regularization one thus

obtains a one-loop contribution to the dilaton/tachyon β -function. In the meantime the consistency of this kind of mechanism has been checked by a number of authors [23 – 27].

It is also the purpose of the present paper to treat string tree and loop contributions to the conformal anomaly, and thus to the equations of motion, on an equal footing. The starting point is the Polyakov path integral [28] and the requirement of its independence of the Liouville mode σ . If short-distance singularities on the world sheet are regularized in a way that reparametrization invariance is maintained (e.g. by dimensional regularization [18] or a cutoff $\epsilon e^{-\sigma}$) an extra σ -dependence arises. The requirement of the cancellation of the σ -dependence even in the background of string fields G^n gives rise to equations of motion for these string fields. These are derived in sect. 2 at string tree level. The results are in fact identical with the ones presented in ref. [18], but here we use a slightly different apparatus, which is more adapted to the inclusion of string loop effects.

In sect. 3 we will show that these can be taken into account by inserting suitable operators (handle and hole operators in the case of the torus and disk, respectively) into the correlation functions. A comparison of the volume of the group of conformal Killing vectors of the different world sheets will show that in the case of the torus (disk) we have to integrate over the size of the handle (hole). This will give rise to new divergences due to the following situations:

- a) The size of the handle (hole) shrinks to zero, but no vertex operator is around.
- b) The size of the handle (hole) shrinks to zero, and simultaneously one vertex operator gets very close to it.

These situations obviously correspond to the ones mentioned before. Due to the power of the apparatus developed in sects. 2 and 3 it will be possible to combine these new sources of anomalous σ -dependence into single loop-corrected string field equations in sect. 4. These will be modified due to a) a new constant term in the dilaton (and tachyon) equation of motion; b) new mass terms proportional to operator product expansion (OPE) coefficients appearing in the OPE of the handle- (hole-) and one vertex operator.

Although the idea of considering loop-corrected string field equations is not new [16, 17, 22 – 27], we find a systematic derivation, based on the Polyakov path integral and including all string modes, quite helpful.

The application of our methods to superstrings still has to be worked out. From the known modifications - the absence of tadpoles and self-energy corrections for massless modes - we can, however, already deduce the modifications of the equations of motion. After relating S -matrix elements to equations of motion it will become clear that n -particle S -matrix elements for massless modes get loop corrections only for $n \geq 4$ in agreement with known non-renormalization theorems [20, 21]. Thus there is no paradox “Fischler-Susskind mechanism versus vanishing tadpoles” in the case of the superstring. The point is simply that massive or off-shell modes do get self-energy corrections, and only for $n \geq 4$ the S -matrix elements of the massless modes do become sensitive to intermediate massive/off-shell modes. These issues, conclusions and an outlook will be presented in sect. 5. Appendix A is devoted to the explicit construction of (multi-) handle and hole operators from Green functions on higher genus Riemann surfaces, and appendix B treats the case of the disk, which is only briefly mentioned in sect. 3, in some detail.

2. Tree Level Equations of Motion from Weyl Invariance

We will consider the closed bosonic string theory defined by the Polyakov path integral [28 – 31]

$$Z = \sum_M g^{2h_M-2} \int D\gamma_{\alpha\beta} DX \frac{1}{V(Diff) \times V(Weyl)} e^{-(S_0 + S_{c.t.} + S_{int})} \quad (2.1)$$

Here M enumerates two-manifolds of different genus h_M endowed with a metric $\gamma_{\alpha\beta}$. g denotes the (renormalized) string coupling constant. $V(Diff)$ denotes the volume of the group of diffeomorphism and $V(Weyl)$ the volume of the group of local Weyl rescalings. S_0 is the free action of the bosonic fields $X^i (i = 1 \dots D$ with D the space-time dimension; we set $2\pi\alpha' = 1$)

$$S_0 = \frac{1}{2} \int d^2z \sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^i(z) \partial_\beta X^i(z). \quad (2.2)$$

Renormalization requires a counter term

$$S_{c.t.} = \mu^2 \int d^2z \sqrt{\gamma}. \quad (2.3)$$

Finally S_{int} depends on the normal ordered vertex operators $V_n(X(z), p)$ multiplied with sources $G^n(p)$,

$$S_{int} = \sum_n \int d^D P \int d^2z G^n(p) V_n(X(z), p), \quad (2.4)$$

such that S -matrix elements are generated by functionally differentiating Z with respect to G^n :

$$\begin{aligned} \int d^2z_1 \dots d^2z_k \langle V_{n_1}(z_1, p_1) \dots V_{n_k}(z_k, p_k) \rangle \\ = \left. \frac{\delta^k Z}{\delta G^{n_1}(p_1) \dots \delta G^{n_k}(p_k)} \right|_{G^n=0} \end{aligned} \quad (2.5)$$

Decomposing variations of the world-sheet metric $\gamma_{\alpha\beta}$ into Weyl rescalings, diffeomorphisms and Teichmüller deformations involving parameters τ_k we may partially carry out the γ -path integral and formally write

$$\int \frac{D\gamma}{V(Diff) \times V(Weyl)} \cong \int [d\tau] \int \frac{d\sigma}{V(Weyl)} e^{\frac{2g}{24\pi} S_L(\sigma)} \frac{1}{V(CKV)}. \quad (2.6)$$

Here we have focused on showing the explicit dependence of the measure on the Liouville mode σ ; per definitionem all other factors are included in the measure of the integral over the Teichmüller parameters $[d\tau]$ [29–31] and will play no role in the following.* $S_L(\sigma)$ denotes the Liouville action [28, 31], and the genus-dependent volume of the group

*. In the case of the sphere $\int [d\tau]$ is to be replaced by a constant.

generated by the conformal Killing vectors $V(CKV)$ is a remnant of $V(Diff)$, which has been incompletely cancelled in the cases $h_M = 0, 1$. Inserting (2.6) into (2.1) we write

$$Z = \sum_M g^{2h_M-2} \int [d\tau] \int \frac{d\sigma}{V(Weyl)} e^{\frac{26}{24\pi} S_L(\sigma) - S_{c.t.}} \frac{1}{V(CKV)} \hat{Z}(\sigma) \quad (2.7)$$

with

$$\hat{Z}(\sigma) = \int DX e^{-(S_0 + S_{int})} \quad (2.8)$$

For $S_{int} = 0$ and $D = 26$ the σ -dependence of $\hat{Z}(\sigma)$ and $\exp\{(26/24\pi)S_L(\sigma) - S_{c.t.}\}$ cancels which allows for a trivial performance of the σ integral and its cancellation with $V(Weyl)$. σ -dependence of $\hat{Z}(\sigma)$ induced by a non-vanishing S_{int} will be the subject of the remaining part of this chapter. We work at string tree level and restrict ourselves to demand independence of $\hat{Z}(\sigma)$ from the zero mode of σ taking the world sheet to be the complex plane, i.e. we write

$$\gamma_{\alpha\beta} = e^\sigma \delta_{\alpha\beta}, \quad \sigma = const. \quad (2.9)$$

For $\hat{Z}$ of eq. (1.10) we may write

$$\hat{Z}(\sigma) \equiv \langle e^{-S_{int}} \rangle. \quad (2.10)$$

The Wick contractions required in the computation of a vacuum expectation value as in (2.10) can be described in terms of the application of an operator e^D with D defined as

$$D = \frac{1}{2} \int d^2 z_1 d^2 z_2 \Delta(z_1, z_2) \frac{\delta}{\delta X^i(z_1)} \frac{\delta}{\delta X^i(z_2)} \quad (2.11)$$

where Δ denotes the Green function of the scalar Laplacian $\square$ on the world sheet,

$$\square_z \Delta(z, z') = -4\pi \delta^2(z - z'). \quad (2.12)$$

(Here we simply have $\square = \partial_\alpha \partial_\alpha$.) Then we may write

$$\langle O(X) \rangle \equiv [e^D O(X)]|_{X=0} \quad (2.13)$$

Generically $O(X)$ stands for a series of normal ordered vertex operators; naturally we have to omit self-contractions within a single normal-ordered operator.

The existence of ultraviolet singularities on the world sheet requires regularization. We will do this by modifying the short distance behavior of the Green function Δ . If this is done in a way which respects conformal invariance (viewed as a subgroup of reparametrization invariance) the Green function Δ becomes σ -dependent. A typical modification consists in replacing $\Delta_0(z - z')$,

$$\Delta_0(z - z') = -\ln|z - z'|^2, \quad (2.14)$$

by $\Delta_{Reg}(z - z')$,

$$\Delta_{Reg}(z - z') = -\ln(|z - z'|^2 + \epsilon e^{-\sigma}), \quad (2.15)$$

with ϵ a short distance cutoff. Note that under $\delta z_\alpha = \beta z_\alpha$, $\delta\sigma = -2\beta$ one has $\delta\Delta_{reg}(z) = \text{const. indep. of } z$. The precise form of the short distance modification of the Green function will play no role in the following, generally one will find, however,

$$\lim_{\epsilon \rightarrow 0} \partial_\sigma \Delta_{Reg}(z) \Big|_{z \neq 0 \text{ indep. of } \epsilon} = 0(\epsilon) \quad (2.16a)$$

$$\lim_{\epsilon \rightarrow 0} \partial_\sigma \Delta_{Reg}(z) \Big|_{|z|^2 \sim 0(\epsilon e^{-\sigma})} = 1 \quad (2.16b)$$

(This also holds in the case of dimensional regularization, where the world sheet dimension $d = 2$ is replaced by $d = 2 - 2\epsilon$, and the Green function essentially becomes

$$\Delta_{Reg}(z) \simeq \frac{1}{\epsilon} e^{\epsilon\sigma} (m^{-2\epsilon} - |z|^{2\epsilon})$$

with m an infrared regulator [18].)

Actually matters are that simple only because we replaced the Liouville mode σ by its zero mode, otherwise a more elaborate form of regularization would be required [32].

Now we are in a position to investigate the dependence of $\hat{Z}(\sigma)$ on the σ zero mode. Thereby we will neglect a contribution due to the path integral measure DX [33] which can be taken care of by an appropriate choice of the parameter μ^2 in $S_{c.t.}$ (2.3). With this in mind we find that $\hat{Z}(\sigma)$ should not depend on the σ zero mode at all in order not to spoil the triviality of the σ -integration. This will lead to constraints on the sources $G_n(p)$ appearing in S_{int} in (2.4) as we will now derive.

Using (2.10) to (2.13) we can write (subsequently D denotes the expression (2.11) with Δ replaced by Δ_{Reg})

$$\begin{aligned} \partial_\sigma \hat{Z} &= \partial_\sigma \langle e^{-S_{int}} \rangle = \partial_\sigma [e^D e^{-S_{int}}] \Big|_{X=0} \\ &= [e^D (-\partial_\sigma S_{int}) e^{-S_{int}} + e^D D' e^{-S_{int}}] \Big|_{X=0} \end{aligned} \quad (2.17)$$

with

$$D' = \frac{1}{2} \int d^2 z_1 d^2 z_2 \partial_\sigma \Delta_{Reg}(z_1, z_2) \frac{\delta}{\delta X^i(z_1)} \frac{\delta}{\delta X^i(z_2)}. \quad (2.18)$$

Let us discuss the first term on the r.h.s. of (2.17). There are two sources of σ -dependence of S_{int} . The first one is due to the fact that S_{int} has to be reparametrization-invariant. The vertex operators $V_n(X, p)$ in (2.4) are of the general form

$$V_n(X, p) =: D_n(\partial X) \exp(ip^\dagger X^i): \quad (2.19)$$

where D_n is a homogeneous polynomial in derivatives ∂_α of X with degree $M_n^2 + 2$ so that M_n^2 has the meaning of the mass squared of the corresponding string mode. These world sheet derivatives have to be contracted either with $\gamma^{\alpha\beta}$ or $\epsilon^{\alpha\beta}/\sqrt{\gamma}$ which, using (2.9), generates a factor $\exp[-(\frac{1}{2}M_n^2 + 1)\sigma]$. Another factor $\exp \sigma$ is a remnant of $\sqrt{\gamma}$ which has to be introduced in order to render the d^2z -integration reparametrization invariant.

A second source of σ -dependence is the fact that the vertex operators in (2.4) are normal ordered; this gives rise to a factor [28, 32, 34]

$$\exp(-\frac{1}{2}\Delta(0)p^2) \cong \exp(-\frac{1}{2}p^2(\sigma - \ln \epsilon) + 0(\epsilon\sigma)). \quad (2.20)$$

All this σ -dependence has tacitly been included in the definition of the sources $G_n(p)$ in (2.4), which thus actually are given by

$$G_n(p, \sigma) = e^{-\frac{1}{2}(p^2 + M_n^2)\sigma} \hat{G}_n(p) \quad (2.21)$$

with $\hat{G}_n(p)$ independent of σ for $\epsilon \rightarrow 0$.

The first term on the r.h.s. of (2.17) thus becomes

$$\left[e^D \frac{1}{2} \sum_n \int d^D p \int d^2 z (p^2 + M_n^2) G^n(p) V_n(X(z), p) e^{-S_{int}} \right]. \quad (2.22)$$

The second term on the r.h.s. of (2.17) can be written as

$$D' e^{-S_{int}} = \frac{1}{2} \int d^2 z_1 d^2 z_2 \partial_\sigma \Delta_{Reg}(z_1, z_2) \frac{\delta S_{int}}{\delta X(z_1)} \frac{\delta S_{int}}{\delta X(z_2)} e^{-S_{int}}. \quad (2.23)$$

(Note that no self-contractions within S_{int} are allowed.) We intend to rewrite this expression in terms of operator product coefficients $C_{mn}^\ell(y, p, p')$ appearing in

$$V_m(X(z + \frac{y}{2}), p) V_n(X(z - \frac{y}{2}), p') = C_{mn}^\ell(y, p, p') V_\ell(X(z), p + p'). \quad (2.24)$$

This has been investigated in detail in ref. [18] with the result that (2.23) can be expressed as

$$\frac{1}{2} \sum_{n, \ell, m} \int d^D p d^D p' \tilde{C}_{\ell m}^n G^\ell(p') G^m(p - p') \int d^2 z V_n(X(z), p) e^{-S_{int}} \quad (2.25)$$

with

$$\tilde{C}_{\ell m}^n = \int d^2 z \partial_\sigma C_{\ell m}^n(y). \quad (2.26)$$

Inserting (2.22) to (2.25) into (2.17), we obtain

$$\begin{aligned} \partial_\sigma \hat{Z} = & \left[e^D e^{-S_{int}} \frac{1}{2} \sum_n \int d^D p \int d^2 z V_n(X(z), p) \times \right. \\ & \left. \times \left\{ (p^2 + M_n^2) G^n(p) + \int d^D p' \tilde{C}_{lm}^n G^\ell(p') G^m(p - p') \right\} \right] \Big|_{X=0} \end{aligned} \quad (2.27)$$

As a sufficient condition for the required vanishing of $\partial_\sigma \hat{Z}$ we thus find the equations of motion for the “string fields” $G^n(p)$ as in [18] :

$$(p^2 + M_n^2) G^n(p) + \int d^D p' \tilde{C}_{lm}^n G^\ell(p') G^m(p - p') = 0. \quad (2.28)$$

Herewith we conclude this section; further discussions and applications of eq. (2.28) have already been published [18].

3. String Loop Effects and the Handle Operator

In this section we will show how string loop effects can be taken into account in the present framework. First we will describe the modifications required in the evaluation of $\hat{Z}(\sigma)$ defined in eqs. (2.7) and (2.8). Our formula (2.13) for the computation of vacuum expectation values remains correct. The Green function $\Delta(z_1, z_2)$ appearing in the definition of the operator D , however, has to be chosen appropriately to correspond to the Riemann surface M under consideration. Thus we write

$$\begin{aligned} \langle O(X) \rangle_M &= [e^{D_M} O(X)] \Big|_{X=0}, \\ D_M &= \frac{1}{2} \int d^2 z_1 d^2 z_2 \Delta_M(z_1, z_2) \frac{\delta}{\delta X^i(z_1)} \frac{\delta}{\delta X^i(z_2)}. \end{aligned} \quad (3.1)$$

The Green function $\Delta_M(z_1, z_2)$ on a Riemann surface M can generally be written as

$$\Delta_M(z_1, z_2) = \Delta_0(z_1, z_2) + \tilde{\Delta}_M(z_1, z_2) \quad (3.2)$$

with Δ_0 given by (2.14), where only Δ_0 is singular in the limit $z_1 \rightarrow z_2$. Accordingly the operator $\exp(D_M)$ can be written as a product,

$$e^{D_M} = e^{\tilde{D}_M} e^{D_0} \quad (3.3)$$

(note that $\tilde{D}_M$ and D_0 commute) with D_0 and $\tilde{D}_M$ defined in terms of Δ_0 and $\tilde{\Delta}_M$ as is obvious from (3.1). Now we are seeking for an operator $H_M(X)$, a functional of the fields X , satisfying

$$e^{D_0} H_M(X) = F(X) e^{\tilde{D}_M} e^{D_0} \quad (3.4)$$

where $F(X)$ is an arbitrary function of X with $F(0) = 1$. Then we can write

$$\begin{aligned} \langle O(X) \rangle_M &= \left[e^{D_M} O(X) \right] \Big|_{X=0} = \left[F(X) e^{\bar{D}_M} e^{D_0} O(X) \right] \Big|_{X=0} \\ &= \left[e^{D_0} H_M(X) O(X) \right] \Big|_{X=0}. \end{aligned} \quad (3.5)$$

i.e. we can express the correlation function of an arbitrary operator $O(X)$ on an arbitrary Riemann surface M in terms of the correlation function of the same operator on the complex plane with an insertion of the operator $H_M(X)$.

In appendix A we explicitly construct $H_M(X)$ in the case of the torus, $M = T_2$. Therefore it will be of help not only that we can accommodate an arbitrary function $F(X)$ in the relation (3.4), which plays no role due to the “ $X=0$ ”-prescription as the last step in the evaluation of vacuum expectation values, but also that terms in $\bar{\Delta}_M(z_1, z_2)$ which do not depend simultaneously on z_1 and z_2 drop out in the computation of expectation values of vertex operators, because they get either differential with respect to z_1, z_2 or multiplied with $\sum p_i$, the sum of all incoming momenta, which vanishes due to momentum conservation. (Note that the mass shell condition is not required; the latter mechanism is familiar from the case of tachyon amplitudes.)

First we need the explicit form of the scalar Green function on the torus. Usually the torus is described by the region inside a parallelogram on the complex plane parametrized by a complex number $\tau = x_\tau + iy_\tau$, x_τ and y_τ real and y_τ positive (see fig. 1a). The complex variable in this parametrization will be called u . By means of two subsequent mappings we intend to switch to a description of the torus in the so-called Schottky parametrization, where the torus consists of the entire complex plane with two circular holes cut out, the boundaries of which are identified. (One can imagine a handle inserted onto the complex plane.) The first mapping is $z' = \exp(2\pi i u)$, after which the image of the fundamental region inside the parallelogram lies between two concentric circles with radii $r_a = 1$, $r_i = \exp(-2\pi y_\tau)$ (see fig. 1b). The second mapping is

$$z' = \frac{z - z_b}{z_c(z - z_a)}, \quad 1 < |z_c| < r_i^{-1} \quad (3.6)$$

after which the image of the fundamental region is the entire complex plane except for two circular holes with radii (see fig. 1c)

$$r_1 = \frac{|z_c(z_a - z_b)|}{|z_c|^2 - 1} \quad r_2 = \frac{|r_i z_c(z_a - z_b)|}{-r_i^2 |z_c|^2 + 1} \quad (3.7)$$

Each of the points z_a, z_b lies within one of the circles (though they are not their mid-points); geometrically these are the fix points of the mapping of one fundamental region into another, which in the u -plane is generated by $u \rightarrow u + 1$, $u \rightarrow u + \tau$. In the z -plane this maps the entire region outside the two circles into part of the inside of the two circles.

The Green function on the z -plane can be obtained from the one in the u -plane[21] and reads

$$\begin{aligned}\Delta_{T_2}(z_i, z_j) &= \Delta_0(z_i, z_j) - \frac{1}{2\pi y_\tau} \ln^2 \left| \frac{(z_i - z_a)(z_j - z_b)}{(z_i - z_b)(z_j - z_a)} \right| \\ &+ \sum_{k=1}^{\infty} \frac{1}{k} \frac{w^k}{1 - w^k} \left\{ \left(\frac{(z_i - z_a)(z_j - z_b)}{(z_i - z_b)(z_j - z_a)} \right)^k + \left(\frac{(z_i - z_b)(z_j - z_a)}{(z_i - z_a)(z_j - z_b)} \right)^k - 2 \right\} + c.c. \\ &+ \ln |(z_i - z_a)(z_i - z_b)(z_j - z_a)(z_j - z_b)|\end{aligned}\quad (3.8)$$

with $w = \exp(2\pi i \tau)$. (The last term as well as the coefficient -2 in the third term are actually irrelevant as remarked below eq. (3.5).) From (3.8) we can easily read off the part $\tilde{\Delta}_{T_2}$ of the full Green function required to define $\tilde{D}_{T_2}$ (see (3.2), (3.3)).

For the operator H_M , which satisfies (3.4) for $M = T_2$, we will make the ansatz

$$H_M =: e^{L_M} :. \quad (3.9)$$

In appendix A we derive the general formula which allows us to compute L_M in terms of the non-trivial part $\tilde{\Delta}_M$ of the Green function on the Riemann surface M :

$$L_M = \frac{1}{2} \int d^2 z_1 d^2 z_2 d^2 z_1' d^2 z_2' X^i(z_1) X^i(z_2) \Delta_0^{-1}(z_1, z_1') \Delta_0^{-1}(z_2, z_2') \tilde{\Delta}_M(z_1', z_2') \quad (3.10)$$

In the present case with $M = T_2$ we get

$$\begin{aligned}L_{T_2} &= \frac{1}{8\pi y_\tau} |X^i(z_a) - X^i(z_b)|^2 \\ &+ \sum_{n=1}^{\infty} \frac{w^n}{(1 - w^n)n!(n-1)!} \frac{\partial^{n-1}}{\partial z_a^{n-1}} \left((z_a - z_b)^n \frac{\partial X^i(z_a)}{\partial z_a} \right) \frac{\partial^{n-1}}{\partial z_b^{n-1}} \left((z_a - z_b)^n \frac{\partial X^i(z_b)}{\partial z_b} \right) \\ &+ c.c.\end{aligned}\quad (3.11)$$

It should be pointed out that a corresponding operator, which we will denote “handle operator”, has already been constructed in ref. [26].

The insertion of this operator into the correlation functions now allows us to compute $\hat{Z}(\sigma)_{T_2}$ in terms of Wick contractions performed by means of the operator $\exp D_0$ as in (3.5). This, however, is not the end of the story; in eq. (2.7) $\hat{Z}$ gets multiplied with $V(CKV)^{-1}$ which also depends on the Riemann surface under consideration. In the case of the sphere we formally have

$$V(CKV)_{S_2} = V(SL(2, C)) = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2} \quad (3.12)$$

whereas for the torus $V(CKV)_{T_2}$ is the (finite) volume of the group of translations corresponding to the area of the fundamental region (the parallelogram) in the u -plane or, after the mapping $z' = \exp(2\pi i u)$,

$$V(CKV)_{T_2} = \frac{1}{4\pi^2} \int_{1 < |z'| < r_i^{-1}} \frac{d^2 z'}{|z'|^2}. \quad (3.13)$$

After performing the DX path integral by the methods described above, $\hat{Z}(\sigma)_{T_2}$ will depend on z_a, z_b and z_c , since the operator H_{T_2} depends on fields X located at z_a, z_b and since the dz integration regions (see fig. 1c) depend on z_a, z_b and z_c which appeared in the mapping (3.6). Subsequently we intend to show that we can formally write

$$V(CKV)_{S_2} \hat{Z}(\sigma)_{T_2} = c_\infty(\tau) V(CKV)_{T_2} \int \frac{d^2 z_a d^2 z_b}{|z_a - z_b|^4} \hat{Z}(\sigma)_{T_2}, \quad (3.14)$$

i.e. we can trade the two integrals by which (3.12) and (3.13) differ for integrals over z_a and z_b . $c_\infty(\tau)$ denotes an infinite factor by which $V(CKV)_{T_2}$ differs from the “volume” of the entire complex plane.

We start with the integrals appearing on the r.h.s. of (3.14):

$$\frac{1}{4\pi^2} \int_{1 < |z_c| < r_1^{-1}} \frac{d^2 z_c}{|z_c|^2} \int \frac{d^2 z_a d^2 z_b}{|z_a - z_b|^4} \quad (3.15)$$

Now we switch from the variable z_c to $\hat{z}$ defined via $z_c = (\hat{z} - z_a)/(\hat{z} - z_b)$, and we let the integral over z_c resp. $\hat{z}$ run over the entire complex plane after dividing by a compensating factor c_∞ . Thus we obtain

$$c_\infty^{-1} \int \frac{d^2 \hat{z} d^2 z_a d^2 z_b}{|\hat{z} - z_a|^2 |\hat{z} - z_b|^2 |z_a - z_b|^2} \quad (3.16)$$

which, in view of (3.10), shows that (3.14) holds. (Actually, instead of working with the infinite quantities in eqs. (3.10) to (3.16), we should have regularized the corresponding integrals, after which c_∞ will obviously diverge when the cutoff is removed.)

In the process of factorizing out $V(SL(2, C))$ in standard string theory in the case of the sphere one has to argue that the integrand does not depend on the three parameters z_a, z_b and z_c of Möbius transformations (cf. (3.6)), which holds since the vertex operators are on-shell. In the present situation on the one hand we deal with generically off-shell vertex operators, on the other hand we have (analytic) reparametrization invariance due to the presence of the Liouville mode σ . Thus we have Möbius invariance in the sense that it is just a subgroup of general coordinate transformations [35].

It will be convenient to define

$$\hat{H}_{T_2} = \int [d\tau] c_\infty \int \frac{d^2 z_a d^2 z_b}{|z_a - z_b|^4} H_{T_2}; \quad (3.17)$$

Then, after dividing (3.14) by $V(CKV)_{S_2} \times V(CKV)_{T_2}$, we can collect all the results of this chapter into the relation

$$\int [d\tau] \frac{1}{V(CKV)_{T_2}} \hat{Z}_{T_2} = \frac{1}{V(CKV)_{S_2}} \left[e^{D_0} \hat{H}_{T_2} e^{-S_{int}} \right] \Big|_{X=0}. \quad (3.18)$$

Up to now we have not considered Riemann surfaces with boundaries. It is fairly straightforward, however, to extend the present formalism to a combined system of closed

and open strings. We may, e.g. consider closed bosonic string fields living on a Riemann surface with the topology of a disc D_2 . The corresponding fundamental region can be taken to be the complex plane except for a circular hole with radius a . As in the case of the torus it is possible to express correlation functions on the disc in terms of correlation functions on the sphere (or the entire complex plane) with an insertion of a "hole operator" $H_{D_2}(X(z_0))$ where z_0 denotes the center of the hole, its explicit form will be derived in appendix B. Again $V(CKV)_{D_2}$ with

$$V(CKV)_{D_2} = V(SU(1, 1)) \quad (3.19)$$

can be related to $V(CKV)_{S_2}$, and in appendix B we will derive the formula corresponding to eq. (3.18):

$$\frac{1}{V(CKV)_{D_2}} \hat{Z}_{D_2} = \frac{1}{V(CKV)_{S_2}} \left[e^{D_0} \hat{H}_{D_2} e^{-S_{int}} \right] \Big|_{X=0}, \quad (3.20)$$

$$\hat{H}_{D_2} = \int d^2 z_0 \int \frac{da}{a^3} : exp \left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{a^{2n}}{n!(n-1)!} \frac{\partial^n}{\partial z_0^n} X(z_0) \frac{\partial^n}{\partial \bar{z}_0^n} X(z_0) \right) :. \quad (3.21)$$

In the next chapter we will use eq. (3.18) to derive loop corrections to the string field equations (2.28).

4. Loop-Corrected String Field Equations

In this chapter we will consider the combined sphere and torus contribution to the partition function. Inserting (3.18) and the corresponding relation for the sphere into (2.7), we can write

$$Z_{S_2+T_2} = g^{-2} \int \frac{d\sigma}{V(Weyl)} e^{\frac{26}{24\pi} S_L(\sigma) - S_{c.t.}} \frac{1}{V(CKV)_{S_2}} \left[e^{D_0} (1 + g^2 \hat{H}_{T_2}) e^{-S_{int}} \right] \Big|_{X=0} \quad (4.1)$$

Actually this procedure was not quite correct. In the case of the sphere the vertex operators present in S_{int} may be located on the entire complex plane, whereas in the case of the torus the corresponding $d^2 z$ -integrals are confined to the region outside the two circular holes as in fig. 1c. The radii of these holes, however, are proportional to $|z_a - z_b|$ (see (3.7)) and subsequently we will be interested in the limit $|z_a - z_b| \rightarrow 0$: In this region the integral inherent in the definition of $\hat{H}_{T_2}$ (see (3.17)) is divergent and requires regularization. As in sect. 2 this will give rise to a σ -dependence, which eventually will modify our equations of motion.

In sect. 2 we confined ourselves to requiring independence of $\hat{Z}(\sigma)$ from the σ zero mode. We will do the same in the present case where the torus is included. Of course we have to specify in which parametrization of the torus (actually also in the case of the sphere) we set $\sigma = \text{const.}$ (Note that we have not done this up to now in order not to spoil the reparametrization invariance of $\hat{Z}(\sigma)_{T_2}$ in (3.14).) It is natural to set $\sigma = \text{const.}$

in the Schottky-parametrization corresponding to fig. 1c, which is actually the one used in deriving (4.1). Thus, as in sect. 2, we will investigate

$$\partial_\sigma \left[e^{D_0} (1 + g^2 \hat{H}_{T_2}) e^{-S_{int}} \right] \Big|_{X=0}. \quad (4.2)$$

σ -dependence of the expression in the square brackets arises from the following sources: the free propagator $\Delta_0(z)$ appearing in D_0 has to be regularized in a σ -dependent way for $z \rightarrow 0$ as in sect. 2. The σ -dependence of S_{int} described in detail in sect. 2 has also remained the same. The new feature is, however, the presence and σ -dependence of $\hat{H}_{T_2}$: as mentioned above, the $d^2 z_a d^2 z_b$ -integral inherent in the definition of $\hat{H}_{T_2}$ requires regularization for $|z_a - z_b| \rightarrow 0$. We will do this by means of a σ -dependent cutoff in order not to spoil (global) conformal invariance, i. e. we write

$$\hat{H}_{T_2} = \int [d\tau] c_\infty \int_{|z_a - z_b|^2 > \epsilon e^{-\sigma}} \frac{d^2 z_a d^2 z_b}{|z_a - z_b|^4} H_{T_2} \quad (4.3)$$

instead of (3.17), thus $\hat{H}_{T_2}$ has become σ -dependent. Collecting these contributions the expression (4.2) becomes

$$\left[e^D \{ (1 + g^2 \hat{H}_{T_2}) (-\partial_\sigma S_{int}) + D' (1 + g^2 \hat{H}_{T_2}) + g^2 \partial_\sigma \hat{H}_{T_2} \} e^{-S_{int}} \right] \Big|_{X=0} \quad (4.4)$$

with D' as in (2.18). The first term in (4.4) can be evaluated as in sect. 2 with the result

$$\partial_\sigma S_{int} = \frac{1}{2} \sum_n \int d^D p \int d^2 z (p^2 + M_n^2) G^n(p) V_n(X(z), p). \quad (4.5)$$

(Note that normal ordering of a vertex operator on an arbitrary Riemann surface M does not modify the σ -dependent factor exhibited in (2.20) since just the M -independent piece $\Delta_0(z)$ of the Green function $\Delta_M(z)$, see (3.2), needs regularization for $z \rightarrow 0$.)

Considering the second term in (4.4) we have to take into account that the two derivatives $\delta/\delta X$ in D' (see (2.18)) are not allowed to act simultaneously on $\hat{H}_{T_2}(X)$ since this operator is already normal ordered (cf. (3.9)). Thus we obtain

$$\begin{aligned} D' (1 + g^2 \hat{H}_{T_2}) e^{-S_{int}} &= -g^2 \int d^2 z_1 d^2 z_2 \partial_\sigma \Delta_{Reg}(z_1, z_2) \frac{\delta \hat{H}_{T_2}}{\delta X(z_1)} \frac{\delta S_{int}}{\delta X(z_2)} e^{-S_{int}} \\ &+ (1 + g^2 \hat{H}_{T_2}) D' e^{-S_{int}} \end{aligned} \quad (4.6)$$

At first sight the first term on the r.h.s. of (4.6) seems to yield a negligible contribution: The operator $\hat{H}_{T_2}$ depends on fields X located at the points z_a, z_b inside the circles of fig. 1c, whereas the vertex operators in S_{int} only live outside of the circles. Thus, after carrying out the functional derivatives $\delta/\delta X$, we obtain an expression proportional to $\partial_\sigma \Delta(z_a, z)$ (or z_a replaced by z_b) where z_a and z can never coincide. In view of eq. (2.16a) one is tempted to argue that one obtains a vanishing result for $\epsilon \rightarrow 0$.

However, one has to be aware of the fact that we have to integrate over $d^2 z_a d^2 z_b$ and that the radii of the circles vanish (or get close to the cutoff $\epsilon e^{-\sigma}$) for $z_a \rightarrow z_b$. In this situation we may have $|z - z_a| \sim 0(\epsilon e^{-\sigma})$ and thus obtain a finite result for $\partial_\sigma \Delta(z_a, z)$ in view of (2.16b).

Thus let us insert (4.3) for $\hat{H}_{T_2}$ and (2.4) for S_{int} into the first term on the r.h.s. of (4.6) which becomes

$$-g^2 \int d[\tau] c_\infty \int \frac{d^2 z_a d^2 z_b}{|z_a - z_b|^4} \int d^2 z \int d^D p \left\{ \int d^2 z_1 d^2 z_2 \partial_\sigma \Delta_{Reg}(z_1, z_2) \frac{\delta H_{T_2}(z_a, z_b)}{\delta X(z_1)} \frac{\delta V_k(z, p)}{\delta X^i(z_2)} \right\} G^k(p) \quad (4.7)$$

Next we intend to relate the expression in the curly bracket to the coefficients appearing in the operator product expansion of H_{T_2} and V_k . Therefore we first normal order the product of these (intrinsically normal ordered) operators according to Wick's theorem:

$$H_{T_2}(z_a, z_b) V_k(z, p) = \sum_{a=0}^{\infty} \frac{1}{a!} \int \prod_{r=1}^a (\Delta(z_r, z'_r) d^2 z_r d^2 z'_r) \times \frac{\delta^a H_{T_2}(z_a, z_b)}{\delta X^{i_1}(z_1) \cdots \delta X^{i_a}(z_a)} \frac{\delta^a V_k(z, p)}{\delta X^{i_1}(z'_1) \cdots \delta X^{i_a}(z'_a)} \quad (4.8)$$

The normal-ordered product of operators located at z_a, z_b and z on the r.h.s. can be expanded into a complete set $V_\ell(z, p)$ of local composite operators by means of a Taylor expansion after which we can write

$$H_{T_2}(z_a, z_b) V_k(z, p) = \sum_{\ell} a_k^\ell(z_a - z, z_b - z, p) V_\ell(z, p). \quad (4.9)$$

Now we compute ∂_σ of eqs. (4.8) and (4.9) and, after some rearrangement on the r.h.s. of (4.8) (where the propagator Δ is σ -dependent) we obtain

$$\begin{aligned} \int d^2 z_1 d^2 z_2 \partial_\sigma \Delta_{Reg}(z_1, z_2) \frac{\delta H_{T_2}(z_a, z_b)}{\delta X^i(z_1)} \frac{\delta V_k(z, p)}{\delta X^i(z_2)} &= \\ &= \sum_{\ell} \partial_\sigma a_k^\ell(z_a - z_1, z_b - z, p) V_\ell(z, p) \end{aligned} \quad (4.10)$$

Defining

$$D_k^\ell(p) = \int d[\tau] c_\infty \int \frac{d^2 z'_a d^2 z'_b}{|z'_a - z'_b|^4} \partial_\sigma a_k^\ell(z'_a, z'_b, p) \quad (4.11)$$

(with $z'_{a,b} = z_{a,b} - z$) the expression (4.7) thus becomes

$$-g^2 \int d^2 z \int d^D p V_\ell(z, p) D_k^\ell(p) G^k(p). \quad (4.12)$$

We note again that the relevance of this contribution is due to overlapping divergences in the situation $|z_a - z_b| \rightarrow 0$, $|z - z_a| \rightarrow 0$ or $|z - z_b| \rightarrow 0$ although the latter divergences are regularized by the condition $|z - z_a| > 0$ ($|z_a - z_b|$). (In a regularization scheme as in (2.15) and with a cutoff dependence as dictated by dimensional analysis the sources $G_n(p, \sigma)$ depend on ϵ , taking into account the effect from normal ordering as in (2.20), as

$$G_n(p, \sigma) \sim \epsilon^{\frac{1}{2}(p^2 + M_n^2)}, \quad (4.13)$$

i.e. they only depend on the combination $\epsilon e^{-\sigma}$. Then it follows again from dimensional analysis that, due to the vanishing canonical dimension of the operator $\hat{H}_{T_2}$, the expression $D_k^\ell G^k(p)$ in (4.12) has the appropriate cutoff dependence of $G^\ell(p)$ [17].)

The contribution of the second term in (4.6) involving $D' \exp(-S_{int})$ is the same as in sect. 2 and we can use the corresponding formulae (2.23) to (2.25).

Thus we come to the last term in (4.4) proportional to $\partial_\sigma \hat{H}_{T_2}$, where the σ -dependence of $\hat{H}_{T_2}$ arises from the need to regularize the divergence near $|z_a - z_b| \rightarrow 0$ of the integral in (4.3). In this limit, and with no other vertex operator nearby, we may safely expand H_{T_2} from (3.9) in powers of $z_a - z_b$ and local operators $h_{n,m}$ located at $\hat{z} = (z_a + z_b)/2$:

$$H_{T_2} = \sum_{n,m=0} (z_a - z_b)^n (\bar{z}_a - \bar{z}_b)^m h_{nm}(X(\hat{z})) \quad (4.14)$$

with

$$h_{00} = 1, h_{11} = \frac{1}{8\pi y_\tau} \partial_z X(\hat{z}) \partial_{\bar{z}} X(\hat{z}) \quad (4.15)$$

Inserting (4.14) into (4.3) one finds that divergences for $|z_a - z_b| \rightarrow 0$ only arise from the terms involving the operators h_{00} and h_{11} , which are proportional to the tachyon vertex operator $V_t(0)$ and the dilaton vertex operator $V_d(0)$,

$$V_t(0) = 1, \quad V_d(0) = \partial_z X(\hat{z}) \partial_{\bar{z}} X(\hat{z}), \quad (4.16)$$

both at vanishing momenta. Defining

$$\begin{aligned} c_t &= \int [d\tau] c_\infty \partial_\sigma \int_{\epsilon e^{-\sigma}} \frac{d^2 \hat{y}}{|\hat{y}|^4}, \\ c_d &= \int [d\tau] c_\infty \partial_\sigma \int_{\epsilon e^{-\sigma}} \frac{d^2 \hat{y}}{|\hat{y}|^2} \frac{1}{8\pi y_\tau} \end{aligned} \quad (4.17)$$

where some infrared regularization has to be understood, $\partial_\sigma \hat{H}_{T_2}$ becomes

$$\partial_\sigma \hat{H}_{T_2} = \int d^2 \hat{z} (c_t V_t(\hat{z}) + c_d V_d(\hat{z})) \quad (4.18)$$

These results obviously correspond to the familiar dilaton/tachyon tadpole contributions mentioned in the introduction.*

* It may be useful to keep the higher operators $h_{n,n}$ in (4.14), which correspond to massive scalar string modes, although their coefficients do not diverge [17]. This is a typical regularization scheme-dependent issue.

Thus we finally can collect all the contributions to the expression (4.4). With (4.5), (4.17), (2.25) and (4.18) we obtain

$$\begin{aligned}
& \left[e^D \left\{ (1 + g^2 \hat{H}_{T_2}) \frac{1}{2} \sum_n \int d^D p \{ (p^2 + M_n^2) G^n(p) + \int d^D p' \right. \right. \\
& \quad \tilde{C}_{tm}^n G^\ell(p') G^m(p - p') \} \int d^2 z V_n(z, p) \\
& \quad - g^2 \int d^D p D_k^n(p) G^k(p) \int d^2 z V_n(z, p) + g^2 \int d^2 z (c_t V_t(z) \\
& \quad \left. \left. + c_d V_d(z)) \right\} e^{-S_{int}} \right] \Big|_{X=0}
\end{aligned} \tag{4.19}$$

In order to extract equations of motion from (4.19) we have to factorize out $(1 + g^2 \hat{H}_{T_2})$ from each term and thereby to neglect effects of $0(g^4)$. Then the external curly bracket in (4.19) can be written as

$$\begin{aligned}
& (1 + g^2 \hat{H}_{T_2}) \frac{1}{2} \sum_n \int d^D p \int d^2 z V_n(z, p) \times \\
& \cdot \{ (p^2 + M_n^2) G^n(p) + \int d^D p' \tilde{C}_{tm}^n G^\ell(p') G^m(p - p') \\
& - 2g^2 D_k^n(p) G^k(p) + 2g^2 \delta^D(p) (c_t \delta_t^n + c_d \delta_d^n) \} \\
& + 0(g^4).
\end{aligned} \tag{4.20}$$

The expression in the curly bracket in (4.20), the vanishing of which is a sufficient condition for the vanishing of $\partial_\sigma \hat{Z}_{S_2+T_2}$, is the one-loop-corrected equation of motion for the “string fields” $G^n(p)$ and has to be compared with the corresponding tree level equation (2.28) in sect. 2.

The term involving $g^2 D_k^n G^k$ corresponds to the one-loop self-energy corrections; in our scheme the coefficients D_k^n are related to the OPE coefficients appearing in the OPE of the handle operator H_{T_2} and the vertex operators. The terms involving c_t and c_d only affect the equations for the zero momentum tachyon and dilaton fields $G^t(0)$, $G^d(0)$ and represent the tadpole effects. Thus all loop effects in (4.20) correspond precisely to our expectations.

The need to neglect terms of $0(g^4)$ in the derivation of eq. (4.20) from eq. (4.19) hints at an interesting extension of the handle operator: One easily verifies that no approximation in the process of deriving the loop-corrected equations of motion would be required if the insertion of $(1 + g^2 \hat{H}_{T_2})$ into the partition function in (4.1) was replaced by the insertion of $\exp(g^2 \hat{H}_{T_2})$. This operator would describe the effects of an arbitrary number of handles in the approximation, however, that all handles are far apart from each other and do not influence each other. It is to be expected that the subset of multiloop effects accounted for by this operator corresponds to the leading log approximation or renormalization group improved perturbation theory in conventional field theory. (Exponentiation

of one-loop self-energy corrections has been found in the framework of Ooguri and Sakai [26].)

5. S-Matrix, Superstrings, and Outlook

The equations of motion derived in sects. 2 and 4 may formally be solved by iteration. The lowest order solution to the tree level equation (2.28) is given by

$$G_0^n(p) = N_n \delta(p^2 + M_n^2) \quad (5.1)$$

with an arbitrary normalization N_n . The full iterative solution is obtained as

$$\begin{aligned} G^n(p) = & G_0^n(p) - \frac{1}{p^2 + M_n^2} \int d^D p' \tilde{C}_{lm}^n G_0^l(p') G_0^m(p - p') \\ & + \frac{2}{p^2 + M_n^2} \int d^D p' d^D p'' \tilde{C}_{lm}^n G_0^l(p') \frac{1}{(p - p')^2 + M_m^2} \tilde{C}_{ij}^m G_0^i(p'') \\ & \cdot G_0^j(p - p' - p'') + \dots \end{aligned} \quad (5.2)$$

A Feynman diagrammatic representation of this solution is shown in fig. 2, where external and internal lines correspond to G_0^n and $(p^2 + M^2)^{-1}$, respectively, and the vertices to the coefficients $\tilde{C}_{lm}^n$.

The reason why we discussed this solution is because we briefly want to comment on the relation to the S -matrix. A close connection between the iterative solution (5.2) and S -matrix elements has been described in [19], which reads essentially (in the absence of wave function renormalization)

$$\left. \frac{\partial^{k-1} G^{n_1}(p)}{\partial G_0^{n_2} \dots \partial G_0^{n_k}} \right|_{G_0=0} = \frac{i}{p^2 + M_{n_1}^2} S_{n_1 n_2 \dots n_k} \quad (5.3)$$

where $S_{n_1 n_2 \dots n_k}$ denotes the k -particle S -matrix element for asymptotic fields $G_0^{n_1} \dots G_0^{n_k}$.

On the other hand it has been argued in refs. [11, 18] that the generating functional $\hat{S}$ of S -matrix elements is given, up to an integral $\int d^2 z \equiv \Omega$ over the world sheet, by the solution $G^n(p)$ of (5.2) corresponding to the tachyon at zero momentum $G^t(0)$:

$$\hat{S} \sim G^t(0) \Omega \quad (5.4)$$

Inserting $G^t(0)$ into (5.3) one observes at first sight that $G^t(0)$ does not generate k -particle S -matrix elements as claimed according to (5.4), but $k + 1$ -particle S -matrix elements involving an additional tachyon at zero momentum. Since, however, the corresponding vertex operator is the identity operator (up to the world sheet integral Ω), the S -matrix elements coincide up to a factor Ω . Thus the prescriptions (5.3) and (5.4) give actually the same results (note that the propagator of a zero momentum tachyon is just a constant).

In order to obtain one-loop S -matrix elements we should consider iterative solutions to the loop-corrected equations of motion (4.20) and take their $O(g^2)$ -part. Loop corrections

will appear in the propagator due to the one-loop self-energy corrections, and due to the tadpole terms in (4.20).

What modification do we have to expect in the case of superstrings? The present apparatus has not yet been extended to include superstrings, but we can guess the effects on the equations of motion from results obtained elsewhere [20, 21]. For superstrings there are no one-loop self-energy corrections for massless modes, and there is no (dilaton) tadpole on the torus. (It follows from the latter fact that the naive Fischler-Susskind mechanism — loop corrections to β -functions from counter terms on the sphere, which cancel divergences from dilaton tadpoles on the torus — does not work for superstrings [23, 25].)

The only one-loop effect for the superstring equations of motion corresponding to (4.20) are thus self-energy corrections for massive or off-shell modes. Studying now k -massless particle- S matrix elements using (5.3), where G^{n+1} is an iterative solution in (5.2), one finds that only for $k \geq 4$ the iterative solution involves a summation over internal (massive and off-shell) modes with a propagator, which feels the loop corrections. Hence only for $k \geq 4$ the massless particle S -matrix elements get loop-corrected in agreement with known non-renormalization theorems [20, 21].

It remains to be shown, however, that in the present framework the massive/off-shell superstring modes do obtain the expected self-energy corrections, i.e. that the integral over the operator product coefficients involving the corresponding vertex operators and the appropriate “super handle operator” is divergent.

Let us summarize the results obtained so far. We have required global Weyl invariance of the partition function in an arbitrary background of string modes G^n , and thereby obtained corresponding equations of motion. Non-trivial σ -dependence arose from regularizing short-distance singularities in a σ -dependent way. In the case of vertex operators getting close to each other it was essential to regularize the divergences by modifying the Green function on the world sheet (and not by a cutoff of the integrals); only then (and with dimensional regularization [18]) one obtains equations of motion involving just a bilinear interaction term. On the other hand, the divergences in the integrals in the handle/hole operators from the region where the non-trivial Riemann surface degenerates (gets close to a lower genus one) have simply been regularized by a cutoff, which is a procedure not directly related to the one above. It remains to find out whether this remains consistent to higher loop order, and whether a regularization scheme applicable to both situations simultaneously can be found.

Also we clearly have not systematically regularized all infinities as, e.g., the τ -integral in the case of the torus; a discussion of the relevance of the different divergences in the multi-loop case to the Weyl anomaly has been given by Ooguri and Sakai [26] by resorting to considerations involving the energy momentum tensor. Our treatment of the divergent volumes of the groups of conformal Killing vectors required to factorize out $V(SL(2, C))$ and thereby to obtain the $d^2 z_a d^2 z_b$ integrals/ da integrals in the cases of the torus and disc, respectively, has also been quite lightheaded, but it has permitted us to obtain the corresponding familiar results [17, 23, 26].

By requiring just global and not local Weyl invariance we have of course lost a large amount of information: our string modes $G_n(p)$ have to be complete in the sense that the

operator product algebra on the corresponding vertex operators closes, but it is well known that not all of them are part of the physical Hilbert space. It follows e.g. from Weinberg's analysis [34] (see also D'Hoker and Phong [31, 32]) that constraints on the vertex operators as transversality conditions of the space-time metric tensor are due to local Weyl invariance, and the string gauge symmetries (as space-time general coordinate invariance) also only become manifest after considering at least local Weyl invariance [11, 19]. (In terms of the Virasoro algebra we just have exploited relations involving the generator $L_0 + \bar{L}_0$.)

It is satisfactory, on the other hand, that the present procedure leads us in a straightforward way towards the perhaps most interesting dynamical relation in string theory, the equations of motion. Although an effective action is lacking, this allows us in principle to find non-trivial string vacua and to compute S -matrix elements.

The most important result of the present paper is the systematic understanding of how loop effects enter the equations of motion. This requires to make explicit the contribution of handles and holes to the Weyl anomaly. An important tool therefore is the notion of handle and hole operators, which we constructed explicitly for the cases of the torus and the disc. This way we found loop corrections to the self-energies and loop-induced tadpole terms in the equations of motion, whereas the basic bilinear interaction term remained unaltered. All this agrees with recent results obtained by different methods [17, 26]. Interestingly we got a hint on what the leading logarithmic approximation (or renormalization group improved perturbation theory) in string theory might look like by observing that the loop-corrected equations of motion follow more systematically (without the need to expand in the string coupling constant) from a certain multi-handle operator.

The previous discussion made it clear which aspects of the subject still deserve a more thorough investigation, and clearly the extension towards superstrings and higher loops is an interesting task for the future.

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Appendix A

It is the purpose of this appendix to present the construction of the operators H_M which, once inserted into the partition function, represent the effects of non-trivial Riemann surfaces:

$$\left[e^{D_M} O(X) \right] \Big|_{X=0} = \left[e^{\tilde{D}_M} e^{D_0} O(X) \right] \Big|_{X=0} = \left[e^{D_0} H_M(X) O(X) \right] \Big|_{X=0} \quad (A.1)$$

where $O(X)$ denotes an arbitrary product of vertex operators and $D_M = D_0 + \tilde{D}_M$ is given in terms of the Green function $\Delta_M = \Delta_0 + \tilde{\Delta}_M$ on a Riemann surface M as in (3.1).

First we make an ansatz

$$H_M =: e^{L_M} : \quad (A.2)$$

where L_M is an operator bilinear in the fields X , and where normal ordering is understood with respect to contractions involving the Green function Δ_0 on the genus zero world sheet. Next we have to solve the combinatoric problem to pull the operator $\exp D_0$ through H_M as in (A.2) with the result*

$$e^{D_0} : e^{L_M} : \Big|_{X=0} = e^{1/2 K_M} e^{D_0} \Big|_{X=0} \quad (A.3)$$

where K_M is given by

$$K_M = \int d^2 z_1 d^2 z'_1 d^2 z_2 d^2 z'_2 \Delta_0(z_1, z'_1) \Delta_0(z_2, z'_2) \left(\frac{\delta}{\delta X^i(z_1)} \frac{\delta}{\delta X^i(z_2)} L_M \right) \frac{\delta}{\delta X^i(z'_1)} \frac{\delta}{\delta X^j(z'_2)}. \quad (A.4)$$

Actually the r.h.s. (A.3) gets multiplied with an X -dependent function $F(X)$ from the left as in (3.4) which, however, can be omitted due to the $\Big|_{X=0}$ prescription. Obviously we have the correspondence $\tilde{D}_M = K_M/2$, and from (A.4) we can read off the relation between the non-trivial part $\tilde{\Delta}_M$ of the Green function on M and the operator L_M :

$$\tilde{\Delta}_M(z'_1, z'_2) = \int d^2 z_1 d^2 z_2 \Delta_0(z_1, z'_1) \Delta_0(z_2, z'_2) \left(\frac{\delta}{\delta X^i(z_1)} \frac{\delta}{\delta X^j(z_2)} L_M \right) \quad (A.5)$$

(Here we anticipated that the last term on the r.h.s. of (A.5) will be proportional to δ_{ij} .) This expression can be inverted after introducing Δ_0^{-1} , the inverse of the Green function Δ_0 , which satisfies

$$\int d^2 z_2 \Delta_0(z_1, z_2) \Delta_0^{-1}(z_2, z_3) = \delta^2(z_1 - z_3). \quad (A.6)$$

* It is possible to draw an analogy to the situation where an additional bilinear part in an action (e.g. a mass term) modifies the corresponding propagator, which can be represented by multiple insertions of a suitable vertex. Here, however, we need only a single insertion into each propagator, which is why the operator H_M is normal ordered.

(In view of (2.12) we have $\Delta_0^{-1}(z, z') = -(1/4\pi)\square_z\delta^2(z - z')$.) Now L_M can be written in terms of $\tilde{\Delta}_M$:

$$L_M = \frac{1}{2} \int d^2 z_1 d^2 z_2 d^2 z'_1 d^2 z'_2 X^i(z_1) X^i(z_2) \Delta_0^{-1}(z_1, z'_1) \Delta_0^{-1}(z_2, z'_2) \tilde{\Delta}_M(z'_1, z'_2, z_a, z_b) \quad (A.7)$$

Here we exhibited explicitly that $\tilde{\Delta}_M$ depends on certain points z_a, z_b on the world sheet. Since $\Delta_0^{-1}(z_1, z'_1)$ contains a δ -function and the application of Δ_0^{-1} to $\tilde{\Delta}_M(z'_1, z'_2, z_a, z_b)$ gives rise to δ -functions of the type $\delta^2(z'_1, z_a)$ etc., the expression (A.7) does not contradict the fact that L_M depends on fields X^i located at z_a, z_b . Using (A.5) and/or (A.7) it is now possible to verify that the operator L_{T_2} in (3.11) corresponds to $\tilde{\Delta}_{T_2}$ as given implicitly in (3.8) up to terms depending solely on z_i or z_j .

Appendix B

In this appendix we will, for completeness, provide the tools for computing the corrections to the equations of motion for the closed bosonic string fields due to the Riemann surface with the topology of a disc D_2 , which is relevant for a combined system of open and closed strings. Thus we intend to construct the “hole operator” H_{D_2} satisfying

$$[e^{D_{D_2}} O(X)] \Big|_{X=0} = [e^{D_0} H_{D_2} O(X)] \Big|_{X=0} \quad (B.1)$$

as in (A.1). Therefore we need an explicit expression for the Green function Δ_{D_2} . In close analogy to the procedure in [21] we will take the world sheet to be the entire complex plane except for a circular hole with radius a centered at z_0 . The corresponding Green function is given by

$$\Delta_{D_2}(z_1, z_2) = -\ln |z_1 - z_2|^2 - \ln \left| 1 - \frac{a^2}{(z_1 - z_0)(\bar{z}_2 - \bar{z}_0)} \right|^2. \quad (B.2)$$

For H_{D_2} we make the same ansatz as in (A.2), i.e. we write

$$H_{D_2} =: e^{L_{D_2}} : \quad (B.3)$$

From the relation between $\tilde{\Delta}_{D_2}$ and L_{D_2} , (A.5) and (A.7), we can now derive L_{D_2} :

$$L_{D_2} = \frac{1}{2} \sum_{n=1} \frac{a^{2n}}{n!(n-1)!} \frac{\partial^n}{\partial z_0^n} X(z_0) \frac{\partial^n}{\partial \bar{z}_0^n} X(z_0) \quad (B.4)$$

As in the case of the torus it now remains to factor out $V(CKV)_{S_2}^{-1}$ from $V(CKV)_{D_2}^{-1} \hat{Z}_{D_2}$ with $V(CKV)_{D_2} = V(SU(1, 1))$. Therefore we will, as in [21], integrate

$\hat{Z}_{D_2}$ over the radius a with an appropriate measure da/a and compensate by dividing by the volume $V(S_c)$ of scale transformations. Introducing

$$V_4 = V(SU(1, 1)) \times V(S_c) = a^2 \int \frac{d^2 z_1 d^2 z_2}{|z_1 - z_2|^2 |a^2 - z_1 \bar{z}_2|^2} \quad (B.5)$$

we thus have [21]

$$\frac{1}{V(SU(1, 1))} \hat{Z}_{D_2} = \int_0^1 \frac{da}{a} \frac{1}{V_4} \hat{Z}_{D_2}. \quad (B.6)$$

In order to factorize out $V(CKV)_{S_2}^{-1}$ with

$$V(CKV)_{S_2} = V(SL(2, C)) = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2} \quad (B.7)$$

from (B.6) we switch from the variable z_3 in (B.7) to z_0 with

$$z_3 = \frac{z_1 e^\gamma - z_2}{e^\gamma - 1}, \quad \gamma = \frac{z_0(z_1 - z_2)}{a^2 - z_1 \bar{z}_2}, \quad (B.8)$$

after which (B.7) becomes

$$c_\infty^{-1} \int \frac{d^2 z_1 d^2 z_2 d^2 z_0}{|z_1 - z_2|^2 |a^2 - z_1 \bar{z}_2|^2}. \quad (B.9)$$

Here c_∞^{-1} compensates for the fact that the z_0 -plane covers the z_3 -plane an infinite number of times. Formally we can thus write in view of (B.5)

$$V(SL(2, C)) \cong c_\infty^{-1} a^{-2} V_4 \int d^2 z_0 \quad (B.10)$$

or, with (B.6),

$$\frac{1}{V(SU(1, 1))} \hat{Z}_{D_2} \cong \frac{1}{V(SL(2, C)) \cdot c_\infty} \int_0^1 \frac{da}{a^3} \int d^2 z_0 \hat{Z}_{D_2}. \quad (B.11)$$

Of course this procedure was only heuristic; for a proper derivation of (B.11) we would have had to carefully regularize all the infinities in (B.5) to (B.11). As in sect. 3, $\hat{Z}_{D_2}$ is invariant under $SU(1, 1) \times$ scale transformations (which is required if one factors out the corresponding volumina) although it involves off-shell vertex operators, since we kept the dependence on the Liouville mode σ and thus have invariance under analytic coordinate transformations, which contain $SU(1, 1) \times$ scale transformations as a subgroup.

As in sect. 3 it is convenient to introduce

$$\hat{H}_{D_2} = c_\infty^{-1} \int_0^1 \frac{da}{a^3} \int d^2 z_0 H_{D_2}(a, z_0) \quad (B.12)$$

and subsequently we can proceed as in sect. 4 in the case of the torus. Now, of course, it is the integral da/a^3 which requires regularization for $a \rightarrow 0$ as $a^2 > \epsilon\epsilon^{-\sigma}$. Expanding H_{D_2} in powers of a ,

$$H_{D_2} = \sum_{n=0} a^{2n} h_n(x(z_0)), \quad (B.13)$$

$$h_0 = 1, \quad h_1 = \frac{1}{2} \partial_{z_0} X(z_0) \partial_{\bar{z}_0} X(z_0),$$

we find as in sect. 3 that divergences for $a \rightarrow 0$ only arise from the terms involving the operators h_0 and h_1 , which are proportional to the zero momentum tachyon and dilaton vertex operators.

Thus the structure of the loop-corrected equations of motion will be the same as in eq. (4.20) including self-energy corrections involving OPE coefficients appearing in the OPE of the hole operator H_{D_2} and the vertex operators, and including tadpole terms for the zero momentum tachyon and dilaton fields.

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Figure Captions

Fig. 1a: Parametrization of the torus in the u -plane

Fig. 1b: Parametrization of the torus in the z' -plane with $z' = \exp(2\pi i n)$

Fig. 1c: Schottky parametrization of the torus in the z -plane

Fig. 2: Diagrammatic representation of the iterative solution of the tree level equation of motion as in eq. (5.2).

Fig. 1a

Fig. 1b

Fig. 1c

Fig. 2