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THE STATIC POTENTIAL OF CLOSED BOSONIC MEMBRANES

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ABSTRACT

The static potential of spherical membranes is calculated in the leading $1/d$ approximation. We stabilize the spherical membrane by considering space times with topology $S^2 \times R^{d-2}$. We find that there is a critical radius R_0 below which the membrane develops tachyons and the quasi static picture breaks down.

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1. Introduction

Amongst the possible candidate rivals of superstrings as theories of unification of all the known forces, the only one which has a high degree of quantum consistency is the theory of the eleven-dimensional supermembrane [1-9]. The intrinsic non-linear dynamics of this extended object is much more complicated than the string dynamics and today very little is known about the mass spectrum of the membranes. Even the question of the massless excitations still remains open [5-8]. We believe that the non-linear dynamics of the supermembranes as well as of the bosonic membranes is an appropriate ground for the known analytic non-perturbative techniques such as $1/d$ expansions, d being the dimensionality of space-time.

In a previous paper [10] we introduced the $1/d$ expansion for the calculation of the mass gap of a rectangular membrane with fixed square boundary, as a first step towards understanding the application of this method to the case of bosonic membranes. In this work we continue up to the case of a spherical membrane.

The interest about the spherical membrane stems from its relation with $SU(\infty)$ Y.M. Quantum mechanics [11], and $SU(N)$ -Y.M. field theories for large N [12]. As in the case of the string with fixed ends [13-15] and the rectangular membrane with fixed boundary [10], we calculate the mass gap of the spherical membrane in a quasi static approximation. Since the spherical membrane does not have a boundary, to keep it static we embed the membrane in a space-time with topology $M = S^2 \times \mathbb{R}^{D-2}$ and we use as a classical solution the wrapped sphere on S^2 . We use functional techniques to sum to all orders, and in the leading $1/d$ approximation, the perturbative corrections around this classical solution. We find that there is a critical radius of the classical background $R = R_c$ below which the membrane develops tachyons and the quasi static picture breaks down.

The action of the membrane moving in a manifold M with metric $G_{\mu\nu}(X)$ is

$$S = \kappa \int d^3\xi \det^{1/2} [\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}] , \quad (1)$$

$$\alpha, \beta : 0, 1, 2 ; \quad \mu, \nu : 0, 1, 2, \dots, D-1$$

In our case $M = S^2 \times \mathbb{R}^{D-2}$ and the metric is:

$$\begin{aligned} G_{00} &= 1, \quad G_{D-1, D-1} = R^2, \quad G_{D-2, D-2} = R^2 \sin^2 \theta \\ G_{ij} &= 1, \quad i = 1, 2, \dots, D-3 \end{aligned} \quad (2)$$

and $G_{\mu\nu} = 0$ for all other μ, ν , where $\bar{\theta}, \bar{\phi}$ are the usual polar coordinates on S^2 , R is the fixed radius on the sphere S^2 and $D-1, D-2$ refer to the $\bar{\theta}$ and $\bar{\phi}$ coordinates respectively. The corresponding equations of motion for the action (1) are:

$$\partial_\alpha \sqrt{g} g^{\alpha\beta} G_{\mu\nu} \partial_\beta X^\nu = 1/2 \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\nu \partial_\beta X^\lambda (\partial G_{\nu\lambda} / \partial X^\mu) \quad (3)$$

$$\mu, \lambda, \nu = 0, 1, 2, \dots, D-1 ; \quad \alpha, \beta = 0, 1, 2 \quad (\tau, \theta, \phi)$$

We check that a solution of eq. (3) is the once wrapped sphere on S^2

$$X_{cl}^0 \equiv \tau = \xi^0, \quad X_{cl}^{D-1} \equiv \bar{\theta} = \theta \equiv \xi^1, \quad X_{cl}^{D-2} \equiv \bar{\phi} = \phi \equiv \xi^2 \quad (4)$$

$$X_{cl}^\perp = X_{cl}^i = 0, \quad i = 1, 2, \dots, D-3.$$

The action for the fluctuations is

$$S = \kappa \int d^3\xi \det^{1/2} [g^c_{\alpha\beta} + \partial_\alpha X^\perp \partial_\beta X^\perp] \quad (5)$$

Here we use the background field gauge

$$X^0 = X_{cl}^0, \quad X^{D-1} = X_{cl}^{D-1}, \quad X^{D-2} = X_{cl}^{D-2} \quad (6)$$

in which there are no F-P ghosts [10]. The classical background metric g^c is:

$$g^c_{\alpha\beta} = \partial_\alpha X_{cl}^\mu \partial_\beta X_{cl}^\nu G_{\mu\nu} \quad (7)$$

$$g^c_{00} = 1, \quad g^c_{\bar{\theta}\bar{\theta}} = R^2, \quad g^c_{\bar{\phi}\bar{\phi}} = R^2 \sin^2 \bar{\theta} \quad (7\alpha)$$

$$g_{\alpha\beta} = 0, \quad \alpha \neq \beta; \quad \alpha, \beta = 0, 1, 2. \quad (7\beta)$$

For convenience we call the fluctuation metric g^q ,

$$g^q_{\alpha\beta} = \partial_\alpha X^\perp \partial_\beta X^\perp \quad (8)$$

To calculate the potential $V(R)$ in the large d approximation we must evaluate

$$V(R) = -\lim_{T \rightarrow \infty} (1/T) \log \int \mathcal{D} X^\perp e^{-S}$$

with the boundary condition for $X^\perp$

$$X^\perp(\tau = 0, \phi, \theta) = 0, \quad (9\alpha)$$

$$X^\perp(\tau = T, \phi, \theta) = 0, \quad (9\beta)$$

and moreover $X^\perp(\tau, \theta, \phi)$ must be functions defined on S^2 . For this purpose we introduce as in [10], Lagrange multipliers, singlets under $O(d-3)$, $\lambda_{\alpha\beta}$, $\sigma_{\alpha\beta}$

$$Z = \int \mathcal{D}X \mathcal{D}\lambda \mathcal{D}\sigma e^{-S(x, \lambda, \sigma)} \quad (10)$$

$$S(x, \lambda, \sigma) = \kappa \int d^3\xi \det^{1/2} g^c \det^{1/2} [1 + \sigma] + \kappa/2 \int d^3\xi \sqrt{h} \lambda^{\alpha\beta} ((g_c^{-1} g_q)_{\alpha\beta} - \sigma_{\alpha\beta}) \quad (11)$$

where $\sqrt{h} = R^2 \sin\theta$.

We integrate over $X^\perp$ and we find

$$Z = \int \mathcal{D}\lambda \mathcal{D}\sigma e^{-S_{\text{eff}}} \quad (12)$$

$$S_{\text{eff}} = \kappa \int d^3\xi \{ \det^{1/2} g^c \det^{1/2} [1 + \sigma] - (1/2) \sqrt{h} \lambda^{\alpha\beta} \sigma_{\alpha\beta} \} \\ + [(D-3)/2] \text{Tr} \log [-h^{-1/2} \partial_\alpha (\lambda g_c^{-1})^{\alpha\beta} \sqrt{h} \partial_\beta] \quad (13)$$

The solution of the saddle point equations

$$\delta S_{\text{eff}} / \delta \sigma_{\alpha\beta} = 0, \quad \delta S_{\text{eff}} / \delta \lambda_{\alpha\beta} = 0 \quad (14)$$

is simplified by the following observations. The first equation,

$$\delta S_{\text{eff}} / \delta \sigma_{\alpha\beta} = (\kappa/2) \{ \det^{1/2} [1 + \sigma] [(1 + \sigma)^{-1}]^{\alpha\beta} - \lambda^{\alpha\beta} \} = 0 \quad (15)$$

implies that the symmetric matrices $\sigma_{\alpha\beta}$, $\lambda^{\alpha\beta}$ are simultaneously diagonalizable. Secondly, the spherical symmetry of the classical solution and the time translational invariance for large times suggest that the ground state configurations of $\sigma_{\alpha\beta}$ and $\lambda^{\alpha\beta}$ must be constant matrices,

$$\sigma_{\alpha\beta} = \text{diag} (\sigma_0, \sigma_1, \sigma_1), \quad \lambda^{\alpha\beta} = \text{diag} (\lambda^0, \lambda^1, \lambda^1) \quad (16)$$

Then the operator inside the Trlog becomes

$$\Delta \equiv [-h^{-1/2} \partial_\alpha (\lambda g_c^{-1})^{\alpha\beta} \sqrt{h} \partial_\beta] = -\lambda^0 \partial_\tau^2 - \lambda^1 [(1/\sin\theta)(\partial_\theta \sin\theta \partial_\theta) + (1/\sin^2\theta) \partial_\phi^2] \quad (17)$$

Under this ansatz for the "vacuum" values of $\sigma_{\alpha\beta}$, $\lambda^{\alpha\beta}$, eqs.(14) become algebraic. We calculate first the Trlog (see ref.[10])

$$\text{Tr} \log \Delta = -\alpha (\lambda^0 / \lambda^1)^{1/2} T/R, \quad \text{for large } T, \quad (18)$$

$$\alpha = -\sum_{l=0}^{\infty} (2l+1) \sqrt{l(l+1)} \quad (19)$$

The constant α has been calculated in a perturbative, lowest order, calculation of the zero point energy of a spherical membrane [16]. We disagree with the numerical value of α given by this reference. We find using zeta-function regularization

$$\alpha = 0.2651 \quad (20)$$

We observe that the sign difference of the $\text{Trlog } \Delta$ with respect to the open membrane [10] is the main factor for the completely different behaviour of the spherical membrane for small radii R and the appearance of tachyons (in contrast with the open membrane).

From now on, the steps are quite similar to the case of the open membrane [10] and we quote only the results. The potential $V(R)$ is expressed as

$$V(R) = V_c(R) f_q(x) \quad (21)$$

$$V_c(R) = \kappa R^2$$

$$f_q(x) = (2/\sqrt{3}) \sqrt{x} \sqrt{(x-\rho)} \quad (22)$$

$$\rho = (3\sqrt{3}\alpha/8\pi) ((d-3)/\kappa R^3) \quad (23)$$

$$x = \cos[-\pi/3 + 1/3 \cos^{-1} \rho] \quad (24)$$

The choice of the solution (24) from the three roots of a cubic equation [10], is dictated by the fact that for large R (small ρ), $f_q(x)$ should approach 1. Indeed for large R , we can apply perturbation theory in $1/R$.

$$\begin{aligned} T \cdot V_{\text{pert.}}(R) &= T \cdot V_c(R) + ((d-3)/2) \text{Trlog} [-\partial_\tau^2 - (1/\sin\theta)(\partial_\theta \sin\theta \partial_\theta) - (1/\sin^2\theta) \partial_\phi^2] \\ &= T \cdot V_c(R) + ((d-3)/2)(T/R)\alpha \end{aligned} \quad (25)$$

From eq. (22) we see that there is a critical distance R_0 , (ρ_0) such that the potential vanishes:

$$x(\rho_0) = \rho_0 = \cos[-\pi/3 + 1/3 \cos^{-1} \rho_0] \quad (26)$$

and below which $x(\rho)$ is real and positive and smaller than one ($x < 1$) (up to some other distance R_1) but $x - \rho$ becomes negative and so $V(R)$ imaginary. This represents a tachyonic instability of the system and the breaking of the quasi static approximation.

From (26) we find

$$\rho_0 = 1/\sqrt{2} \quad (27)$$

or

$$R_0 = (3^3/2^5)^{1/6} ((d-3)/\kappa)^{1/3} \quad (28)$$

The corresponding tachyonic mass is:

$$M_0 = \kappa R_0^2 = (3/2^{5/3}) \cdot ((d-3)^2 \kappa)^{1/3} \quad (29)$$

We conclude with the following observations:

In the saddle point approximation for large d there is a complex solution, for the ground state Lagrange multiplier fields, which gives, for small R , negative repulsive potential with a finite maximum at $R = 0$. This solution is not continuously connected with the perturbative regime of large R . Instead this solution for large R gives $V(R) \rightarrow -V_c(R)$. At this moment we do not have an answer for the possibility of a physical mechanism which will connect the inner solution ($R < R_0$) and the exterior one ($R > R_0$).

We calculated the fluctuation matrix of second derivatives of the effective action with respect to the eigenvalues and we found that there are two positive and two negative eigenvalues. Since the Lagrange multiplier fields $\sigma_{\alpha\beta}$ and $\lambda^{\alpha\beta}$ are real and imaginary respectively, the above result guarantees the stability of one saddle point solution for the potential.

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